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Study of weak solutions for parabolic variational inequalities with nonstandard growth conditions

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Abstract

In this paper, we study the degenerate parabolic variational inequality problem in a bounded domain. First, the weak solutions of the variational inequality are defined. Second, the existence and uniqueness of the solutions in the weak sense are proved by using the penalty method and the reduction method.

MSC: 35B40; 35K35

Keywords: Parabolic variational inequality; Weak solution; Penalty method; Existence

1 Introduction

This article concerned with initial-boundary problem whose model is

$$\begin{cases} \min\{Lu, u(x,0)-u_0(x)\}=0, & (x,t)\in Q_T, \\ u(x,t)=0, & (x,t)\in \Gamma_T, \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

with

$$Lu = u_t - \operatorname{div}\big(a(u)|\nabla u|^{p(x,t)-2}\nabla u\big) - f(x,t), \quad a(u) = u^\sigma + d_o,$$

where $\Omega \subset \mathbb{R}^+$ is a bounded simply connected domain, $Q_T = \Omega \times (0,T]$, and Γ_T denotes the lateral boundary of the cylinder Q_T.

This type of variational inequality was studied initially by Chen and Yi [1], who proposed the equation

$$\begin{cases} \frac{\partial}{\partial\tau}V - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V - (r-\frac{1}{2}\sigma^2)\frac{\partial}{\partial x}V + rV \geq 0 & \text{in } \Omega_T, \\ V \geq g(x), & \text{in } \Omega_T, \\ (\frac{\partial}{\partial\tau}V - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V - (r-\frac{1}{2}\sigma^2)\frac{\partial}{\partial x}V + rV)(V-g(x)) = 0 & \text{in } \Omega_T, \\ V(t,x) = 0 & \text{on } \partial\Omega_T, \\ V(x,0) = g(x) & \text{in } \Omega \end{cases} \tag{2}$$

for modeling the American option. When r and σ are positive constant, the existence and uniqueness of solutions to problem (4) were also studied in [2–4].

In 2014, the authors in [5] discussed the problem

$$\begin{cases} u_t - Lu - F(x,t,u,\nabla u) \geq 0 & \text{in } Q_T, \\ u(x,t) \geq u_0(x) & \text{in } Q_T, \\ (u_t - Lu - F(x,t,u,\nabla u))(u - u_0(x)) = 0 & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ u(x,t) = g(x) & \text{on } \partial\Omega \times (0,T) \end{cases}$$

with second-order elliptic operator

$$L(x,t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a^{ij}(x,t)\frac{\partial}{\partial x_i}\right) - \sum_{i,j=1}^{d} b^i(x,t)\frac{\partial}{\partial x_i} - c(x,t).$$

They proved the existence and uniqueness of a solution to this problem with some conditions on u_0, F, and L. Later, the authors in [6, 7] extended the relative conclusions with the assumption that $a(u)$ and $p(x)$ are two positive constants. The author discussed the existence and uniqueness of a solution by the penalty method.

The existence and uniqueness of such a problem with the assumption that $p(x)$ and $a(u)$ are variables were less studied.

The aim of this paper is to study the existence and uniqueness of solutions for a degenerate parabolic variational inequality problem. Throughout the paper, we assume that the exponent $p(x,t)$ is continuous in $Q = \overline{Q_T}$ with logarithmic module of continuity:

$$1 < p^- = \inf_{(x,t)\in Q} p(x,t) \leq p(x,t) \leq p^+ = \sup_{(x,t)\in Q} p(x,t) < \infty, \tag{3}$$

$$\forall z = (x,t) \in Q, \qquad \xi = (y,s) \in Q_T, \qquad |z - \xi| < 1, \qquad \bigl|p(z) - p(\xi)\bigr| \leq \omega\bigl(|z - \xi|\bigr), \tag{4}$$

where

$$\limsup_{\tau\to 0^+} \omega(\tau) \ln\frac{1}{\tau} = C < +\infty.$$

The outline of this paper is as follows. In Section 2, we introduce the function spaces of Orlicz-Sobolev type, give the definition of a weak solution to the problem, and prove the existence and uniqueness. Section 3 is devoted to the proof of the existence and uniqueness of the solution obtained in Section 2.

2 Basic spaces and the main results

To study our problems, let us introduce the Banach spaces:

$$L^{p(x,t)}(Q_T) = \left\{ u(x,t) \middle| u \text{ is measurable in } Q_T, A_{p(\cdot)}(u) = \int\int_{Q_T} |u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t < \infty \right\},$$

$$\|u\|_{p(\cdot)} = \inf\bigl\{\lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1\bigr\},$$

$$V_t(\Omega) = \bigl\{u | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p(x,t)}(\Omega)\bigr\}, \qquad \|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + |\nabla u|_{p(\cdot)\Omega},$$

$$W(Q_t) = \{u : [0,T] \mapsto V_t(\Omega) | u \in L^2(Q_t), |\nabla u| \in L^{p(x,t)}(Q_T), u = 0 \text{ on } \Gamma_T\},$$

$$\|u\|_{W(Q_t)} = \|u\|_{2,Q_T} + |\nabla u|_{p(x,t),Q_T}$$

and denote by $W'(Q_T)$ the dual of $W(Q_T)$ with respect to the inner product in $L^2(Q_T)$.

