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The uniqueness of a nonlinear diffusion equation related to the p -Laplacian

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Abstract

Consider a nonlinear diffusion equation related to the p -Laplacian. Different from the usual evolutionary p -Laplacian equation, the equation is degenerate on the boundary due to the fact that the diffusion coefficient is dependent on the distance function. Not only the existence of the weak solution is established, but also the uniqueness of the weak solution is proved.

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1 Introduction and the main results

Recently, we noticed that Benedikt *et al.* [1] had studied the equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) + q(x)u^\gamma, \quad (x, t) \in Q_T \Omega \times (0, T), \quad (1.1)$$

and shown that the uniqueness of the solutions of equation (1.1) is not true. Here, Ω is an open bounded domain with a smooth boundary, $0 < \gamma < 1$, $p > 1$, $q(x) \in C^1(\overline{\Omega})$, $q(x) \geq 0$ and there exists at least a point $x_0 \in \Omega$, $q(x_0) > 0$. This comes more or less as a surprise. In general, we may think that the source term $q(x)u^\gamma$ only affects the existence of the weak solutions. At the same time, in [2], we have considered the following equation:

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2} \nabla u) + f(u, x, t), \quad (x, t) \in Q_T, \quad (1.2)$$

and we have shown that the uniqueness of the weak solution is true when $f(u, \cdot, \cdot)$ is a Lipschitz function, here $\alpha > 0$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary. Certainly, since $0 < \gamma < 1$ in equation (1.1), $f(u, x, t) = q(x)u^\gamma$ is not a Lipschitz function about the variable u . Consequently, the results in [1] and [2] are compatible.

If $\alpha = 0$, there are a great deal of papers devoted to equations (1.2), many of them are important and interesting. But it is impossible to list all these papers, and we only list a few of them [3–7] here.

In this paper, we assume that $q(x) \in C^1(\overline{\Omega})$. We will consider a nonlinear convection-diffusion equation related to the p -Laplacian,

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2} \nabla u) + \sum_{i=1}^N \frac{\partial b_i(u)}{\partial x_i} + q(x)|u|^{\gamma-1}u, \quad (x, t) \in Q_T, \quad (1.3)$$

where $0 < \gamma < 1$. The initial value condition

$$u(x, t) = u_0(x), \quad x \in \Omega, \quad (1.4)$$

is always necessary. Different from the usual evolutionary p -Laplacian equation or equation (1.1), an obvious feature of equations (1.2), (1.3) lies in that the diffusion coefficient ρ^α depends on the distance to the boundary. By this feature, instead of the usual boundary value condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.5)$$

only a partial boundary condition,

$$u(x, t) = 0, \quad (x, t) \in \Sigma_p \times (0, T), \quad (1.6)$$

should be imposed generally, where $\Sigma_p \subseteq \partial\Omega$ is a relatively open subset in $\partial\Omega$. One can refer to our previous work [2, 8].

Since equation (1.3) is a nonlinear equation, it is difficult to depict Σ_p as the linear degenerate parabolic equation by the Fichera function. The main aim of this paper is to prove the uniqueness of the solutions without any boundary value condition.

In the first place, since we had known the interesting result of [1] (*i.e.* the nonuniqueness of the weak solution of equation (1.1)), we should clarify why the uniqueness of the weak solutions of equation (1.3) can be obtained. Let us introduce some basic functional spaces. For every fixed $t \in [0, T]$, we define the Banach space

$$V_t(\Omega) = \{u(x, t) : u(x, t) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u(x, t)|^p \in L^1(\Omega)\},$$

$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p,\Omega},$$

and we denote by $V'_t(\Omega)$ its dual. Also, we denote the Banach space

$$\begin{cases} \mathbf{W}(Q_T) = \{u : [0, T] \rightarrow V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^p \in L^1(Q_T), u = 0 \text{ on } \Gamma = \partial\Omega\}, \\ \|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{p,Q_T} + \|u\|_{2,Q_T}, \end{cases}$$

and we denote by $\mathbf{W}'(Q_T)$ its dual. According to Antontsev-Shmarev [9], we know

$$w \in \mathbf{W}'(Q_T) \iff \begin{cases} w = w_0 + \sum_{i=1}^N D_i w_i, & w_0 \in L^2(Q_T), w_i \in L^{p'}(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T), & \langle w, \phi \rangle = \iint_{Q_T} (w_0 \phi + \sum_{i=1}^N w_i D_i \phi) dx dt. \end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\{\langle v, \phi \rangle | \phi \in \mathbf{W}(Q_T), \|\phi\|_{\mathbf{W}(Q_T)} \leq 1\}.$$

Basing on these functional spaces, we can give the definition of the weak solution.

