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# Discussion of some inequalities via fractional integrals

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## Abstract

Recently, many generalizations and extensions of well-known inequalities were obtained via different kinds of fractional integrals. In this paper, we show that most of those results are particular cases of (or equivalent to) existing inequalities from the literature. As consequence, such results are not real generalizations.

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**Keywords:** inequalities; fractional integrals

## 1 Introduction

Fractional calculus has received a great attention from many researchers in different disciplines. In particular, there has been an important interest in studying inequalities involving different kinds of fractional integrals. Unfortunately, as we will show later, most of the obtained results in this direction are particular cases of (or equivalent to) existing inequalities from the literature.

At first, let us recall briefly some definitions on fractional calculus that will be used later.

**Definition 1.1** (see [1]) Let  $f \in L^1((a, b); \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ . The Riemann-Liouville fractional integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

**Definition 1.2** (see [2]) Let  $f \in L^1((a, b); \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ . The fractional integrals  $I_a^\alpha f$  and  $I_b^\alpha f$  of order  $\alpha \in (0, 1)$  are defined by

$$I_a^\alpha f(x) = \frac{1}{\alpha} \int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-t)\right) f(t) dt, \quad x > a,$$

and

$$I_b^\alpha f(x) = \frac{1}{\alpha} \int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(t-x)\right) f(t) dt, \quad x < b.$$

The paper is organized as follows. Section 2 is devoted to results and discussions. More precisely, in Section 2.1, we discuss some recent Hermite-Hadamard-type inequalities via different kinds of fractional integrals. We show that such inequalities are particular cases of (or equivalent to) Fejér inequality. In Section 2.2, we discuss a Grüss-type inequality involving fractional integrals, which was obtained by Dahmani *et al.* [3]. We show that such inequality is a particular case of a weighted version of Grüss inequality, which was established by Dragomir [4]. In Section 2.3, we discuss a fractional-type inequality related to weighted Chebyshev's functional, which was presented by Dahmani [5]. We show that such an inequality is not new, and it is equivalent to an existing inequality proved by Dragomir [4]. We end the paper with a conclusion in Section 3.

## 2 Results and discussions

In this section, we discuss several recent inequalities involving different types of fractional integrals, and we prove that these inequalities are particular cases of (or equivalent to) previous existing results from the literature.

### 2.1 On Hermite-Hadamard-type inequalities involving fractional integrals

Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ , be a convex function. Then (see [6–8])

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Inequality (1) was known in the literature as Hermite-Hadamard inequality.

In [9], Fejér established the following result, which contains a weighted generalization of (1).

**Theorem 2.1** *Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ , be a convex function. Let  $w \in L^1((a, b); \mathbb{R})$  be non-negative and symmetric to  $\frac{a+b}{2}$ . Then*

$$f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b f(t)w(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b w(t) dt. \quad (2)$$

Observe that (1) follows from (2) by taking  $w \equiv 1$ .

Recently, many generalizations and extensions of (1) were derived by many authors using fractional integrals. In this direction, we refer the reader to [2, 10–14], and the references therein. In this section, we show that most of those results are particular cases of (or equivalent to) Theorem 2.1. To simplify the presentation, we will consider only the results obtained in [2, 12, 14].

In [14], Sarikaya *et al.* established the following Hermite-Hadamard-type inequality via Riemann-Liouville integrals.

**Theorem 2.2** *Let  $f \in L^1([a, b]; \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ , be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2}, \quad (3)$$

where  $\alpha > 0$ .

Note that in [14], it is supposed that  $a \geq 0$  and  $f$  is a non-negative function. We will show later that such assumptions are superfluous.

In [12], İşcan presented the following result.

**Theorem 2.3** *Let  $f \in L^1([a, b]; \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ , be a convex function. Let  $g \in L^1((a, b); \mathbb{R})$  be non-negative and symmetric to  $\frac{a+b}{2}$ . Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] &\leq [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \\ &\leq \frac{f(a) + f(b)}{2} [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)], \end{aligned} \quad (4)$$

where  $\alpha > 0$ .

