Li et al. *Journal of Inequalities and Applications* (2018) 2018:27
https://doi.org/10.1186/s13660-018-1620-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Sharp Smith's bounds for the gamma function

Xi-Qiao Li[1], Zhi-Ming Liu[1], Zhen-Hang Yang[2] and Shen-Zhou Zheng[3*]

*Correspondence:
shzhzheng@bjtu.edu.cn
[3]College of Science, Beijing Jiaotong University, Beijing, China
Full list of author information is available at the end of the article

Abstract

Among various approximation formulas for the gamma function, Smith showed that

$$\Gamma\left(x+\frac{1}{2}\right) \sim S(x) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2}, \quad x\to\infty,$$

which is a little-known but accurate and simple one. In this note, we prove that the function $x \mapsto \ln\Gamma(x+1/2) - \ln S(x)$ is strictly increasing and concave on $(0,\infty)$, which shows that Smith's approximation is just an upper one.

MSC: Primary 33B15; 26A48; secondary 26D15; 26A51

Keywords: Gamma function; Approximation formula; Inequality

1 Introduction

The Stirling formula

$$n! \sim \sqrt{2\pi n}n^n e^{-n} \tag{1.1}$$

has many important applications in statistical physics, probability theory and number theory. Due to its practical importance, it has attracted much interest of many mathematicians and has also motivated a large number of research papers concerning various generalizations and improvements; see for example, Burnside's [1], Gosper [2], Batir [3], Mortici [4].

The gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$ is closely related to the Stirling formula, since $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. This inspired some authors to also pay attention to find various better approximations for the gamma function; see, for instance, Ramanujan [5, p. 339], Windschitl (see Nemes [6, Corollary 4.1]), Yang and Chu [7], Chen [8].

More results involving the approximation formulas for the factorial or gamma function can be found in [9–23] and the references cited therein.

In this note, we are interested in Smith's approximation formula (see [24, equation (42)]):

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2} := S(x), \quad \text{as } x\to\infty. \tag{1.2}$$

It is easy to check that

$$\Gamma\left(x+\frac{1}{2}\right)=\sqrt{2\pi}\left(\frac{x}{e}\right)^{x}\left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1+O\left(\frac{1}{x^5}\right)\right),$$

which shows that the rate of $S(x)$ converging to $\Gamma(x+1/2)$ as $x\to\infty$ is like x^{-5}. According to the comment in [8, (3.5)–(3.10)], it is well known that Smith's approximation is an accurate but simple one for gamma function.

The aim of this short note is to further prove the Smith approximation $S(x)$ is an upper one. Our main result is stated as follows.

Theorem 1 *The function*

$$f(x)=\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\sqrt{2\pi}-x\ln x+x-\frac{x}{2}\ln\left(2x\tanh\frac{1}{2x}\right)$$

is strictly increasing and concave from $(0,\infty)$ *onto* $(-\ln\sqrt{2},0)$.

2 Proof of Theorem 1

To prove Theorem 1 we need the following two lemmas.

Lemma 1 *The inequality*

$$\psi'\left(x+\frac{1}{2}\right)<\frac{4}{3}\frac{15x^2+4}{x(20x^2+7)} \tag{2.1}$$

holds for all $x>0$.

Proof Let

$$f_1(x)=\psi'\left(x+\frac{1}{2}\right)-\frac{4}{3}\frac{15x^2+4}{x(20x^2+7)}. \tag{2.2}$$

Using the recurrence formula [25, pp. 258–260]:

$$\psi^{(n)}(x+1)-\psi^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}},$$

we have

$$\begin{aligned}
f_1(x+1)-f_1(x)&=\psi'\left(x+\frac{3}{2}\right)-\frac{4}{3(x+1)}\frac{15x^2+30x+19}{20x^2+40x+27}\\
&\quad-\psi'\left(x+\frac{1}{2}\right)+\frac{4}{3}\frac{15x^2+4}{x(20x^2+7)}\\
&\quad-\frac{1}{(x+1/2)^2}-\frac{4}{3(x+1)}\frac{15x^2+30x+19}{20x^2+40x+27}+\frac{4}{3}\frac{15x^2+4}{x(20x^2+7)}\\
&=\frac{144}{x(x+1)(2x+1)^2(20x^2+7)(20x^2+40x+27)}>0.
\end{aligned}$$

It then follows that

$$f_1(x) < f_1(x+1) < \cdots < \lim_{n\to\infty} f_1(x+n) = 0,$$

which proves the desired inequality (2.1). □

Lemma 2 *The inequality*

$$\frac{\sinh^2 t}{\cosh t} > \frac{t^2(21t^2+60)}{31t^2+60} \tag{2.3}$$

holds for all $t > 0$.

