RESEARCH

Open Access

CrossMark

k-fractional integral trapezium-like inequalities through (h, m)-convex and (α, m)-convex mappings

Hao Wang, Tingsong Du* and Yao Zhang

*Correspondence:
tingsongdu@ctgu.edu.cn
Department of Mathematics,
College of Science, China Three
Gorges University, Yichang, 443002,
China

Abstract

In this paper, a new general identity for differentiable mappings via *k*-fractional integrals is derived. By using the concept of (h, m)-convexity, (α, m)-convexity and the obtained equation, some new trapezium-like integral inequalities are established. The results presented provide extensions of those given in earlier works.

MSC: 26A33; 26A51; 26D07; 26D20; 41A55

Keywords: (h, m)-convex functions; (α, m)-convex functions; *k*-fractional integrals

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ along with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a)+f(b)}{2}, \qquad (1.1)$$

named Hermite-Hadamard's inequality, is one of the most famous results for convex mappings. This inequality (1.1) is also known as trapezium inequality.

The trapezium-type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Many researchers generalized and extended it via mappings of different classes. For recent results, for example, see [1–7] and the references mentioned in these papers.

In 2013, Sarikaya et al. [8] established the following theorem by utilizing Riemann-Liouville fractional integrals.

Theorem 1.1 *Let $f : [a, b] \to \mathbb{R}$ be a positive function along with $0 \le a < b$, and let $f \in L^1[a, b]$. Suppose that f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold*:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\mu+1)}{2(b-a)^\mu}\left[J_{a^+}^\mu f(b) + J_{b^-}^\mu f(a)\right] \le \frac{f(a)+f(b)}{2}, \qquad (1.2)$$

Springer

where the symbols $J_{a^+}^{\mu}f$ and $J_{b^-}^{\mu}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu > 0$ defined by

$$J_{a^+}^{\mu}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1}f(t)\,dt, \quad a < x$$

and

$$J_{b^-}^{\mu}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1}f(t)\,dt, \quad x < b.$$

Here, $\Gamma(\mu)$ is the gamma function and its definition is $\Gamma(\mu) = \int_0^\infty e^{-t}t^{\mu-1}\,dt$. It is to be noted that $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral recaptures the classical integral.

Because of the extensive application of Riemann-Liouville fractional integrals, some authors extended their studies to fractional trapezium-type inequalities via mappings of different classes. For example, refer to [9–12] for convex mappings, to [13] for s-convex mappings, to [14] for (s, m)-convex mappings, to [15] for r-convex mappings, to [16] for harmonically convex mappings, to [17] for s-Godunova-Levin mappings, to [18, 19] for preinvex mappings, to [20] for MT_m-preinvex mappings, to [21] for h-convex mappings and to references cited therein.

In [22], Mubeen and Habibullah introduced the following class of fractional derivatives.

Definition 1.1 ([22]) Let $f \in L^1[a, b]$, then k-Riemann-Liouville fractional derivatives ${}_kJ_{a^+}^{\mu}f(x)$ and ${}_kJ_{b^-}^{\mu}f(x)$ of order $\mu > 0$ are given as

$$\,_kJ_{a^+}^{\mu}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1}f(t)\,dt \quad (0 \le a < x < b)$$

and

$$\,_kJ_{b^-}^{\mu}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1}f(t)\,dt \quad (0 \le a < x < b),$$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k-gamma function defined by $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} \times e^{-\frac{t^k}{k}}\,dt$. Furthermore, $\Gamma_k(\mu + k) = \mu\Gamma_k(\mu)$ and ${}_kJ_{a^+}^0 f(x) = {}_kJ_{b^-}^0 f(x) = f(x)$.

The concept of k-Riemann-Liouville fractional integral is an important extension of Riemann-Liouville fractional integrals. We want to stress here that for $k \ne 1$ the properties of k-Riemann-Liouville fractional integrals are quite dissimilar from those of general Riemann-Liouville fractional integrals. For this, the k-Riemann-Liouville fractional integrals have aroused the interest of many researchers. Properties concerning this operator can be sought out [23–26], and for the bounds for integral inequality related to this operator, the reader can refer to [27–29] and the references mentioned in these papers.

Motivated and inspired by the recent research in this field, we obtain some k-Riemann-Liouville fractional integral of trapezium-type inequalities for (h, m)-convex mappings and (α, m)-convex mappings. The results presented in this paper provide extensions of those given in earlier works.

To end this section, we restate some special functions and definitions.