In [2], Kirane and Torebek presented the following result.

**Theorem 2.4** *Let  $f \in L^1([a, b]; \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ , be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2(1-\exp(-\mathcal{A}))} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}, \quad (5)$$

where  $\alpha \in (0, 1)$  and  $\mathcal{A} = \frac{1-\alpha}{\alpha}(b-a)$ .

In [2], it is supposed that  $a \geq 0$  and  $f$  is a non-negative function. We will show later that such assumptions are superfluous.

Our first observation is formulated by the following theorem.

**Theorem 2.5** *Theorem 2.1  $\implies$  Theorem 2.2.*

*Proof* Let us suppose that all assumptions of Theorem 2.2 are satisfied. Let us define the function  $w$  by

$$w(t) = \frac{1}{\Gamma(\alpha)} ((b-t)^{\alpha-1} + (t-a)^{\alpha-1}), \quad a < t < b.$$

Clearly,  $w \in L^1((a, b); \mathbb{R})$ , and it is a non-negative function. Moreover, for all  $t \in (a, b)$ , we have

$$\begin{aligned} \Gamma(\alpha)w(a+b-t) &= (b-(a+b-t))^{\alpha-1} + ((a+b-t)-a)^{\alpha-1} \\ &= (t-a)^{\alpha-1} + (b-t)^{\alpha-1} = \Gamma(\alpha)w(t). \end{aligned}$$

Therefore,  $w$  is symmetric to  $\frac{a+b}{2}$ . Now, by Theorem 2.1, it follows from (2) that

$$f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b f(t)w(t) dt \leq \frac{f(a) + f(b)}{2} \int_a^b w(t) dt. \quad (6)$$

On the other hand, we have

$$\Gamma(\alpha) \int_a^b w(t) dt = \int_a^b (b-t)^{\alpha-1} dt + \int_a^b (t-a)^{\alpha-1} dt = \frac{2}{\alpha}(b-a)^\alpha. \quad (7)$$

Moreover, we have

$$\begin{aligned}\int_a^b f(t)w(t) dt &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t) dt + \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} f(t) dt \\ &= J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a).\end{aligned}\quad (8)$$

Therefore, using (6), (7), and (8), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2},$$

which is inequality (3). Therefore, we proved that Theorem 2.1  $\implies$  Theorem 2.2.  $\square$

Next, we have the following observation concerning Theorem 2.3.

**Theorem 2.6** *Theorem 2.1  $\Leftrightarrow$  Theorem 2.3.*

*Proof* Let us suppose that all assumptions of Theorem 2.3 are satisfied. Let us define the function  $w$  by

$$w(t) = \frac{g(t)}{\Gamma(\alpha)} ((b-t)^{\alpha-1} + (t-a)^{\alpha-1}), \quad a < t < b.$$

Clearly,  $w \in L^1((a, b); \mathbb{R})$ , and it is non-negative and symmetric to  $\frac{a+b}{2}$  (since  $g$  is symmetric to  $\frac{a+b}{2}$ ). By Theorem 2.1, it follows from (2) that

$$f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b f(t)w(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b w(t) dt. \quad (9)$$

On the other hand, we have

$$\begin{aligned}\int_a^b w(t) dt &= \frac{1}{\Gamma(\alpha)} \int_a^b g(t)((b-t)^{\alpha-1} + (t-a)^{\alpha-1}) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} g(t) dt + \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} g(t) dt \\ &= J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a).\end{aligned}\quad (10)$$

Moreover,

$$\begin{aligned}\int_a^b f(t)w(t) dt &= \frac{1}{\Gamma(\alpha)} \int_a^b f(t)g(t)((b-t)^{\alpha-1} + (t-a)^{\alpha-1}) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t)g(t) dt + \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} f(t)g(t) dt \\ &= J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a).\end{aligned}\quad (11)$$

Combining (9), (10), and (11), we obtain

$$f\left(\frac{a+b}{2}\right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] \leq [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \leq \frac{f(a)+f(b)}{2} [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)],$$

which is inequality (4). Therefore, we proved that Theorem 2.1  $\implies$  Theorem 2.3.