Definition 1.1 A nonnegative function $u(x, t)$ is said to be a weak solution of equation (1.3) with the initial value (1.4), if

$$u \in L^\infty(Q_T), \quad \rho^\alpha |\nabla u|^p \in L^1(Q_T), \quad u_t \in \mathbf{W}'(Q_T), \quad (1.7)$$

and, for any function $\varphi \in L^\infty(0, T; W_0^{1,p}(\Omega)) \cap \mathbf{W}(Q_T)$,

$$\begin{aligned} \langle u_t, \varphi \rangle + \iint_{Q_T} \left(\rho^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \sum_{i=1}^N b_i(u) \cdot \varphi_{x_i} \right) dx dt \\ = \iint_{Q_T} q(x) |u|^{\gamma-1} u \varphi dx dt. \end{aligned} \quad (1.8)$$

The initial value is satisfied in the sense that

$$\lim_{t \rightarrow 0} \int_{\Omega} |u(x, t) - u_0(x)| dx = 0. \quad (1.9)$$

The most important of Definition 1.1 lies in $u_t \in \mathbf{W}'(Q_T)$. Once the weak solution comes with this property, then we have Lemma 3.1 below, and just by this lemma, we can prove the uniqueness. By comparing the analysis in [1], we know the weak solution defined in [1] does not have this property.

Second, we introduce the existence result.

Theorem 1.2 If $p > 1$, $0 < \gamma < 1$, $b_i(s)$ is a C^1 function, and

$$u_0 \in L^\infty(\Omega), \quad \rho^\alpha |\nabla u_0|^p \in L^1(\Omega), \quad (1.10)$$

then equation (1.1) with initial value (1.4) has a weak solution.

Last but not least we will prove the following local stability.

Theorem 1.3 Let $p > 1$, $\gamma > 0$, $b_i(s)$ be a Lipschitz function. If u, v are two solutions of equation (1.3) with the initial values $u_0(x)$, $v_0(x)$, respectively, then there exists a positive constant $\beta \geq \max\{\frac{p-\alpha}{p-1}, 2, \alpha p\}$ such that

$$\int_{\Omega} \rho^\beta |u(x, t) - v(x, t)|^2 dx \leq c \int_{\Omega} \rho^\beta |u_0(x) - v_0(x)|^2 dx. \quad (1.11)$$

In particular, for any small enough constant $\lambda > 0$,

$$\int_{\Omega_\lambda} |u(x, t) - v(x, t)|^2 dx \leq c \lambda^{-\beta} \int_{\Omega} |u_0(x) - v_0(x)|^2 dx. \quad (1.12)$$

Here, $\Omega_\lambda = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \lambda\}$, by the arbitrariness of λ , we have the uniqueness of the solution. This conclusion implies that the degeneracy of the diffusion coefficient can take place of the usual boundary value condition.

We would like to suggest that, if ρ^α is substituted by a nonnegative diffusion coefficient $a(x) \in C^1(\overline{\Omega})$ with

$$a(x)|_{x \in \Omega} > 0, \quad a(x)|_{x \in \partial\Omega} = 0,$$

a similar conclusion to Theorem 1.3 is still true. For some special cases, one can see our recent work [10]. Actually, we had used some ideas of [10] to prove Theorem 1.3.

This paper is arranged as follows. In Section 1, we give the basic definition and introduce the main results. In Section 2, we prove the existence of the solution to equation (1.1) with initial value (1.4). In Section 3, we prove Theorem 1.3 and obtain the uniqueness of the solution.

2 The weak solutions dependent on the initial value

We consider the weak solution of the initial value problem for equation (1.3) in this section. It is supposed that u_0 satisfies

$$u_0 \in L^\infty(\Omega), \quad |\nabla u_0|^p \in L^1(\Omega).$$

Let $u_{\varepsilon,0} \in C_0^\infty(\Omega)$ and $\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^p \in L^1(\Omega)$ be uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W_0^{1,p}(\Omega)$. Here $\rho_\varepsilon = \rho * \delta_\varepsilon + \varepsilon$, $\varepsilon > 0$, δ_ε is the mollifier as usual.

By the results of [11, Section 8], we have the following important lemma.

Lemma 2.1 *If $u_\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap \mathbf{W}(Q_T)$, $\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c$, $\|\nabla(|u_\varepsilon|^{q-1}u_\varepsilon)\|_{p,Q_T} \leq c$, then there is a subsequence of $\{u_\varepsilon\}$ which is relatively compact in $L^s(Q_T)$ with $s \in (1, \infty)$. Here $q \geq 1$.*

We now consider the following regularized problem:

$$\begin{aligned} u_{\varepsilon t} - \operatorname{div}(\rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon) - \sum_{i=1}^N \frac{\partial b_i(u_\varepsilon)}{\partial x_i} \\ = q(x)|u_\varepsilon|^{\gamma-1}u_\varepsilon, \quad (x, t) \in Q_T, \end{aligned} \quad (2.1)$$

$$u_\varepsilon(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (2.2)$$

$$u_\varepsilon(x, 0) = u_{\varepsilon,0}(x), \quad x \in \Omega, \quad (2.3)$$

since $0 < \gamma < 1$, it is well known that the above problem has an unique classical solution [12, 13].