In spirit of [3] and [4], we introduce the following maximal monotone graph:

$$G(\lambda) = \begin{cases} 0, & \lambda > 0, \\ [0, +\infty), & \lambda = 0. \end{cases}$$

In addition, we define the following function class for the solution:

$$B = \{u \in W(Q_T) \cap L^\infty(0, T; L^\infty(\Omega))\}.$$

Definition 2.1 A pair $(u, \xi) \in B \times L^\infty(\Omega_T)$ is called a weak solution of problem (1) if (a) $u(x,t) \le u_0(x)$, (b) $u(x,0) = u_0(x)$, (c) $\xi \in G(u - u_0)$, (d) for all $t_1, t_2 \in [0,T]$, the following identity holds:

$$\int_{t_1}^{t_2} \int_\Omega \left[u\varphi_t - (u^\sigma + d_0)|\nabla u|^{p(x,t)-2} \nabla u \nabla \varphi + f(x,t)\varphi - \xi\varphi\right] \mathrm{d}x\,\mathrm{d}t = \int_\Omega u\varphi\,\mathrm{d}x\Big|_{t_1}^{t_2}.$$

The main theorem in this section is the following:

Theorem 2.1 *Let* $p(x,t)$ *satisfy conditions* (3)–(4). *Suppose also that the following conditions hold*:

(H_1) $\max\{1, \frac{2N}{N+2}\} < p^- < N$, $2 \le \sigma < \frac{2p^+}{p^+-1}$,

(H_2) $u_0 \ge 0, f \ge 0$, $\|u_0\|_{\infty,\Omega} + \int_0^T \|f(x,t)\|_{\infty,\Omega}\,\mathrm{d}t = K(T) < \infty$.

Then problem (1) *has at least one weak solution in the sense of Definition* 2.1.

Theorem 2.2 *Suppose that the conditions in Theorem* 2.1 *are fulfilled and* $p^+ \ge 2$. *Then problem* (1) *admits a unique solution in the sense of Definition* 2.1.

3 Proof of the main results

In this section, we consider the family of auxiliary parabolic problems

$$\begin{cases} L_\varepsilon u_\varepsilon + \beta_\varepsilon(u_\varepsilon - u_0) = 0, & (x,t) \in Q_T, \\ u(x,t) = \varepsilon, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x) + \varepsilon, & x \in \Omega. \end{cases} \tag{5}$$

Here, M is a positive parameter to be chosen later. Moreover,

$$L_\varepsilon u_\varepsilon = \partial_t u_\varepsilon - \operatorname{div}\left(a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\right) - f(x,t),$$

$$0 < d_0 \le a_{\varepsilon,M}(u) = \left(\min(|u|^2, M^2) + \varepsilon^2\right)^{\frac{\sigma}{2}} + d_0 \le \left(M^2 + 1\right) + d_0, \quad 0 < \varepsilon < 1,$$

and $\beta_\varepsilon(\cdot)$ is the penalty function satisfying

$$0 < \varepsilon \le 1, \qquad \beta_\varepsilon(x) \in C^2(R), \qquad \beta_\varepsilon(x) \le 0, \qquad \beta_\varepsilon(0) = -1,$$
$$\beta'_\varepsilon(x) \ge 0, \qquad \beta''_\varepsilon(x) \le 0, \qquad \lim_{\varepsilon\to 0} \beta_\varepsilon(x) = \begin{cases} 0, & x > 0, \\ -\infty, & x < 0. \end{cases} \tag{6}$$

Following a similar method as in [6], we can prove that the regularized problem has a unique weak solution $u_\varepsilon(x,t) \in W(Q_T) \cap L^2(Q_T)$, $\partial_t u_\varepsilon(x,t) \in W'(Q_T)$ satisfying the following integral identities:

$$\int_{t_1}^{t_2} \int_\Omega \bigl[u_\varepsilon \varphi_t - a_\varepsilon, M(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2} \nabla u_\varepsilon \nabla\varphi + f(x,t)\varphi\bigr] \, dx \, dt$$
$$= \int_{t_1}^{t_2} \int_\Omega \beta_\varepsilon(u_\varepsilon - u_0)\varphi \, dx \, dt + \int_\Omega u_\varepsilon \varphi \, dx \bigg|_{t_1}^{t_2} \tag{7}$$

and

$$\int_{t_1}^{t_2} \int_\Omega \bigl[(\partial_t u_\varepsilon)\varphi + a_{\varepsilon,M}(u_\varepsilon)|\nabla u|^{p(x,t)-2} \nabla u_\varepsilon \nabla\varphi - f(x,t)\varphi + \beta_\varepsilon(u_\varepsilon - u_0)\varphi\bigr] \, dx \, dt = 0. \tag{8}$$

We start with two preliminary results that will be used several times.