Proof It is obvious that the inequality what we consider is equivalent to

$$f_2(t) = \bigl(31t^2+60\bigr)(\sinh t)^2 - t^2\bigl(21t^2+60\bigr)\cosh t > 0.$$

Simplifying and expanding it in power series lead us to

$$\begin{aligned} 2f_2(t) &= 60\cosh 2t + 31t^2\cosh 2t - 120t^2\cosh t - 42t^4\cosh t - 31t^2 - 60 \\ &= 60\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n} + 31\sum_{n=1}^{\infty}\frac{2^{2n-2}}{(2n-2)!}t^{2n} \\ &\quad - 120\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}t^{2n} - 42\sum_{n=2}^{\infty}\frac{1}{(2n-4)!}t^{2n} - 31t^2 - 60 \\ &:= \sum_{n=2}^{\infty}\frac{a_n}{(2n)!}t^{2n}, \end{aligned}$$

where

$$a_n = \bigl(62n^2-31n+120\bigr)2^{2n-1} - 24n(2n-1)\bigl(14n^2-35n+31\bigr).$$

It is easy to check that $a_2 = a_3 = 0$ and $a_4 = 49\,184 > 0$. It remains to prove $a_n > 0$ for $n \ge 5$.

To this end, it suffices to prove $b_n = 2^{2n-1} - 6n(2n-1) > 0$ for $n \ge 5$, because the inequality

$$\bigl(62n^2-31n+120\bigr) > 4\bigl(14n^2-35n+31\bigr)$$

is clearly valid for $n \ge 5$. We easily obtain

$$b_{n+1} - 4b_n = 6\bigl(6n^2-7n-1\bigr) > 0$$

for $n \ge 5$, which in combination with $b_5 = 242 > 0$ yields $b_n > 0$ for $n \ge 5$. This completes the proof. □

Now we are in a position to prove Theorem 1.

Theorem 1 Differentiating and simplifying yields

$$f'(x) = \psi\left(x + \frac{1}{2}\right) - \ln x - \frac{1}{2}\ln\left(2x\tanh\frac{1}{2x}\right) + \frac{1}{2x\sinh(1/x)} - \frac{1}{2},$$
$$f''(x) = \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^3}\frac{\cosh(1/x)}{\sinh^2(1/x)} - \frac{3}{2x}.$$

As an application of inequalities (2.1) and (2.3) it gives

$$\begin{aligned} f''(x) &< \frac{4}{3}\frac{15x^2+4}{x(20x^2+7)} + \frac{1}{2x^3}\frac{\cosh(1/x)}{\sinh^2(1/x)} - \frac{3}{2x} \\ &= \frac{1}{2x^3}\frac{\cosh(1/x)}{\sinh^2(1/x)} - \frac{1}{6}\frac{60x^2+31}{x(20x^2+7)} \\ &\overset{x=1/t}{=} \frac{t^3}{2}\left(\frac{\cosh t}{\sinh^2 t} - \frac{31t^2+60}{t^2(21t^2+60)}\right) < 0. \end{aligned}$$

Then it is deduced that

$$f'(x) > \lim_{x\to\infty} f'(x) = 0,$$

which in turn implies that

$$-\frac{1}{2}\ln 2 = \lim_{x\to 0^+} f(x) < f(x) < \lim_{x\to\infty} f(x) = 0.$$

This completes the proof. □

3 Corollaries and remarks

Using the increasing property of $f(x + 1/2)$ given in Theorem 1 and noting that

$$f\left(\frac{1}{2}\right) = \ln\frac{\sqrt{e}}{\sqrt{\pi}(\tanh 1)^{1/4}} \quad \text{and} \quad f\left(\frac{3}{2}\right) = \ln\left(\frac{2e\sqrt{e}3^{3/4}}{27\sqrt{\pi}\tanh^{3/4}(1/3)}\right),$$

we have the corollaries.