By the maximum principle, there is a constant c only dependent on $\|u_0\|_{L^\infty(\Omega)}$ but independent on ε , such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c. \quad (2.4)$$

Multiplying (2.1) by u_ε and integrating it over Q_T , we get

$$\begin{aligned} \frac{1}{2} \int_\Omega u_\varepsilon^2 dx + \iint_{Q_T} \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 dx dt \\ + \sum_{i=1}^N \iint_{Q_T} u_\varepsilon \frac{\partial b_i(u_\varepsilon)}{\partial x_i} dx dt \\ = \frac{1}{2} \int_\Omega u_0^2 dx + \iint_{Q_T} q(x)|u_\varepsilon|^{\gamma-1}u_\varepsilon dx dt. \end{aligned}$$

By the fact

$$\begin{aligned} \sum_{i=1}^N \iint_{Q_T} u_\varepsilon \frac{\partial b_i(u_\varepsilon)}{\partial x_i} dx dt &= - \sum_{i=1}^N \iint_{Q_T} \frac{\partial u_\varepsilon}{\partial x_i} b_i(u_\varepsilon) dx dt \\ &= - \sum_{i=1}^N \int_\Omega \frac{\partial}{\partial x_i} \int_0^{u_\varepsilon} b_i(s) ds dx = 0, \end{aligned}$$

then

$$\frac{1}{2} \int_\Omega u_\varepsilon^2 dx + \iint_{Q_T} \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 dx dt \leq c. \quad (2.5)$$

It is also easy to show that

$$\iint_{Q_T} \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^p dx dt \leq c \iint_{Q_T} \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^p dx dt \leq c. \quad (2.6)$$

Now, for any $v \in \mathbf{W}(Q_T)$, $\|v\|_{\mathbf{W}(Q_T)} = 1$,

$$\begin{aligned} \langle u_{\varepsilon t}, v \rangle &= - \iint_{Q_T} \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla v dx dt \\ &\quad - \iint_{Q_T} \frac{\partial v}{\partial x_i} b_i(u_\varepsilon) dx dt + \iint_{Q_T} q(x) |u_\varepsilon|^{\gamma-1} u_\varepsilon v dx dt, \end{aligned} \quad (2.7)$$

by Young inequality, we can show that

$$|\langle u_{\varepsilon t}, v \rangle| \leq c \left[\iint_{Q_T} \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^p dx dt + \iint_{Q_T} (|v|^p + |\nabla v|^p) dx dt + 1 \right] \leq c,$$

then

$$\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c. \quad (2.8)$$

Now, let $\varphi \in C_0^1(\Omega)$, $0 \leq \varphi \leq 1$ such that

$$\varphi|_{\Omega_{2\lambda}} = 1, \quad \varphi|_{\Omega \setminus \Omega_\lambda} = 0.$$

Then

$$|\langle (\varphi u_\varepsilon)_t, v \rangle| = |\langle \varphi u_{\varepsilon t}, v \rangle| \leq |\langle u_{\varepsilon t}, v \rangle|,$$

we have

$$\|(\varphi(x)u)_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq \|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c, \quad (2.9)$$

$$\iint_{Q_T} |\nabla(\varphi u_\varepsilon)|^p dx dt \leq c(\lambda) \left(1 + \int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^p dx dt \right) \leq c(\lambda), \quad (2.10)$$

and so

$$\|\nabla(\varphi u_\varepsilon)\|_{p, Q_T} \leq c. \quad (2.11)$$

By Lemma 2.1, φu_ε is relatively compact in $L^s(Q_T)$ with $s \in (1, \infty)$. Then $\varphi u_\varepsilon \rightarrow \varphi u$ a.e. in Q_T . In particular, due to the arbitrariness of λ , $u_\varepsilon \rightarrow u$ a.e. in Q_T .