Now, suppose that all the assumptions of Theorem 2.1 are satisfied. Taking  $g = w$  and  $\alpha = 1$  in (4), we obtain (2). Therefore, we proved that Theorem 2.3  $\implies$  Theorem 2.1.  $\square$

Our comment on Theorem 2.4 is formulated by the following theorem.

**Theorem 2.7** *Theorem 2.1  $\implies$  Theorem 2.4.*

*Proof* Let us suppose that all assumptions of Theorem 2.4 are satisfied. Let us define the function  $w$  by

$$w(t) = \frac{1}{\alpha} \left( \exp\left(-\frac{1-\alpha}{\alpha}(b-t)\right) + \exp\left(-\frac{1-\alpha}{\alpha}(t-a)\right) \right), \quad a \leq t \leq b.$$

It can be easily seen that  $w \in L^1((a, b); \mathbb{R})$ , and it is non-negative and symmetric to  $\frac{a+b}{2}$ . By Theorem 2.1, it follows from (2) that

$$f\left(\frac{a+b}{2}\right) \int_a^b w(t) dt \leq \int_a^b f(t)w(t) dt \leq \frac{f(a)+f(b)}{2} \int_a^b w(t) dt. \quad (12)$$

On the other hand, we have

$$\begin{aligned} \int_a^b w(t) dt &= \frac{1}{\alpha} \int_a^b \left( \exp\left(-\frac{1-\alpha}{\alpha}(b-t)\right) + \exp\left(-\frac{1-\alpha}{\alpha}(t-a)\right) \right) dt \\ &= \frac{1}{\alpha} \int_a^b \exp\left(-\frac{1-\alpha}{\alpha}(b-t)\right) dt + \frac{1}{\alpha} \int_a^b \exp\left(-\frac{1-\alpha}{\alpha}(t-a)\right) dt \\ &= \frac{1}{1-\alpha} \left( 1 - \exp\left(-\frac{1-\alpha}{\alpha}(b-a)\right) \right) - \frac{1}{1-\alpha} \left( \exp\left(-\frac{1-\alpha}{\alpha}(b-a)\right) - 1 \right) \\ &= \frac{2}{1-\alpha} \left( 1 - \exp\left(-\frac{1-\alpha}{\alpha}(b-a)\right) \right) \\ &= \frac{2}{1-\alpha} (1 - \exp(-\mathcal{A})). \end{aligned} \quad (13)$$

Moreover, we have

$$\begin{aligned} \int_a^b f(t)w(t) dt &= \frac{1}{\alpha} \int_a^b \left( \exp\left(-\frac{1-\alpha}{\alpha}(b-t)\right) + \exp\left(-\frac{1-\alpha}{\alpha}(t-a)\right) \right) f(t) dt \\ &= \frac{1}{\alpha} \int_a^b \exp\left(-\frac{1-\alpha}{\alpha}(b-t)\right) f(t) dt + \frac{1}{\alpha} \int_a^b \exp\left(-\frac{1-\alpha}{\alpha}(t-a)\right) f(t) dt \\ &= I_a^\alpha f(b) + I_b^\alpha f(a). \end{aligned} \quad (14)$$

Combining (12), (13), and (14), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2(1-\exp(-\mathcal{A}))} [I_a^\alpha f(b) + I_b^\alpha f(a)] \leq \frac{f(a)+f(b)}{2},$$

which is inequality (5). Therefore, we proved that Theorem 2.1  $\implies$  Theorem 2.4.  $\square$

## 2.2 Discussion of Gruss-type inequalities involving fractional integrals

In 1935, Gruss [15] proved the following result.

**Theorem 2.8** *Let  $f, g \in L^1((a, b); \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ . Suppose that there exist constants  $m, M, p, P \in \mathbb{R}$  such that*

$$m \leq f(x) \leq M, \quad p \leq g(x) \leq P, \quad a \leq x \leq b.$$

*Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right) \left( \frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{(M-m)(P-p)}{4}. \quad (15)$$

Inequality (15) has evoked the interest of many researchers, and several generalizations of this inequality have appeared in the literature. In particular, in 1998, Dragomir [4] established the following interesting generalization, which provides a weighted version of the Gruss inequality.