Corollary 1 *The double inequality*

$$\alpha_1 < \frac{e^{x+1/2}\Gamma(x+1)}{\sqrt{2\pi}(x+1/2)^{x+1/2}[(2x+1)\tanh(1/(2x+1))]^{(2x+1)/4}} < 1$$

holds for all $x > 0$ *with the best constants* 1 *and* $\alpha_1 = \sqrt{e/\pi}/(\tanh 1)^{1/4} \approx 0.99573$.

Corollary 2 *The double inequality*

$$\alpha_2 < \frac{n!}{\sqrt{2\pi}((n+1/2)/e)^{n+1/2}[(2n+1)\tanh(1/(2n+1))]^{(2n+1)/4}} < 1$$

holds for all $n \in \mathbb{N}$ *with the best constants* 1 *and*

$$\alpha_2 = \frac{2e\sqrt{e}3^{3/4}}{27\sqrt{\pi}\tanh^{3/4}(1/3)} \approx 0.99994.$$

By the decreasing property of $f'(x+1/2)$ given in Theorem 1 and the facts that

$$f'\left(\frac{1}{2}\right)=\frac{1}{\sinh 2}-\frac{1}{2}\ln(\tanh 1)+\ln 2-\frac{1}{2}-\gamma\approx 0.027823,$$

$$f'\left(\frac{3}{2}\right)=\frac{1}{3\sinh(2/3)}-\frac{1}{2}\ln\left(3\tanh\frac{1}{3}\right)-\ln\frac{3}{2}+\psi(1)+\frac{1}{2}\approx 0.00016946,$$

the following corollaries are immediate.

Corollary 3 *For $x>0$, the inequalities*

$$\begin{aligned}&\frac{1}{2}+\frac{1}{2}\ln\left((2x+1)\tanh\frac{1}{2x+1}\right)-\frac{1}{(2x+1)\sinh(2/(2x+1))}\\&\quad<\psi(x+1)-\ln\left(x+\frac{1}{2}\right)\\&\quad<\beta_1+\frac{1}{2}\ln\left((2x+1)\tanh\frac{1}{2x+1}\right)-\frac{1}{(2x+1)\sinh(2/(2x+1))}\end{aligned}$$

hold, where the constants $1/2$ and

$$\beta_1=\ln 2-\frac{1}{2}\ln(\tanh 1)+\frac{1}{\sinh 2}-\gamma\approx 0.52782$$

are the best possible.

Corollary 4 *Let $H_n=\sum_{k=1}^{n}$ for $n\in\mathbb{N}$. The inequalities*

$$\begin{aligned}&\left(\frac{1}{2}+\gamma\right)+\frac{1}{2}\ln\left((2n+1)\tanh\frac{1}{2n+1}\right)-\frac{1}{(2n+1)\sinh(2/(2x+1))}\\&\quad<H_n-\ln\left(n+\frac{1}{2}\right)\\&\quad<\beta_2+\frac{1}{2}\ln\left((2n+1)\tanh\frac{1}{2n+1}\right)-\frac{1}{(2n+1)\sinh(2/(2n+1))}\end{aligned}$$

hold, where $1/2+\gamma\approx 1.0772$ and

$$\beta_2=\frac{1}{3\sinh(2/3)}-\frac{1}{2}\ln\left(3\tanh\frac{1}{3}\right)-\ln\frac{3}{2}+1\approx 1.0774$$

are the best possible constants.

Finally, as a by-product of Lemma 1, we draw the following conclusion.

Theorem 2 *Let g be defined on $(0,\infty)$ by*

$$g(x)=\ln\Gamma\left(x+\frac{1}{2}\right)-\left[\frac{1}{2}\ln 2\pi+\frac{16}{21}x\ln x+\frac{5x}{42}\ln\left(x^2+\frac{7}{20}\right)-x-\frac{\sqrt{35}}{42}\operatorname{arccot}\left(\sqrt{\frac{20}{7}}x\right)\right].$$

Then g is strictly increasing and concave on $(0,\infty)$.