(1) The beta function:

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x, y > 0.$$

(2) The incomplete beta function:

$$\beta(a, x, y) = \int_0^a t^{x-1}(1-t)^{y-1}\, \mathrm{d}t, \quad 0 < a < 1, x, y > 0.$$

Definition 1.2 ([30]) The function $f : [0, b] \to \mathbb{R}$ is named (α, m)-convex if, for every $x, y \in [0, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f\big(tx + m(1-t)y\big) \leq t^\alpha f(x) + m\big(1 - t^\alpha\big)f(y),$$

where $(\alpha, m) \in (0, 1] \times (0, 1]$.

Definition 1.3 ([31]) The function $f : [0, b] \to \mathbb{R}$ is called m-MT-convex if f is non-negative and, for all $x, y \in [0, b]$ and $t \in (0, 1)$, with $m \in (0, 1]$, it satisfies the following inequality:

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Definition 1.4 ([32]) Let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I \to \mathbb{R}$ is said to be h-convex if f is non-negative and

$$f\big(tx + (1-t)y\big) \leq h(t)f(x) + h(1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5 ([33]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. A function $f : I \to \mathbb{R}$ is said to be tgs-convex if the inequality

$$f\big(tx + (1-t)y\big) \leq t(1-t)\big[f(x) + f(y)\big]$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.6 ([34]) Let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : [0, b] \to \mathbb{R}$ is named (h, m)-convex if f is non-negative and

$$f\big(tx + m(1-t)y\big) \leq h(t)f(x) + mh(1-t)f(y)$$

holds for all $x, y \in [0, b]$, $t \in (0, 1)$ and some fixed $m \in (0, 1]$.

Clearly, when putting $h(t) = t(1-t)$ in Definition 1.6, f becomes an (m, tgs)-convex function on $[0, b]$ as follows.

Definition 1.7 The function $f : [0, b] \to \mathbb{R}$ is named (m, tgs)-convex if f is non-negative and

$$f(tx + m(1 - t)y) \leq t(1 - t)[f(x) + mf(y)]$$

holds for all $x, y \in [0, b]$, $t \in (0, 1)$ and some fixed $m \in (0, 1]$.

Note that, if we choose $m = 1$ in Definition 1.7, f reduces to a *tgs*-convex function in Definition 1.5.

2 A lemma

To prove our main results, we consider the following new lemma.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o (the interior of I) with $0 \leq a < mr$, $a, r \in I$, for some fixed $m \in (0, 1]$. If $f' \in L^1[a, mr]$, then the following equality for k-fractional integral along with $\lambda \in (0, 1] \backslash \frac{1}{2}$, $k > 0$ and $\mu > 0$ exists:*

$$
\mathcal{T}_{k,\mu}(m, \lambda, r)
$$
$$
= \int_0^1 \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right) f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) dt, \quad (2.1)
$$

where

$$
\mathcal{T}_{k,\mu}(m, \lambda, r)
$$
$$
:= -\frac{f(m\lambda r + (1-\lambda)a) + f(\lambda a + m(1-\lambda)r)}{(1-2\lambda)(mr-a)} + \frac{\Gamma_k(\mu + k)}{(1-2\lambda)^{\frac{\mu}{k}+1}(mr-a)^{\frac{\mu}{k}+1}}
$$
$$
\times \left[{}_k J^{\mu}_{(m\lambda r + (1-\lambda)a)^+} f(\lambda a + m(1-\lambda)r) + {}_k J^{\mu}_{(\lambda a + m(1-\lambda)r)^-} f(m\lambda r + (1-\lambda)a) \right]. \quad (2.2)
$$

Proof It suffices to note that

$$
I^* = \int_0^1 \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right) f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) dt
$$
$$
= \left[\int_0^1 (1-t)^{\frac{\mu}{k}} f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) dt \right]
$$
$$
+ \left[-\int_0^1 t^{\frac{\mu}{k}} f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) dt \right]
$$
$$
:= I_1 + I_2. \quad (2.3)
$$

Integrating by parts, we get

$$
I_1 = \left[\int_0^1 (1-t)^{\frac{\mu}{k}} f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) dt \right]
$$
$$
= \frac{f(t(\lambda a + m(1-\lambda)r) + m(1-t)(\lambda r + (1-\lambda)\frac{a}{m}))(1-t)^{\frac{\mu}{k}}}{(1-2\lambda)(mr-a)} \Big|_0^1
$$
$$
+ \frac{\frac{\mu}{k}}{(1-2\lambda)(mr-a)} \left[\int_0^1 (1-t)^{\frac{\mu}{k}-1} f\left(t(\lambda a + m(1-\lambda)r) \right. \right.
$$

$$+ m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right) dt\right]$$

$$= -\frac{f(m\lambda r + (1-\lambda)a)}{(1-2\lambda)(mr-a)} + \frac{\frac{\mu}{k}}{(1-2\lambda)(mr-a)}$$

$$\times \left[\int_0^1 (1-t)^{\frac{\mu}{k}-1}f\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right) dt\right]. \qquad (2.4)$$