**Theorem 2.9** *Let  $f, g \in L^1((a, b); \mathbb{R})$ ,  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ . Suppose that there exist constants  $m, M, p, P \in \mathbb{R}$  such that*

$$m \leq f(x) \leq M, \quad p \leq g(x) \leq P, \quad a \leq x \leq b.$$

*Let  $h \in L^1((a, b); \mathbb{R})$  be a non-negative function such that  $\int_a^b h(x) dx > 0$ . Then*

$$\left| \int_a^b h(x) dx \int_a^b f(x)g(x)h(x) dx - \int_a^b f(x)h(x) dx \int_a^b g(x)h(x) dx \right| \leq \frac{(M-m)(P-p)}{4} \left( \int_a^b h(x) dx \right)^2. \quad (16)$$

Observe that (15) follows from (16) by taking  $h \equiv 1$ .

After the publication of reference [4], in 2010, Dahmani and Tabharit [3] presented the following result.

**Theorem 2.10** *Let  $f$  and  $g$  be two integrable functions on  $[0, \infty)$ . Suppose that there exist constants  $m, M, p, P \in \mathbb{R}$  such that*

$$m \leq f(x) \leq M, \quad p \leq g(x) \leq P, \quad x \geq 0.$$

*Let  $p \in L^1([0, \infty); \mathbb{R})$  be a non-negative function such that  $J_{0+}^\alpha p(T) > 0$ , for all  $T > 0$ . Then*

$$\left| J_{0+}^\alpha p(T) J_{0+}^\alpha (pfg)(T) - J_{0+}^\alpha (pf)(T) J_{0+}^\alpha (pg)(T) \right| \leq (J_{0+}^\alpha p(T))^2 \frac{(M-m)(P-p)}{4} \quad (17)$$

*for all  $\alpha > 0$  and  $T > 0$ .*

We have the following observation.

**Theorem 2.11** *Theorem 2.9  $\implies$  Theorem 2.10.*

*Proof* Suppose that all assumptions of Theorem 2.10 are satisfied. Let us fix  $\alpha > 0$  and  $T > 0$ . Let

$$h(x) = \frac{1}{\Gamma(\alpha)}(T-x)^{\alpha-1}p(x), \quad 0 \leq x < T.$$

Clearly,  $h$  is a non-negative function. Moreover, we have

$$\int_0^T h(x) dx \geq \frac{1}{\Gamma(\alpha)} \int_0^T (T-x)^{\alpha-1} p(x) dx = J_{0+}^\alpha p(T) > 0. \quad (18)$$

Therefore, using (18), by Theorem 2.9 with  $(a, b) = (0, T)$ , it follows from (16) that

$$\begin{aligned} & \left| J_{0+}^\alpha p(T) \int_0^T f(x)g(x)h(x) dx - \int_0^T f(x)h(x) dx \int_0^T g(x)h(x) dx \right| \\ & \leq \frac{(M-m)(P-p)}{4} (J_{0+}^\alpha p(T))^2. \end{aligned} \quad (19)$$

On the other hand, observe that

$$\int_0^T f(x)g(x)h(x) dx = J_{0+}^\alpha (pfg)(T), \quad (20)$$

$$\int_0^T f(x)h(x) dx = J_{0+}^\alpha (pf)(T), \quad (21)$$

$$\int_0^T g(x)h(x) dx = J_{0+}^\alpha (pg)(T). \quad (22)$$

Combining (19), (20), (21), and (22), we obtain inequality (17). Therefore, we proved that Theorem 2.9  $\implies$  Theorem 2.10.  $\square$

### 2.3 Discussion of fractional-type inequalities related to the weighted Chebyshev's functional

Let us introduce Chebyshev functional

$$M(f, g, p) = \int_0^T p(x) dx \int_0^T p(x)f(x)g(x) dx - \int_0^T p(x)f(x) dx \int_0^T p(x)g(x) dx,$$

where  $T > 0$ ,  $f$  and  $g$  are two integrable functions on  $[0, T]$ , and  $p$  is a non-negative and integrable function on  $[0, T]$ .