Proof Differentiation yields

$$g'(x) = \psi\left(x+\frac{1}{2}\right) - \left[\frac{5}{42}\ln\left(x^2+\frac{7}{20}\right) + \frac{16}{21}\ln x\right],$$
$$g''(x) = \psi'\left(x+\frac{1}{2}\right) - \frac{4}{3}\frac{15x^2+4}{x(20x^2+7)} = f_1(x) < 0,$$

where the inequality holds due to Lemma 1. This completes the proof. □

Remark 1 Theorem 2 gives a new approximation for the gamma function

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}x^{26x/21}\left(x^2+\frac{7}{20}\right)^{5x/42}\exp\left[-x-\frac{\sqrt{35}}{42}\operatorname{arccot}\left(\sqrt{\frac{20}{7}}x\right)\right],$$

as $x\to\infty$, which satisfies

$$\Gamma\left(x+\frac{1}{2}\right) = \sqrt{2\pi}x^{26x/21}\left(x^2+\frac{7}{20}\right)^{5x/42}\exp\left[-x-\frac{\sqrt{35}}{42}\operatorname{arccot}\left(\sqrt{\frac{20}{7}}x\right)\right]\left(1+O\left(x^{-5}\right)\right).$$

Remark 2 Theorem 2 also offers an asymptotic formula for the psi function

$$\psi\left(x+\frac{1}{2}\right) \sim \frac{5}{42}\ln\left(x^2+\frac{7}{20}\right) + \frac{16}{21}\ln x \quad \text{as } x\to\infty.$$

Furthermore, by replacing x with $x + 1/2$, we have the following sharp inequalities:

$$\begin{aligned}&\frac{5}{42}\ln\left(x^2+x+\frac{3}{5}\right) + \frac{16}{21}\ln\left(x+\frac{1}{2}\right)\\ &\quad < \psi(x+1) < \lambda_0 + \frac{5}{42}\ln\left(x^2+x+\frac{3}{5}\right) + \frac{16}{21}\ln\left(x+\frac{1}{2}\right)\end{aligned} \tag{3.1}$$

for $x > 0$ with the best constant

$$\lambda_0 = \frac{16}{21}\ln 2 - \frac{5}{42}\ln\frac{3}{5} - \gamma \approx 0.011709;$$
$$\begin{aligned}&\gamma + \frac{5}{42}\ln\left(n^2+n+\frac{3}{5}\right) + \frac{16}{21}\ln\left(n+\frac{1}{2}\right)\\ &\quad < H_n < \lambda_0 + \gamma + \frac{5}{42}\ln\left(n^2+n+\frac{3}{5}\right) + \frac{16}{21}\ln\left(n+\frac{1}{2}\right)\end{aligned}$$

for $n \in \mathbb{N}$ with the best constant

$$\lambda_1 = 1 - \frac{16}{21}\ln\frac{3}{2} - \frac{5}{42}\ln\frac{13}{5} - \gamma \approx 0.00010718.$$

Inequalities (3.1) first appeared in [26, Corollary 3.4].

4 Conclusions

In this note, we mainly presented an upper bound of Smith's approximation in accordance with the fact that the function $x \mapsto \ln\Gamma(x+1/2) - \ln S(x)$ is strictly increasing and concave on $(0,\infty)$. As a consequence, we get some new sharp estimates to various classical inequalities concerning the gamma function and hyperbolic functions.

Acknowledgements
This paper is supported by the National Science Foundation of China grant No. 11371050.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

Author details
[1]School of Mechanical, Electronic and Control Engineering, Beijing Jiaotong University, Beijing, China. [2]Department of Science and Technology, State Grid Zhejiang Electric Power Company Research Institute, Hangzhou, China. [3]College of Science, Beijing Jiaotong University, Beijing, China.

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Received: 2 October 2017 Accepted: 15 January 2018 Published online: 31 January 2018

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