Lemma 3.1 *Let $M(s) = |s|^{p(x,t)-2}s$. Then for all $\xi, \eta \in \mathrm{R}^N$,*

$$\bigl(M(\xi) - M(\eta)\bigr)(\xi - \eta)$$
$$\ge \begin{cases} 2^{-p(x,t)}|\xi - \eta|^{p(x,t)}, & 2 \le p(x,t) < \infty, \\ (p(x,t) - 1)|\xi - \eta|^2(|\xi|^{p(x,t)} + |\eta|^{p(x,t)})^{\frac{p(x,t)-2}{p(x,t)}}, & 1 < p(x,t) < 2. \end{cases}$$

Lemma 3.2 (Comparison principle) *Assume that $2 < \sigma < \frac{2p^+}{p^+-1}$, $p^+ \ge 2$, and u and v are in $W(Q_T) \cap L^\infty(0,T;L^\infty(\Omega))$. If $L_\varepsilon u \ge L_\varepsilon v$ in Q_T and if $u(x,t) \le v(x,t)$ on ∂Q_T, then $u(x,t) \le v(x,t)$ in Q_T.*

Proof We argue by contradiction. Suppose $u(x,t)$ and $v(x,t)$ satisfy $L_\varepsilon u \ge L_\varepsilon v$ in Q_T and there is $\delta > 0$ such that for $0 < \tau \le T$, $w = u - v > \delta$ on the set $\Omega_\delta = \Omega \cap \{x : w(x,t) > \delta\}$, and $\mu(\Omega_\delta) > 0$. Let

$$F_\varepsilon(\xi) = \begin{cases} \frac{1}{\alpha-1}\varepsilon^{1-\alpha} - \frac{1}{\alpha-1}\xi^{1-\alpha} & \text{if } \xi > \varepsilon, \\ 0 & \text{if } \xi \le \varepsilon, \end{cases}$$

where $\delta > 2\varepsilon > 0$ and $\alpha = \frac{\sigma}{2}$. Let a test-function $\xi = F_\varepsilon(w) \in Z$ in (8). Then

$$0 \ge \int\int_{Q_T} \bigl[w_t F_\varepsilon(w) + \bigl(v^\sigma + d_0\bigr)\bigl(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla F_\varepsilon(w)\bigr] \, dx \, dt$$
$$+ \int\int_{Q_T} \bigl(u^\sigma - v^\sigma\bigr)|\nabla u|^{p(x,t)-2}\nabla u \nabla F_\varepsilon(w) \, dx \, dt = J_1 + J_2 + J_3, \tag{9}$$

where $Q_{\varepsilon,\tau} = Q_\tau \cap \{(x,t) \in Q_\tau | w > \varepsilon\}$,

$$J_1 = \int\int_{Q_T} w_t F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t, \qquad J_2 = \int\int_{Q_T} \big(u^\sigma - v^\sigma\big) w^{-\alpha} |\nabla u|^{p(x,t)-2} \nabla u \nabla w\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_3 = \int\int_{Q_T} \big(v^\sigma + d_0\big) w^{-\alpha} \big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla w\,\mathrm{d}x\,\mathrm{d}t.$$

Now, let $t_0 = \inf\{t \in (0,\tau] : w > \varepsilon\}$. Then we estimate J_1, J_2, and J_3 as follows:

$$\begin{aligned} J_1 &= \int\int_{Q_{\varepsilon,\tau}} w_t F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left(\int_0^{t_0} w_t F_\varepsilon(w)\,\mathrm{d}t + \int_{t_0}^{\tau} w_t F_\varepsilon(w)\,\mathrm{d}t\right)\mathrm{d}x \\ &\geq \int_\Omega \int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,\mathrm{d}s\,\mathrm{d}x \geq \int_{\Omega_\delta}\int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\geq \int_{\Omega_\delta} (w - 2\varepsilon) F_\varepsilon(2\varepsilon)\,\mathrm{d}x \geq (\delta - 2\varepsilon) F_\varepsilon(2\varepsilon)\mu(\Omega_\delta). \end{aligned} \tag{10}$$

Let us first consider the case $p^- \geq 2$. By the first inequality of Lemma 3.1 we get

$$\begin{aligned} J_2 &= \int\int_{Q_{\varepsilon,\tau}} \big(v^\sigma + d_0\big) w^{-\alpha}\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla w\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int\int_{Q_{\varepsilon,\tau}} \big(v^\sigma + d_0\big) w^{-\alpha} 2^{-p(x,t)} |\nabla w|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \\ &\geq 2^{-p^+} \int\int_{Q_{\varepsilon,\tau}} \big(v^\sigma + d_0\big) w^{-\alpha} |\nabla w|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \geq 0. \end{aligned} \tag{11}$$