Let $x = t(\lambda a + m(1-\lambda)r) + m(1-t)(\lambda r + (1-\lambda)\frac{a}{m})$, $t \in [0,1]$, equality (2.4) can be written as

$$I_1 = -\frac{f(m\lambda r + (1-\lambda)a)}{(1-2\lambda)(mr-a)}$$

$$+ \frac{\frac{\mu}{k}}{(1-2\lambda)^{\frac{\mu}{k}}(mr-a)^{\frac{\mu}{k}+1}} \int_{m\lambda r + (1-\lambda)a}^{\lambda a + m(1-\lambda)r} \left(\lambda a + m(1-\lambda)r - x\right)^{\frac{\mu}{k}-1} f(x)\, dx$$

$$= -\frac{f(m\lambda r + (1-\lambda)a)}{(1-2\lambda)(mr-a)}$$

$$+ \frac{\Gamma_k(\mu+k)}{(1-2\lambda)^{\frac{\mu}{k}+1}(mr-a)^{\frac{\mu}{k}+1}} {}_kJ^{\mu}_{(m\lambda r + (1-\lambda)a)^+} f\left(\lambda a + m(1-\lambda)r\right), \qquad (2.5)$$

and similarly we get

$$I_2 = \int_0^1 t^{\frac{\mu}{k}} f'\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right) dt$$

$$= -\frac{t^{\frac{\mu}{k}} f(t(\lambda a + m(1-\lambda)r) + m(1-t)(\lambda r + (1-\lambda)\frac{a}{m}))}{(1-2\lambda)(mr-a)}\Big|_0^1$$

$$+ \frac{\frac{\mu}{k}}{(1-2\lambda)(mr-a)}$$

$$\times \int_0^1 t^{\frac{\mu}{k}-1} f\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right) dt$$

$$= -\frac{f(\lambda a + m(1-\lambda)r)}{(1-2\lambda)(mr-a)}$$

$$+ \frac{\Gamma_k(\mu+k)}{(1-2\lambda)^{\frac{\mu}{k}+1}(mr-a)^{\frac{\mu}{k}+1}} {}_kJ^{\mu}_{(\lambda a + m(1-\lambda)r)^-} f\left(m\lambda r + (1-\lambda)a\right). \qquad (2.6)$$

Hence, using (2.5) and (2.6) in (2.3), we can obtain the desired result. $\qquad \square$

Corollary 2.1 *In Lemma* 2.1, *for* $k = 1$, *we can get the result for Riemann-Liouville fractional integral.*

Corollary 2.2 *In Lemma* 2.1, *if we put* $\lambda = 0$, *we get*

$$-\frac{f(a) + f(mr)}{mr-a} + \frac{\Gamma_k(\mu+k)}{(mr-a)^{\frac{\mu}{k}+1}}\left[{}_kJ^{\mu}_{a^+} f(mr) + {}_kJ^{\mu}_{mr^-} f(a)\right]$$

$$= \int_0^1 \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right) f'\left(tmr + (1-t)a\right) dt. \qquad (2.7)$$

Similarly, taking λ = 1 in Lemma 2.1, we obtain

$$\frac{f(a) + f(mr)}{mr - a} + \frac{\Gamma_k(\mu + k)}{(-1)^{\frac{\mu}{k}+1}(mr - a)^{\frac{\mu}{k}+1}} \left[{}_kJ^{\mu}_{mr^+}f(a) + {}_kJ^{\mu}_{a^-}f(mr) \right]$$

$$= \int_0^1 \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right) f'\left(ta + (1-t)mr \right) \mathrm{d}t. \tag{2.8}$$

Note that ${}_kJ^{\mu}_{mr^+}f(a) + {}_kJ^{\mu}_{a^-}f(mr) = (-1)^{\frac{\mu}{k}} [{}_kJ^{\mu}_{a^+}f(mr) + {}_kJ^{\mu}_{mr^-}f(a)]$, it is easy to see that identity (2.8) is equal to identity (2.7).