In [4], Dragomir proved the following interesting result.

**Theorem 2.12** *Suppose that  $f$  and  $g$  are two differentiable functions,  $f', g' \in L^\infty((0, T); \mathbb{R})$ , and  $p$  is a non-negative and integrable function on  $[0, T]$ . Then*

$$|M(f, g, p)| \leq \|f'\|_\infty \|g'\|_\infty \left( \int_0^T p(x) dx \int_0^T x^2 p(x) dx - \left( \int_0^T x p(x) dx \right)^2 \right).$$

Observe that if we assume also that  $f$  and  $g$  have the same monotony, then

$$M(f, g, p) \geq 0.$$

Indeed, in this case, we have

$$F(x, y) := \frac{(f(x) - f(y))(g(x) - g(y))}{2} \geq 0, \quad (x, y) \in (0, T) \times (0, T).$$

Therefore,

$$\int_0^T \int_0^T p(x)p(y)F(x, y) dx dy \geq 0.$$

On the other hand, it can be easily seen that

$$M(f, g, p) = \int_0^T \int_0^T p(x)p(y)F(x, y) dx dy \geq 0.$$

Then we can state the following result.

**Theorem 2.13** *Suppose that  $f$  and  $g$  are two differentiable functions having the same monotony,  $f', g' \in L^\infty((0, T); \mathbb{R})$ , and  $p$  is a non-negative and integrable function on  $[0, T]$ . Then*

$$0 \leq M(f, g, p) \leq \|f'\|_\infty \|g'\|_\infty \left( \int_0^T p(x) dx \int_0^T x^2 p(x) dx - \left( \int_0^T xp(x) dx \right)^2 \right). \quad (23)$$

In [5], Dahmani presented the following fractional version of Theorem 2.13.

**Theorem 2.14** *Let  $p$  be a non-negative function on  $[0, \infty)$  and let  $f$  and  $g$  be two differentiable functions having the same monotony on  $[0, \infty)$ . If  $f', g' \in L^\infty((0, \infty); \mathbb{R})$ , then*

$$\begin{aligned} 0 &\leq J_{0+}^\alpha p(T) J_{0+}^\alpha (pfg)(T) - J_{0+}^\alpha (pf)(T) J_{0+}^\alpha (pg)(T) \\ &\leq \|f'\|_\infty \|g'\|_\infty (J_{0+}^\alpha p(T) J_{0+}^\alpha (T^2 p)(T) - (J_{0+}^\alpha (Tp)(T))^2), \end{aligned} \quad (24)$$

where  $\alpha > 0$ .

We have the following observation concerning Theorem 2.14.

**Theorem 2.15** *Theorem 2.13  $\Leftrightarrow$  Theorem 2.14.*

*Proof* Suppose that all assumptions of Theorem 2.14 are satisfied. Let us introduce the function

$$\tilde{p}(x) = (T - x)^{\alpha-1} p(x), \quad 0 \leq x < T.$$

Clearly,  $\tilde{p}$  is non-negative and

$$\int_0^T \tilde{p}(x) dx = J_{0+}^\alpha p(T). \quad (25)$$



By Theorem 2.13, it follows from (23) that

$$0 \leq M(f, g, \tilde{p}) \leq \|f'\|_{\infty} \|g'\|_{\infty} \left( \int_0^T \tilde{p}(x) dx \int_0^T x^2 \tilde{p}(x) dx - \left( \int_0^T x \tilde{p}(x) dx \right)^2 \right). \quad (26)$$