Hence, by (2.4), (2.7), there exists a function u and an n -dimensional vector function $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$ satisfying

$$u \in L^\infty(Q_T), \quad |\vec{\zeta}| \in L^{\frac{p}{p-1}}(Q_T),$$

and

$$u_\varepsilon \rightharpoonup^* u, \quad \text{in } L^\infty(Q_T), \quad u_\varepsilon \rightarrow u, \quad \text{a.e. in } Q_T.$$

$$\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2} u_{\varepsilon x_i} \rightharpoonup \zeta_i \quad \text{in } L^{\frac{p}{p-1}}(Q_T).$$

In order to prove that u satisfies equation (1.3), we notice that, for any function $\varphi \in C_0^\infty(Q_T)$,

$$\begin{aligned} & \iint_{Q_T} \left(-u_\varepsilon \varphi_t + \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \varphi + \sum_{i=1}^N b_i(u_\varepsilon) \cdot \varphi_{x_i} \right) dx dt \\ &= \iint_{Q_T} q(x) |u_\varepsilon|^{\gamma-1} u_\varepsilon \varphi dx dt, \end{aligned} \quad (2.12)$$

and $u_\varepsilon \rightarrow u$ is almost everywhere convergent, so $b_i(u_\varepsilon) \rightarrow b_i(u)$, $|u_\varepsilon|^{\gamma-1} u_\varepsilon \rightarrow |u|^{\gamma-1} u$. Then

$$\begin{aligned} & \iint_{Q_T} \left(\frac{\partial u}{\partial t} \varphi + \vec{\zeta} \cdot \nabla \varphi + \sum_{i=1}^N b_i(u) \cdot \varphi_{x_i} \right) dx dt \\ &= \iint_{Q_T} q(x) |u|^{\gamma-1} u \varphi dx dt. \end{aligned} \quad (2.13)$$

Now, if we can prove that

$$\iint_{Q_T} \rho^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_1 dx dt = \iint_{Q_T} \vec{\zeta} \cdot \nabla \varphi_1 dx dt \quad (2.14)$$

for any function $\varphi_1 \in C_0^\infty(Q_T)$, then u satisfies equation (1.3). In what follows, we will use a similar method to that in [14] to prove (2.14).

Let $0 \leq \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ in $\text{supp } \varphi_1$. Let $v \in L^\infty(Q_T)$, $\rho^\alpha |\nabla v|^p \in L^1(Q_T)$. It is well known that

$$\iint_{Q_T} \psi \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla v|^{p-2} \nabla v) \cdot (\nabla u_\varepsilon - \nabla v) dx dt \geq 0. \quad (2.15)$$

By choosing $\varphi = \psi u_\varepsilon$ in (2.12), we can obtain

$$\begin{aligned} & \iint_{Q_T} \psi \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 dx dt \\ &= \frac{1}{2} \iint_{Q_T} \psi_t u_\varepsilon^2 dx dt - \iint_{Q_T} \rho_\varepsilon^\alpha u_\varepsilon (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi dx dt \\ & \quad - \sum_{i=1}^N \iint_{Q_T} b_i(u_\varepsilon) (u_{\varepsilon x_i} \psi + u_\varepsilon \psi_{x_i}) dx dt + \iint_{Q_T} q(x) |u_\varepsilon|^{\gamma-1} u_\varepsilon \varphi dx dt. \end{aligned} \quad (2.16)$$

Noticing that, when $p \geq 2$,

$$\begin{aligned} & (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 \geq |\nabla u_\varepsilon|^p, \\ & (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon| \leq (|\nabla u_\varepsilon|^{p-1} + 1), \end{aligned}$$

and, when $1 < p < 2$,

$$\begin{aligned} & (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 \geq (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} - \varepsilon^{\frac{p}{2}}, \\ & (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon| \leq (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-1}{2}}, \end{aligned}$$

then in both cases, by (2.15), we have

$$\begin{aligned} & \frac{1}{2} \iint_{Q_T} \psi_t u_\varepsilon^2 dx dt - \iint_{Q_T} \rho_\varepsilon^\alpha u_\varepsilon (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi dx dt \\ & \quad - \sum_{i=1}^N \iint_{Q_T} b_i(u_\varepsilon) (u_{\varepsilon x_i} \psi + u_\varepsilon \psi_{x_i}) dx dt + \iint_{Q_T} q(x) |u_\varepsilon|^{\gamma-1} u_\varepsilon \varphi dx dt \\ & \quad + \varepsilon^{\frac{p}{2}} c(\Omega) - \iint_{Q_T} \rho_\varepsilon^\alpha \psi |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla v dx dt \\ & \quad - \iint_{Q_T} \rho_\varepsilon^\alpha \psi |\nabla v|^{p-2} \nabla (u_\varepsilon - v) dx dt \geq 0. \end{aligned} \quad (2.17)$$