Noting that $\frac{p(x,t)}{p(x,t)-1} \geq \frac{p^+}{p^+-1} \geq \frac{\sigma}{2} = \alpha > 1$ and applying Young's inequality, we can estimate the integrand of J_3 in the following way:

$$\begin{aligned} &\big|\big(u^\sigma - v^\sigma\big) w^{-\alpha} |\nabla w|^{p(x,t)-2}\nabla u \nabla w\big| \\ &\quad = \left|\sigma w \int_0^1 \big(\theta u + (1-\theta)v\big)^{\sigma-1}\,d\theta\, w^{-\alpha}|\nabla w|^{p(x,t)-2}\nabla u\nabla w\right| \\ &\quad \leq \frac{C}{w^\alpha}\left[\frac{v^\sigma + d_0}{C}\right]|\nabla w|^{p(x,t)} + C_1\big(\sigma, d_0, K(T), p^\pm\big)|w|^{p'(x,t)}|\nabla u|^{p(x,t)}] \\ &\quad = \frac{(v^\sigma + d_0)}{2^{p^+ +1} w^\alpha}|\nabla w|^{p(x,t)} + C_1\big(\sigma, d_0, K(T), p^\pm\big)|w|^{p'(x,t)-\alpha}|\nabla u|^{p(x,t)} \\ &\quad \leq \frac{(v^\sigma + d_0)}{2^{p^+ +1} w^\alpha}|\nabla w|^{p(x,t)} + C_1\big(\sigma, d_0, K(T), p^\pm\big)|\nabla u|^{p(x,t)}. \end{aligned} \tag{12}$$

Substituting (12) into J_3, we get

$$J_3 \leq \frac{1}{2} J_2 + C \int\int_{Q_{\varepsilon,\tau}} |\nabla u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Second, we consider the case $1 < p^- \le p(x,t) < 2$, $p^+ \ge 2$. According to the second inequality of Lemma 3.1, it is easily seen that the following inequalities hold:

$$\begin{aligned} J_2 &= \int\int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} \bigl(|\nabla w|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\bigr)\nabla w \,\mathrm{d}x\,\mathrm{d}t \\ &\ge (p^- - 1)\int\int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} \bigl(|\nabla w| + |\nabla v|\bigr)^{p(x,t)-2} |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{14}$$

Using the conditions $1 < \alpha \le \frac{p^+}{p^+-1} \le 2$ and Young's inequality, we can evaluate the integrand of J_3 as follows:

$$\begin{aligned} &\bigl|(u^\sigma - v^\sigma) w^{-\alpha} |\nabla w|^{p(x,t)-2}\nabla u \nabla w\bigr| \\ &\quad = \left|\sigma w \int_0^1 \bigl(\theta u + (1-\theta)v\bigr)^{\sigma-1}\,d\theta\, w^{-\alpha}|\nabla w|^{p(x,t)-2}\nabla u\nabla w\right| \\ &\quad \le \frac{(v^\sigma + d_0)(p^- - 1)}{2w^\alpha}\bigl(|\nabla w| + |\nabla v|\bigr)^{p(x,t)-2}|\nabla w|^2 \\ &\qquad + C_1\bigl(\sigma, d_0, K(T), p^\pm\bigr)|w|^{2-\alpha}|\nabla w| + |\nabla v|)^{p(x,t)}. \end{aligned} \tag{15}$$

Plugging (15) into J_3, we get

$$J_3 \le \frac{1}{2}J_2 + C\int\int_{Q_{\varepsilon,\tau}} |\nabla w| + |\nabla v|)^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t. \tag{16}$$

Plugging estimates (10), (11), (13) and (10), (14), (16) into (9) and dropping the nonnegative terms, we arrive at the inequality

$$(\delta - 2\varepsilon)\bigl(1 - 2^{1-\alpha}\bigr)\varepsilon^{1-\alpha}\mu(\Omega_\delta) \le \tilde{C} \tag{17}$$

with a constant $\tilde{C}$ independent of ε.

Notice that $\lim_{\varepsilon\to 0}(\delta - 2\varepsilon)(1 - 2^{1-\alpha})\varepsilon^{1-\alpha}\mu(\Omega_\delta) = +\infty$, a contradiction. This means that $\mu(\Omega_\delta) = 0$ and $w \le 0$ a.e. in Q_τ. □

Lemma 3.3 *Let u_ε be weak solutions of* (5). *Then*

$$u_{0\varepsilon} \le u_\varepsilon \le |u_0|_\infty + \varepsilon, \tag{18}$$

$$u_{\varepsilon_1} \le u_{\varepsilon_2} \quad \textit{for } \varepsilon_1 \le \varepsilon_2. \tag{19}$$

Proof First, we prove $u_\varepsilon \ge u_{0\varepsilon}$ by contradiction. Assume that $u_\varepsilon \le u_{0\varepsilon}$ in Q_T^0, $Q_T^0 \subset Q_T$. Noting that $u_\varepsilon \ge u_{0\varepsilon}$ on ∂Q_T, we may assume that $u_\varepsilon = u_{0\varepsilon}$ on ∂Q_T^0. With (5) and letting $t = 0$, we deduce that

$$Lu_{0\varepsilon} = \beta_\varepsilon(u_{0\varepsilon} - u_{0\varepsilon}) = -1, \tag{20}$$

$$Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon - u_{0\varepsilon}) \le -1. \tag{21}$$

From Lemma 3.2 we conclude that

$$u(x,t) \le u_{0\varepsilon}(x) \quad \text{for any } (x,t) \in \Omega_T, \tag{22}$$

obtaining a contradiction.