On the other hand, it can be easily seen that

$$\int_0^T \tilde{P}(x) f(x) g(x) dx = J_{0+}^{\alpha} (pfg)(T), \quad (27)$$

$$\int_0^T \tilde{P}(x) f(x) dx = J_{0+}^{\alpha} (pf)(T), \quad (28)$$

$$\int_0^T \tilde{P}(x) g(x) dx = J_{0+}^{\alpha} (pg)(T), \quad (29)$$

$$\int_0^T x^2 \tilde{P}(x) dx = J_{0+}^{\alpha} (T^2 p)(T), \quad (30)$$

$$\int_0^T x \tilde{P}(x) dx = J_{0+}^{\alpha} (Tp)(T). \quad (31)$$

Using (25), (27), (28), and (29), we obtain

$$M(f, g, \tilde{p}) = J_{0+}^{\alpha} p(T) J_{0+}^{\alpha} (pfg)(T) - J_{0+}^{\alpha} (pf)(T) J_{0+}^{\alpha} (pg)(T). \quad (32)$$

Next, using (26), (25), (30), and (31), we obtain

$$\begin{aligned} 0 &\leq J_{0+}^{\alpha} p(T) J_{0+}^{\alpha} (pfg)(T) - J_{0+}^{\alpha} (pf)(T) J_{0+}^{\alpha} (pg)(T) \\ &\leq \|f'\|_{\infty} \|g'\|_{\infty} \left( J_{0+}^{\alpha} p(T) J_{0+}^{\alpha} (T^2 p)(T) - \left( J_{0+}^{\alpha} (Tp)(T) \right)^2 \right), \end{aligned}$$

which is inequality (24). Therefore, we proved that Theorem 2.13  $\implies$  Theorem 2.14.

Finally, taking  $\alpha = 1$  in Theorem 2.14, we obtain the result given by Theorem 2.13. Therefore, we have Theorem 2.14  $\implies$  Theorem 2.13.  $\square$

### 3 Conclusion

Recently, a lot of papers are published concerning inequalities involving different kinds of fractional integrals. In this paper, we proved that most of those inequalities are just particular cases of (or equivalent to) existing results from the literature. We discussed only three types of inequalities: Hermite-Hadamard-type inequalities, Grüss-type inequalities, and an inequality related to Chebyshev's functional. However, the used technique can be also applied for many other published results.

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The authors declare that they have no competing interests.

#### Authors' contributions

MK and BS worked jointly. All the authors read and approved the final manuscript.

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**References**

- Gorenflo, R, Mainardi, F: Fractional Calculus: Integral and Differential Equations of Fractional Order, pp. 223-276. Springer, Wien (1997)
- Kirane, M, Torebek, BT: Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via fractional integrals (2016). arXiv:1701.00092v1
- Dahmani, Z, Tabharit, L: On weighted Gruss type inequalities via fractional integrals. *J. Adv. Res. Pure Math.* **2**(4), 31-38 (2010)
- Dragomir, SS: Some integral inequalities of Gruss type. *RGMA Res. Rep. Collect.* **1**(2), 95-111 (1998)
- Dahmani, Z: The Riemann-Liouville operator to generate some new inequalities. *Int. J. Nonlinear Sci.* **12**(4), 452-455 (2011)
- Dragomir, SS, Pearce, CEM: Selected topics on Hermite-Hadamard inequalities and applications (2000)
- Hadamard, J: Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **58**, 171-215 (1893)
- Hermite, Ch: Sur deux limites d'une integrale définie. *Mathesis* **3**, 82 (1883)
- Fejér, L: Über die Fourierreihen, II. *Math. Naturwiss. Anz. Ungar. Akad. Wiss.* **24**, 369-390 (1906) (in Hungarian)
- Agarwal, P, Jleli, M, Tomar, M: Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. *J. Inequal. Appl.* **2017**, Article ID 55 (2017)
- Chen, H, Katugampola, UN: Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals. *J. Math. Anal. Appl.* **446**(2), 1274-1291 (2017)
- Işcan, I: Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals (2014) arXiv:1404.7722v1
- Jleli, M, Samet, B: On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function. *J. Nonlinear Sci. Appl.* **9**, 1252-1260 (2016)
- Sarikaya, MZ, Set, E, Yaldiz, H, Başak, N: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **57**, 2403-2407 (2013)
- Gruss, D: Über das maximum des absoluten Betrages von  $\frac{1}{b-a} \int_a^b f(x)g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx$ . *Math. Z.* **39**, 215-226 (1935)

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