Thus

$$\begin{aligned} & \frac{1}{2} \iint_{Q_T} \psi_t u_\varepsilon^2 dx dt - \iint_{Q_T} \rho_\varepsilon^\alpha u_\varepsilon (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi dx dt \\ & \quad - \sum_{i=1}^N \iint_{Q_T} b_i(u_\varepsilon) (u_{\varepsilon x_i} \psi + u_\varepsilon \psi_{x_i}) dx dt + \iint_{Q_T} q(x) |u_\varepsilon|^{\gamma-1} u_\varepsilon \varphi dx dt \\ & \quad + \varepsilon^{\frac{p}{2}} c(\Omega) - \iint_{Q_T} \rho_\varepsilon^\alpha \psi |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla v dx dt \\ & \quad - \iint_{Q_T} \psi \rho_\varepsilon^\alpha |\nabla v|^{p-2} \nabla v \cdot (\nabla u_\varepsilon - \nabla v) dx dt \\ & \quad + \iint_{Q_T} \psi (\rho^\alpha - \rho_\varepsilon^\alpha) |\nabla v|^{p-2} \nabla v \cdot (\nabla u_\varepsilon - \nabla v) dx dt \geq 0. \end{aligned} \quad (2.18)$$

Notice

$$\begin{aligned}
 & \left| \iint_{Q_T} \psi (\rho^\alpha - \rho_\varepsilon^\alpha) |\nabla v|^{p-2} \nabla v \cdot (\nabla u_\varepsilon - \nabla v) dx dt \right| \\
 & \leq \sup_{(x,t) \in Q_T} \frac{|\psi (\rho^\alpha - \rho_\varepsilon^\alpha)|}{\rho^\alpha} \iint_{Q_T} \rho^\alpha |\nabla v|^{p-1} |\nabla u_\varepsilon - \nabla v| dx dt \\
 & \leq \sup_{(x,t) \in Q_T} \frac{|\psi (\rho^\alpha - \rho_\varepsilon^\alpha)|}{\rho^\alpha} \left(\iint_{Q_T} \rho^\alpha |\nabla v|^p dx dt + \iint_{Q_T} \rho^\alpha |\nabla v|^{p-1} |\nabla u_\varepsilon| dx dt \right) \quad (2.19)
 \end{aligned}$$

and by the Hölder inequality

$$\begin{aligned}
 & \iint_{Q_T} \rho^\alpha |\nabla v|^{p-1} |\nabla u_\varepsilon| dx dt \\
 & \leq \left(\iint_{Q_T} (\rho^m |\nabla v|^{p-1})^s dx dt \right)^{1/s} \cdot \left(\iint_{Q_T} (\rho^n |\nabla u_\varepsilon|)^p dx dt \right)^{1/p},
 \end{aligned}$$

where $m = \frac{\alpha(p-1)}{p}$, $n = \frac{\alpha}{p}$, $s = \frac{p}{p-1}$. Due to $\rho^\alpha |\nabla u|^p, \rho^\alpha |\nabla v|^p \in L^1(Q_T)$, we have

$$\iint_{Q_T} \rho^\alpha |\nabla v|^p dx dt + \iint_{Q_T} \rho^\alpha |\nabla v|^{p-1} |\nabla u_\varepsilon| dx dt \leq c.$$

Let $\varepsilon \rightarrow 0$ in (2.19). It converges to 0.

Once more, we notice that

$$\begin{aligned}
 & (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \\
 & = |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon + \frac{p-2}{2} \varepsilon \int_0^1 (|\nabla u_\varepsilon|^2 + \varepsilon s)^{\frac{p-4}{2}} ds \nabla u_\varepsilon, \\
 & \lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \frac{p-2}{2} \varepsilon \int_0^1 (|\nabla u_\varepsilon|^2 + \varepsilon s)^{\frac{p-4}{2}} ds \nabla u_\varepsilon \nabla \psi u_\varepsilon dx dt = 0.
 \end{aligned} \quad (2.20)$$

Let $\varepsilon \rightarrow 0$ in (2.18), we have

$$\begin{aligned}
 & \frac{1}{2} \iint_{Q_T} \psi_t u^2 dx dt - \iint_{Q_T} u \vec{\zeta} \cdot \nabla \psi dx dt - \sum_{i=1}^N \iint_{Q_T} b_i(u) (u_{x_i} \psi + u \psi_{x_i}) dx dt \\
 & - \iint_{Q_T} \psi \vec{\zeta} \cdot \nabla v dx dt - \iint_{Q_T} \psi \rho^\alpha |\nabla v|^{p-2} \nabla v \cdot (\nabla u - \nabla v) dx dt \\
 & + \iint_{Q_T} q(x) |u|^{\gamma-1} u \varphi dx dt \geq 0.
 \end{aligned}$$

Let $\varphi = \psi u$ in (2.13), we get

$$\begin{aligned}
 & \iint_{Q_T} \psi \vec{\zeta} \cdot \nabla u dx dt - \frac{1}{2} \iint_{Q_T} u^2 \psi_t dx dt + \iint_{Q_T} u \vec{\zeta} \cdot \nabla \psi dx dt \\
 & + \sum_{i=1}^N \iint_{Q_T} b_i(u) (u_{x_i} \psi + u \psi_{x_i}) dx dt + \iint_{Q_T} q(x) |u|^{\gamma-1} u \psi u dx dt = 0.
 \end{aligned}$$