Second, we pay attention to $u_\varepsilon(t,x) \le |u_0|_\infty + \varepsilon$. Applying the definition of $\beta_\varepsilon(\cdot)$, we have

$$L\big(|u_0|_\infty + \varepsilon\big) = 0, \quad Lu_\varepsilon \le 0. \tag{23}$$

From (5) it is easy to prove that $u_\varepsilon(x,t) \ge \varepsilon$ on $\partial\Omega \times (0,T)$ and $u_{0\varepsilon}(x) \ge \varepsilon$ in Ω. Thus, combining (21) and (23) and repeating Lemma 3.3, we have

$$u_\varepsilon(x,t) \ge \varepsilon \quad \text{in } \Omega \times (0,T). \tag{24}$$

Third, we aim to prove (19). Since

$$Lu_{\varepsilon_1} - \beta_{\varepsilon_1}(u_{0\varepsilon_1} - u_{\varepsilon_1}) = 0, \tag{25}$$

$$Lu_{\varepsilon_2} - \beta_{\varepsilon_2}(u_{0\varepsilon_2} - u_{\varepsilon_2}) = 0. \tag{26}$$

It follows by $\varepsilon_1 \le \varepsilon_2$ and the definition of $\beta_\varepsilon(\cdot)$ that

$$\begin{aligned} &\partial_t u_{\varepsilon_2} - Lu_{\varepsilon_2} - \beta_{\varepsilon_1}(u_{0\varepsilon_1} - u_{\varepsilon_2}) \\ &\quad = \beta_{\varepsilon_2}(u_{0\varepsilon_2} - u_{\varepsilon_2}) - \beta_{\varepsilon_1}(u_{0\varepsilon_1} - u_{\varepsilon_2}) \ge \beta_{\varepsilon_2}(u_{0\varepsilon_2} - u_{\varepsilon_2}) - \beta_{\varepsilon_1}(u_{0\varepsilon_2} - u_{\varepsilon_2}) \ge 0. \end{aligned} \tag{27}$$

Thus, Lemma 3.3 can be proved by combining initial and boundary conditions in (5). □

Moreover, with (18), we assert that there exists a subsequence ε (still denoted by ε) such that

$$u_\varepsilon \to u \in L^p\big(0,T; W_0^{1,p}(\Omega_T)\big) \quad \text{as } \varepsilon \to 0, \tag{28}$$

$$u_\varepsilon \ge u \ge 0 \quad \text{for any } \varepsilon > 0. \tag{29}$$

Lemma 3.4 *Let u_ε be a solution of problem* (5). *For any $\varepsilon > 0$, we have*

$$\|u_\varepsilon\|_{\infty,Q_T} \le \|u_0\|_{\infty,\Omega} + \int_0^T \big\|f(x,t)\big\|_{\infty,\Omega}\,\mathrm{d}t = K(T) < \infty. \tag{30}$$

Proof Define

$$u_{\varepsilon M} = \begin{cases} M & \text{if } u_\varepsilon > M, \\ u_\varepsilon & \text{if } |u_\varepsilon| \le M, \\ -M & \text{if } u_\varepsilon < -M. \end{cases}$$

Choosing $u_{\varepsilon M}^{2k-1}$ as a test-function in (8) and letting $t_1 = t$ and $t_2 = t + h$, we conclude that

$$\frac{1}{2k}\int_t^{t+h}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_\Omega u_{\varepsilon M}^{2k}\,dx\biggr)\,\mathrm{d}t + \int_t^{t+h}\int_\Omega (2k-1)a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon M}|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_t^{t+h}\int_\Omega f u_{\varepsilon M}^{2k-1}\,\mathrm{d}x\,\mathrm{d}t - \int_t^{t+h}\int_\Omega \beta_\varepsilon u_{\varepsilon M}^{2k-1}\,\mathrm{d}x. \tag{31}$$

Letting $h \to 0$ and applying Lebesgue's dominated convergence theorem, we have that, for all $t \in (0, T)$,

$$\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{\varepsilon M}^{2k}\,\mathrm{d}x + \int_\Omega (2k-1)a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon M}|^{p(x,t)}\,\mathrm{d}x$$
$$= \int_\Omega f u_{\varepsilon M}^{2k-1}\,\mathrm{d}x - \int_\Omega \beta_\varepsilon u_{\varepsilon M}^{2k-1}\,\mathrm{d}x. \tag{32}$$

Using Holder's inequality, we obtain

$$\left|\int_\Omega f u_{\varepsilon M}^{2k-1}\,dx\right| \le \bigl\|u_{\varepsilon M}(\cdot,t)\bigr\|_{2k,\Omega}^{2k-1}\cdot\bigl\|f(\cdot,t)\bigr\|_{2k,\Omega}, \quad k = 1,2,\ldots,$$
$$\left|\int_\Omega \beta_\varepsilon u_{\varepsilon M}^{2k-1}\,\mathrm{d}x\right| \le \int_\Omega u_{\varepsilon M}^{2k-1}\,\mathrm{d}x \le \bigl\|u_{\varepsilon M}(\cdot,t)\bigr\|_{2k,\Omega}^{2k-1},$$

whence

$$\|u_{\varepsilon M}\|_{2k,\Omega}^{2k-1}\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\|u_{\varepsilon M}\|_{2k,\Omega}\bigr) + (2k-1)\int_\Omega a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x$$
$$\le \|u_{\varepsilon M}\|_{2k,\Omega}^{2k-1}\cdot\bigl\|f(\cdot,t)\bigr\|_{2k,\Omega} + C(T)\|u_{\varepsilon M}\|_{2k,\Omega}^{2k-1}, \quad k = 1,2,\ldots. \tag{33}$$