Thus

$$\iint_{Q_T} \psi (\vec{\zeta} - \rho^\alpha |\nabla v|^{p-2} \nabla v) \cdot (\nabla u - \nabla v) dx dt \geq 0. \quad (2.21)$$

Let $v = u - \lambda \varphi_1$, $\lambda > 0$, $\varphi_1 \in C_0^\infty(Q_T)$ is given in (2.14), then

$$\iint_{Q_T} \psi [\vec{\zeta} - \rho^\alpha |\nabla(u - \lambda \varphi_1)|^{p-2} \nabla(u - \lambda \varphi_1)] \cdot \nabla \varphi_1 dx dt \geq 0.$$

If $\lambda \rightarrow 0$, then

$$\iint_{Q_T} \psi (\vec{\zeta} - \rho^\alpha |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi_1 dx dt \geq 0.$$

Moreover, if $\lambda < 0$, similarly we can get

$$\iint_{Q_T} \psi (\vec{\zeta} - \rho^\alpha |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi_1 \leq 0.$$

Thus

$$\iint_{Q_T} \psi (\vec{\zeta} - \rho^\alpha |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi_1 dx dt = 0.$$

Noticing that $\psi = 1$ on $\text{supp } \varphi_1$, (2.14) holds.

At same time, we are able to prove (1.9) as in [15], thus we have Theorem 1.2.

3 The uniqueness without the boundary value condition

Lemma 3.1 *Let $u \in \mathbf{W}(Q_T)$, $u_t \in \mathbf{W}'(Q_T)$. Then \forall a.e. $t_1, t_2 \in (0, T)$,*

$$\int_{t_1}^{t_2} \int_{\Omega} uu_t dx dt = \frac{1}{2} \left[\int_{\Omega} (u^2(x, t_2) - u^2(x, t_1)) dx \right]. \quad (3.1)$$

This is Corollary 2.1 of [9].

Proof of Theorem 1.3 Let u, v be two solutions of equation (1.3) with the initial values $u_0(x), v_0(x)$, respectively. Denote $\Omega_\lambda = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \lambda\}$, let the constant $\beta \geq \max\{\frac{p-\alpha}{p-1}, 2, \alpha p\}$ and

$$\xi_\lambda = [\text{dist}(x, \Omega \setminus \Omega_\lambda)]^\beta = d_\lambda^\beta. \quad (3.2)$$

We may choose $\chi_{[\tau, s]}(u_\varepsilon - v_\varepsilon)\xi_\lambda$ as a test function, where u_ε and v_ε are the mollified function of the solutions u and v , respectively. Then

$$\begin{aligned} & \langle (u - v)_t, \chi_{[\tau, s]}(u_\varepsilon - v_\varepsilon)\xi_\lambda \rangle \\ &= \iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon)\xi_\lambda \frac{\partial(u - v)}{\partial t} dx dt \\ &= - \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla [(u_\varepsilon - v_\varepsilon)\xi_\lambda] dx dt \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^N \iint_{Q_{\tau s}} [b_i(u) - b_i(v)] [(u_\varepsilon - v_\varepsilon) \xi_\lambda]_{x_i} dx dt \\
& + \iint_{Q_{\tau s}} q(x) (|u|^{\gamma-1} u - |v|^{\gamma-1} v) (u_\varepsilon - v_\varepsilon) \xi_\lambda dx dt,
\end{aligned} \quad (3.3)$$

where $Q_{\tau s} = \Omega \times (\tau, s)$. For any give $\lambda > 0$, since $\nabla u \in L^p(\Omega_\lambda)$, $\nabla v \in L^p(\Omega_\lambda)$, according to the definition of the mollified function u_ε and v_ε , we have

$$u_\varepsilon \in L^\infty(Q_T), \quad v_\varepsilon \in L^\infty(Q_T), \quad u_\varepsilon \rightarrow u, \quad v_\varepsilon \rightarrow v, \quad \text{a.e. in } Q_T, \quad (3.4)$$

$$\begin{aligned}
& \|\nabla u_\varepsilon\|_{p, \Omega_\lambda} \leq \|\nabla u\|_{p, \Omega_\lambda}, \quad \|\nabla v_\varepsilon\|_{p, \Omega_\lambda} \leq \|\nabla v\|_{p, \Omega_\lambda}, \\
& u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v, \quad \text{in } W^{1,p}(\Omega_\lambda).
\end{aligned} \quad (3.5)$$

Let us analyze every term in (3.3). For a start, we deal with the first term on the right hand side of (3.3). Since on Ω_λ ,

$$|\rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v)| \in L^{\frac{p}{p-1}}(\Omega_\lambda)$$

by the weak convergency of (3.5)

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla (u_\varepsilon - v_\varepsilon) dx dt \\
& = \iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla (u - v) dx dt.
\end{aligned}$$