By integration over $(0, t)$, for all t, we have

$$\bigl\|u_{\varepsilon M}(\cdot,t)\bigr\|_{2k,\Omega} \le \bigl\|u_{\varepsilon M}(\cdot,0)\bigr\|_{2k,\Omega} + \int_0^T \|f\|_{2k,\Omega}\,\mathrm{d}t + C(T), \quad \forall k \in \mathbb{N}.$$

Then, as $k \to \infty$,

$$\bigl\|u_{\varepsilon M}(\cdot,t)\bigr\|_{\infty,\Omega} \le \bigl\|u_{\varepsilon M}(\cdot,0)\bigr\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega}\,\mathrm{d}t$$
$$\le \|u_0\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega}\,\mathrm{d}t + C(T) = K(T).$$

□

If we chose $M > K(T)$ then $u_{\varepsilon M}(\cdot,t) \le \sup|u_{\varepsilon M}(\cdot,t)| \le K(T) < M$, and therefore $u_{\varepsilon M}(\cdot,t) = u_\varepsilon(\cdot,t)$.

Corollary 3.1 *Choosing M large enough, we have*

$$\min\bigl\{u_\varepsilon^2, M^2\bigr\} = u_\varepsilon^2 \quad \textit{and} \quad a_{\varepsilon,M}\bigl(u_{\varepsilon M}\bigr) = a_{\varepsilon,M}(a_\varepsilon) = \bigl(\varepsilon^2 + u_\varepsilon^2\bigr)^{\sigma/2} + d_0.$$

Corollary 3.2 *If $u_0 \ge 0$ and $f \ge 0$, then the solution $u_\varepsilon(x,t)$ is nonnegative in Q_T.*

Proof Set $u_\varepsilon^- = \min\{u_\varepsilon, 0\}$. Then $u_\varepsilon^-(x,0) = 0$, $u_\varepsilon^- | \Gamma_T = 0$, and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|u_\varepsilon^-(x,t)\right\|_{2,\Omega}^2\right) + \int_\Omega a_{\varepsilon,M}(u_\varepsilon)\left|\nabla u_\varepsilon^-\right|^{p(x,t)}\,\mathrm{d}x \le 0.$$

It follows that, for every $t > 0$,

$$\left\|u_\varepsilon^-(x,t)\right\|_{2,\Omega} \le \left\|u_\varepsilon^-(\cdot,0)\right\|_{2,\Omega} = 0,$$

and the required assertion follows. □

Lemma 3.5 *The solution of problem* (5) *satisfies the estimates*

$$\int\int_{Q_T} u_\varepsilon^\sigma |\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}, \tag{34}$$

$$\varepsilon^\sigma \int\int_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}, \tag{35}$$

$$d_0 \int\int_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}. \tag{36}$$

Proof Similarly as in Lemma 3.4, we take $k = 1$ in (32) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_\varepsilon(\cdot,t)\right\|_{2,\Omega} + \int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x \le \|f\|_{2,\Omega} + C(T), \quad \forall t \in (0,T).$$

Clearly, integrating over $(0,t)$, we have

$$\left\|u_\varepsilon(\cdot,t)\right\|_{2,\Omega} + \int_0^t\int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le \left\|u_\varepsilon(\cdot,t)\right\|_{2,\Omega} + \int_0^T \|f\|_{2,\Omega}\,\mathrm{d}t + C(T). \tag{37}$$

Note that the first term on the left-hand side is nonnegative. We drop the nonpositive term in (37) to get

$$\int_0^t\int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}.$$

If $a_{\varepsilon,M}(u_\varepsilon) \ge d_0$, then we have inequality (36), and if $a_{\varepsilon,M}(u_\varepsilon) \ge \varepsilon^\sigma$, then we have inequality (35) such that $M > K(T)$, and we have $a_{\varepsilon,M}(u_\varepsilon) \ge u_\varepsilon^\sigma$. Furthermore, we get inequality (34). □

Lemma 3.6 *The solution of problem* (5) *satisfies the estimate*

$$\|u_{\varepsilon t}\|_{W'(Q_T)} \le C\left(\sigma, p^\pm, K(T), |\Omega|\right). \tag{38}$$

Proof From identity (7) we get

$$\int\int_{Q_T} u_{\varepsilon t}\xi\,\mathrm{d}x\,\mathrm{d}t = -\int\int_{Q_T}\left[\left(u_\varepsilon^\sigma + \varepsilon^2\right)^{\sigma/2} + d_0\right]|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,\mathrm{d}x\,\mathrm{d}t + \int\int_{Q_T} f(x,t)\xi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int\int_{Q_T}\beta_\varepsilon(x,t)\xi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Applying the fact that $\beta_\varepsilon(x,t) \in (0,1)$, we get

$$\int\int_{Q_T} u_{\varepsilon t}\xi \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_\Omega \big[\big(u_\varepsilon^\sigma + \varepsilon^2\big)^{\sigma/2} + d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\nabla u_\varepsilon \nabla\xi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega |f+1|\cdot|\xi|\,\mathrm{d}x\,\mathrm{d}t.$$