By (3.4)-(3.5), using the Lebesgue dominated convergence theorem,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) (u_\varepsilon - v_\varepsilon) \nabla \xi_\lambda dx dt \\
& = \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) (u - v) \nabla \xi_\lambda dx dt.
\end{aligned}$$

So

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla [(u_\varepsilon - v_\varepsilon) \xi_\lambda] dx dt \\
& = \iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla (u - v) dx dt \\
& \quad + \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) (u - v) \nabla \xi_\lambda dx dt.
\end{aligned} \quad (3.6)$$

We have

$$\iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla (u - v) dx dt \geq 0 \quad (3.7)$$

and

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \left| \iint_{Q_{\tau s}} \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) (u - v) \nabla \xi_\lambda \, dx \, dt \right| \\
&= \left| \iint_{Q_{\tau s}} (u - v) \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla \rho^\beta \, dx \, dt \right| \\
&\leq \iint_{Q_{\tau s}} |u - v| \rho^\alpha (|\nabla u|^{p-1} + |\nabla v|^{p-1}) |\nabla \rho^\beta| \, dx \, dt \\
&\leq c \left(\int_\tau^s \int_\Omega \rho^\alpha (|\nabla u|^p + |\nabla v|^p) \, dx \, dt \right)^{\frac{p-1}{p}} \cdot \left(\int_\tau^s \int_\Omega \rho^\alpha |\nabla \rho^\beta|^p |u - v|^p \, dx \, dt \right)^{\frac{1}{p}} \\
&\leq c \left(\int_\tau^s \int_\Omega \rho^\alpha (|\nabla u|^p + |\nabla v|^p) \, dx \, dt \right)^{\frac{p-1}{p}} \cdot \left(\int_\tau^s \int_\Omega \rho^{\alpha+p(\beta-1)} |u - v|^p \, dx \, dt \right)^{\frac{1}{p}} \\
&\leq c \left(\int_\tau^s \int_\Omega \rho^{\alpha+p(\beta-1)} |u - v|^p \, dx \, dt \right)^{\frac{1}{p}}. \tag{3.8}
\end{aligned}$$

Here, we have used the fact that $|\nabla \rho| = 1$ is true almost everywhere. Now, by $\beta \geq \frac{p-\alpha}{p-1}$, we have

$$\begin{aligned}
& \left| \iint_{Q_{\tau s}} (u - v) \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \nabla \rho^\beta \, dx \, dt \right| \\
&\leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^p \, dx \, dt \right)^{\frac{1}{p}}. \tag{3.9}
\end{aligned}$$

If $p \geq 2$,

$$\left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^p \, dx \, dt \right)^{\frac{1}{p}} \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 \, dx \, dt \right)^{\frac{1}{p}}. \tag{3.10}$$

If $1 < p < 2$, by the Hölder inequality

$$\left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^p \, dx \, dt \right)^{\frac{1}{p}} \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{3.11}$$

Now we deal the second term on the right hand side of (3.3). By the Lebesgue dominated convergence theorem and the Hölder inequality

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} [b_i(u) - b_i(v)] [(u_\varepsilon - v_\varepsilon) \xi_\lambda]_{x_i} \, dx \, dt \\
&= \lim_{\lambda \rightarrow 0} \iint_{Q_{\tau s}} [b_i(u) - b_i(v)] [(u - v) \xi_\lambda]_{x_i} \, dx \, dt \\
&= \lim_{\lambda \rightarrow 0} \left(\iint_{Q_{\tau s}} [b_i(u) - b_i(v)] (u - v) \xi_{\lambda, x_i} \, dx \, dt \right. \\
&\quad \left. + \iint_{Q_s} [b_i(u) - b_i(v)] (u - v)_{x_i} \xi_\lambda \, dx \, dt \right) \\
&= \iint_{Q_{\tau s}} [b_i(u) - b_i(v)] (u - v) \rho_{x_i}^\beta \, dx \, dt + \iint_{Q_s} [b_i(u) - b_i(v)] (u - v)_{x_i} \rho^\beta \, dx \, dt. \tag{3.12}
\end{aligned}$$

Since $\beta \geq 2$, $|\rho_{x_i}| \leq |\nabla \rho| = 1$, by the Hölder inequality,

$$\begin{aligned} & \iint_{Q_{\tau s}} [b_i(u) - b_i(v)](u - v) \rho_{x_i}^\beta dx dt \\ &= \int_\tau^s \int_{\Omega_\lambda} [b_i(u) - b_i(v)](u - v) \rho^{\beta-1} |\rho_{x_i}| dx \\ &\leq \int_\tau^s \int_\Omega |u - v| \rho^{\beta-1} dx \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned} \quad (3.13)$$

Since $\beta \geq \alpha p$, we have

$$\left(\beta - \frac{\alpha}{p} \right) \frac{p}{p-1} \geq \beta,$$

by this result, we have

$$\begin{aligned} & \left| \iint_{Q_{\tau s}} [b_i(u) - b_i(v)](u - v)_{x_i} \rho^\beta dx dt \right| \\ &\leq \sum_{i=1}^N \left(\int_\tau^s \int_\Omega \rho^{(\beta - \frac{\alpha}{p})p'} (|b_i(u) - b_i(v)|)^{p'} dx dt \right)^{\frac{1}{p'}} \\ &\quad \times \left(\int_\tau^s \int_\Omega \rho^\alpha (|\nabla u|^p + |\nabla v|^p) dx dt \right)^{\frac{1}{p}} \\ &\leq c \sum_{i=1}^N \left(\int_\tau^s \int_\Omega \rho^\beta (|b_i(u) - b_i(v)|)^{p'} dx dt \right)^{\frac{1}{p'}} \\ &\leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^{p'} dx dt \right)^{\frac{1}{p'}}. \end{aligned} \quad (3.14)$$

If $p > 2$, then $1 < p' < 2$. By the Hölder inequality,

$$\left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^{p'} dx dt \right)^{\frac{1}{p'}} \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 dx dt \right)^{\frac{1}{2}}, \quad (3.15)$$

If $1 < p \leq 2$, then $p' \geq 2$,

$$\left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^{p'} dx dt \right)^{\frac{1}{p'}} \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 dx dt \right)^{\frac{1}{p'}}. \quad (3.16)$$

Again, for the third term on the right hand side of (3.3),

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} q(x) (|u|^{\gamma-1} u - |v|^{\gamma-1} v) (u_\varepsilon - v_\varepsilon) \xi_\lambda dx dt \\ &\leq \iint_{Q_{\tau s}} q(x) |u^\gamma - v^\gamma| |u - v| \rho^\beta dx dt \leq c \iint_{Q_{\tau s}} q(x) \rho^\beta |u - v| dx dt \\ &\leq c \iint_{Q_{\tau s}} \sqrt{\rho^\beta} |u - v| dx dt \leq c \left(\int_\tau^s \int_\Omega \rho^\beta |u - v|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned} \quad (3.17)$$

At last, by Lemma 3.1,

$$\begin{aligned}
 & \lim_{\lambda \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon) \xi_\lambda \frac{\partial(u-v)}{\partial t} dx dt \\
 &= \lim_{\lambda \rightarrow 0} \iint_{Q_{\tau s}} (u-v) \sqrt{\xi_\lambda} \frac{\partial \sqrt{\xi_\lambda}(u-v)}{\partial t} dx dt \\
 &= \frac{1}{2} \lim_{\lambda \rightarrow 0} \int_{\Omega} \xi_\lambda [(u-v)^2(x, s) - (u-v)^2(x, \tau)] dx \\
 &= \frac{1}{2} \left\{ \int_{\Omega} \rho^\beta [u(x, s) - v(x, s)]^2 dx - \int_{\Omega} \rho^\beta [u(x, \tau) - v(x, \tau)]^2 dx \right\}. \quad (3.18)
 \end{aligned}$$

Now, after letting $\varepsilon \rightarrow 0$, let $\lambda \rightarrow 0$ in (3.3). Then, by (3.7)-(3.18),

$$\begin{aligned}
 & \int_{\Omega} \rho^\beta [u(x, s) - v(x, s)]^2 dx - \int_{\Omega} \rho^\beta [u(x, \tau) - v(x, \tau)]^2 dx \\
 & \leq c \left(\int_{\tau}^s \int_{\Omega} \rho^\beta |u(x, t) - v(x, t)|^2 dx dt \right)^k, \quad (3.19)
 \end{aligned}$$

where $k < 1$. By this inequality, we are able to show that

$$\int_{\Omega} \rho^\beta |u(x, s) - v(x, s)|^2 dx \leq \int_{\Omega} \rho^\beta |u(x, \tau) - v(x, \tau)|^2 dx. \quad (3.20)$$

Thus, by the arbitrariness of τ , we have

$$\int_{\Omega} \rho^\beta |u(x, s) - v(x, s)|^2 dx \leq \int_{\Omega} \rho^\beta |u_0 - v_0|^2 dx. \quad (3.21)$$

By (3.21), we clearly have (1.10). The proof is complete. \square

4 Conclusion

The equations considered in this paper come from many applied fields such as mechanics, biology, etc. The main points of focus of this paper are two aspects. One is that the weak solution defined in this paper satisfies $u_t \in W'(Q_T)$, then the uniqueness can be proved. The other one is to show that the degeneracy of the diffusion coefficient ρ^α can take place with the usual boundary value condition.

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Authors' contributions

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