Using the Hölder inequality repeatedly, we have that

$$\begin{aligned}\int\int_{Q_T} u_{\varepsilon t}\xi \,\mathrm{d}x\,\mathrm{d}t &\le 2\big\|\big[\big(u_\varepsilon^\sigma + \varepsilon^2\big)^{\sigma/2} + d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\|_{p'(x,t)}\|\nabla\xi\|_{p(x,t)} \\ &\quad + 2\|f+1\|_{p'(x,t)}\cdot\|\xi\|_{p(x,t)} \\ &\le 2\max\{F_1,F_2\}\|\nabla\xi\|_{p(x,t)} + 2\max\{F_3,F_4\}\|\xi\|_{p(x,t)} \\ &\le \big(2\big(\big(K^2(T)+1\big)^{\sigma/2} + d_0\big)^{\frac{1}{p^{\pm}-1}} K(T)|\Omega| + 2|f+1|_\infty|T|\big)\|\xi\|_{W(Q_T)},\end{aligned}$$

where

$$F_1 = \left(\int_0^T \int_\Omega \big\{\big[\big(u_\varepsilon^\sigma + \varepsilon^2\big)^{\sigma/2} + d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\}^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p'^+}},$$

$$F_2 = \left(\int_0^T \int_\Omega \big\{\big[\big(u_\varepsilon^\sigma + \varepsilon^2\big)^{\sigma/2} + d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\}^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p'^-}},$$

$$F_3 = \left(\int_0^T \int_\Omega |f|^{p'(x,t)} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p'^+}}, \qquad F_4 = \left(\int_0^T \int_\Omega |f+1|^{p'(x,t)} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p'^-}}.$$

Then (38) follows from Lemma 3.5. □

From [6] we can get the following inclusions:

$$u_\varepsilon \in W(Q_T) \subseteq L^{p^-}\big(0,T;W_0^{1,p^-}(\Omega)\big), \qquad u_{\varepsilon t} \in W'(Q_T) \subseteq L^{\frac{p^+}{p^+-1}}\big(0,T;V_+(\Omega)\big),$$
$$W_0^{1,p^-}(\Omega) \subset L^2(\Omega) \subset V_+{}'(\Omega) \quad \text{with } V_+(\Omega) = \big\{u(x) | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p^+}\big\}.$$

These conclusions, together with the uniform estimates in ε, allow us to extract from the sequence $\{u_\varepsilon\}$ a subsequence (for simplicity, we assume that it merely coincides with the whole sequence) such that

$$\begin{cases} u_\varepsilon \to u & \text{a.e. in } Q_T, \\ \nabla u_\varepsilon \to \nabla u & \text{weakly in } L^{p(x,t)}(Q_T), \\ u_\varepsilon^\sigma |\nabla u_\varepsilon|^{p(x,t)-2} D_i u_\varepsilon \to A_i(x,t) & \text{weakly in } L^{p'(x,t)}(Q_T), \\ |\nabla u_\varepsilon|^{p(x,t)-2} D_i u_\varepsilon \to W_i(x,t) & \text{weakly in } L^{p'(x,t)}(Q_T) \end{cases} \tag{39}$$

for some functions $u \in W(Q_T)$, $A_i(x,t) \in L^{p'(x,t)}(Q_T)$, and $W_i(x,t) \in L^{p'(x,t)}(Q_T)$.

Lemma 3.7 *For almost all* $(x,t) \in Q_T$,

$$\lim_{\varepsilon\to 0^+} \int\int_{Q_T} \big(\big(u_\varepsilon^2 + \varepsilon^2\big)^{\frac{\sigma}{2}} - u_\varepsilon^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall \xi = W(Q_T).$$

Proof We first compute

$$\begin{aligned}
I &\triangleq \int\int_{Q_T} \bigl((u_\varepsilon^2+\varepsilon^2)^{\frac{\sigma}{2}} - u_\varepsilon^\sigma\bigr)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi \,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\sigma}{2}\varepsilon^2 \int\int_{Q_T} \biggl(\int_0^1 (u_\varepsilon^2+s\varepsilon^2)^{\frac{\sigma-2}{2}}\,ds\biggr)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\le \sigma\varepsilon^2\bigl(K^2(T)+1\bigr)^{\frac{\sigma-2}{2}} \bigl\||\nabla u_\varepsilon|^{p(x,t)-1}\bigr\|_{p'(x,t)}\|\nabla\xi\|_{p(x,t)} \\
&\le C\varepsilon^2 \max\biggl\{\biggl(\int\int_{Q_T}|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t\biggr)^{\frac{p^+-1}{p^+}}, \biggl(\int\int_{Q_T}|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t\biggr)^{\frac{p^--1}{p^-}}\biggr\}\|\nabla\xi\|_{p(x,t)}.
\end{aligned}$$

By (35) we get

$$I \le C\varepsilon^{2-\frac{\sigma(p^+-1)}{p^+}}\|\nabla\xi\|_{p(x,t)}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain Lemma 3.7. □

Lemma 3.8 *For almost all* $(x,t) \in Q_T$, *we have*

$$A_i(x,t) = u^\sigma W_i(x,t), \quad i = 1,2,\dots,N.$$

Proof In (39), letting $\varepsilon \to 0$, we have

$$\int\int_{Q_T} u_\varepsilon^\sigma|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \to \sum\int\int_{Q_T} A_i(x,t)D_i\xi\,\mathrm{d}x\,\mathrm{d}t, \tag{40}$$

$$\int\int_{Q_T} u_\varepsilon^\sigma|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \to \sum\int\int_{Q_T} W_i(x,t)D_i\xi\,\mathrm{d}x\,\mathrm{d}t. \tag{41}$$

By Lebesgue's dominated convergence theorem we have

$$\lim_{\varepsilon\to 0}\sum_{i=1}^N\int\int_{Q_T}\bigl(u_\varepsilon^\sigma - u^\sigma\bigr)A_i(x,t)D_i\xi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{42}$$

So

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum\int\int_{Q_T}\bigl(u_\varepsilon^\sigma|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon - u^\sigma W_i(x,t)\bigr)D_i\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \lim_{\varepsilon\to 0}\int\int_{Q_T}\bigl(u_\varepsilon^\sigma - u^\sigma\bigr)|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon + u^\sigma\bigl(|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon - W_i(x,t)\bigr)D_i\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= 0.
\end{aligned}$$

By (40)–(42) and the previous inequalities, we complete the proof of Lemma 3.8. □

Lemma 3.9 *For almost all* $(x,t) \in Q_T$, *we have*

$$W_i(x,t) = |\nabla u_\varepsilon|^{p(x,t)-2}D_iu, \quad i = 1,2,\dots,N.$$

Proof In (8), choosing $\xi = (u_\varepsilon - u)\Phi$ with $\Phi \in W(Q_T)$, $\Phi \geq 0$, we have

$$\int\int_{Q_T} \big[u_{\varepsilon t}(u_\varepsilon - u)\Phi + \Phi\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\big]\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int\int_{Q_T} \big[(u_\varepsilon - u)\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\Phi - f(x,t)(u_\varepsilon - u)\Phi\big]\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int\int_{Q_T} \big(\big(u_\varepsilon^\sigma - \varepsilon^2\big)^{\frac{\sigma}{2}} - u_\varepsilon^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

It follows that

$$\lim_{\varepsilon\to 0}\int\int_{Q_T} \Phi\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{43}$$

On the other hand, from $u_\varepsilon, u \in L^\infty(Q_T)$ and $|\nabla u| \in L^{p(x,t)}(Q_T)$ we get

$$\lim_{\varepsilon\to 0}\int\int_{Q_T} \Phi\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u \nabla(u_\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{44}$$
$$\lim_{\varepsilon\to 0}\int\int_{Q_T} \Phi\big(u_\varepsilon^\sigma + u^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{45}$$

Note that

$$0 \leq \big(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u\big)\nabla(u_\varepsilon - u)$$
$$\leq \frac{1}{d_0}\big[\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - \big(u_\varepsilon^\sigma - u^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u\big]\nabla(u_\varepsilon - u)$$
$$- \frac{1}{d_0}\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u \nabla(u_\varepsilon - u). \tag{46}$$

By (44)–(46) we have

$$\lim_{\varepsilon\to 0}\int\int_{Q_T} \Phi\big(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u\big)\nabla(u_\varepsilon - u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{47}$$

□

Lemma 3.10 *As $\varepsilon \to 0$, we have*

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \in G(u - u_0). \tag{48}$$

Proof Using (7) and the definition of β_ε, we have

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \quad \text{as } \varepsilon \to 0.$$

Now we prove that $\xi \in G(u - u_0)$. According to the definition of $G(\cdot)$, we only need to prove that if $u(x_0, t_0) > u_0(x_0)$, then $\xi(x_0, t_0) = 0$. In fact, if $u(x_0, t_0) > u_0(x)$, there exist a constant $\lambda > 0$ and a δ neighborhood $B_\delta(x_0, t_0)$ such that if ε is small enough, we have

$$u_\varepsilon(x,t) \geq u_0(x) + \lambda, \quad \forall (x,t) \in B_\delta(x_0, t_0).$$

Thus, if ε is small enough, then we have

$$0 \geq \beta_\varepsilon(u_\varepsilon - u_0) \geq \beta_\varepsilon(\lambda) = 0, \quad \forall (x,t) \in B_\delta(x_0,t_0).$$

Furthermore, it follows by $\varepsilon \to 0$ that

$$\xi(x,t) = 0, \quad \forall (x,t) \in B_\delta(x_0,t_0).$$

Hence, (48) holds, and the proof of Lemma 3.10 is completed. □

Applying (28), (29), and Lemma 3.10, it is clear that

$$u(x,t) \leq u_0(x) \quad \text{in } \Omega_T, \qquad u(x,0) = u_0(x) \quad \text{in } \Omega, \xi \in G(u - u_0),$$

and thus (a), (b), and (c) hold. The remaining arguments of the existence part are the same as those of Theorem 2.1 in [8], and we omit the details. Moreover, the uniqueness of solutions can be proved by repeating Lemma 3.1.

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The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The author read and approved the final manuscript.

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