Remark 2.1

(i) In Corollary 2.1, if we put $r = b$, then one can obtain Lemma 3.1 which is proved in [35]. Further, if we take $m = 1$, then we obtain Lemma 2.1 in [12].

(ii) In Corollary 2.2,

    (a) if we put $k = 1 = m$, then we obtain Lemma 3 in [11],

    (b) if we put $k = 1 = m$ and $r = b$, then we obtain Lemma 2 in [8],

    (c) if we put $k = m = \mu = 1$ and $r = b$, then we obtain Lemma 2.1 in [36].

3 *k*-fractional integral inequalities for (*h*, *m*)-convex functions

In what follows, we establish some k-fractional integral inequalities for (h, m)-convex functions by using Lemma 2.1.

Theorem 3.1 *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0, 1] \subseteq J$) be a non-negative function, and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o along with $a, r \in I$, $0 \le a < mr$, for some fixed $m \in (0, 1]$. If $f' \in L^1[a, mr]$ and $|f'|^q$ for $q \ge 1$ is (h, m)-convex on $[a, mr]$, then the following inequality exists:*

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right| \le \left[\frac{2k}{\mu + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}}} \right) \right]^{1-\frac{1}{q}} \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right) \left(h(t) + h(1-t) \right) \mathrm{d}t \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + m(1-\lambda)r) \right|^q + m \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}, \tag{3.1}$$

where $\lambda \in (0, 1] \backslash \frac{1}{2}$, $k > 0$ and $\mu > 0$.

Proof Case 1: $q = 1$. Applying Lemma 2.1 and the (h, m)-convexity of $|f'|$, we have

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$= \left| \int_0^1 \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right) f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) \mathrm{d}t \right|$$

$$\le \int_0^1 \left| (1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right| \left| f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) \right| \mathrm{d}t$$

$$\le \int_0^1 \left| (1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right|$$

$$\times \left[h(t) \left| f'(\lambda a + m(1-\lambda)r) \right| + mh(1-t) \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right| \right] \mathrm{d}t$$

$$= \int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right)$$

$$\times \left[h(t)\big|f'\big(\lambda a + m(1-\lambda)r\big)\big| + mh(1-t)\Big|f'\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big| \right] dt$$

$$+ \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}} \right)$$

$$\times \left[h(t)\big|f'\big(\lambda a + m(1-\lambda)r\big)\big| + mh(1-t)\Big|f'\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big| \right] dt,$$

where we use the fact that

$$\int_{\frac{1}{2}}^1 t^{\frac{\mu}{k}} h(t)\,dt = \int_0^{\frac{1}{2}} (1-t)^{\frac{\mu}{k}} h(1-t)\,dt,$$

$$\int_{\frac{1}{2}}^1 t^{\frac{\mu}{k}} h(1-t)\,dt = \int_0^{\frac{1}{2}} (1-t)^{\frac{\mu}{k}} h(t)\,dt,$$

$$\int_{\frac{1}{2}}^1 (1-t)^{\frac{\mu}{k}} h(t)\,dt = \int_0^{\frac{1}{2}} t^{\frac{\mu}{k}} h(1-t)\,dt$$

and

$$\int_{\frac{1}{2}}^1 (1-t)^{\frac{\mu}{k}} h(1-t)\,dt = \int_0^{\frac{1}{2}} t^{\frac{\mu}{k}} h(t)\,dt.$$

By calculation,

$$\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left[h(t)\big|f'\big(\lambda a + m(1-\lambda)r\big)\big| + mh(t)\Big|f'\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big| \right] dt$$

$$\le \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right)\big(h(t) + h(1-t)\big)\,dt \right]$$

$$\times \left[\big|f'\big(\lambda a + m(1-\lambda)r\big)\big| + m\Big|f'\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big| \right].$$

Case 2: $q > 1$. Employing Lemma 2.1, the power mean inequality and the (h, m)-convexity of $|f'|^q$ leads to

$$\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left| f'\Big(t\big(\lambda a + m(1-\lambda)r\big) + m(1-t)\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big) \right| dt$$

$$\le \left[\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| dt \right]^{1-\frac{1}{q}}$$

$$\times \left[\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left| f'\Big(t\big(\lambda a + m(1-\lambda)r\big) + m(1-t)\Big(\lambda r + (1-\lambda)\frac{a}{m}\Big)\Big) \right|^q dt \right]^{\frac{1}{q}}$$

$$\le \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right)dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}} \right)dt \right]^{1-\frac{1}{q}}$$

$$\times \left\{ \int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left[h(t)\big|f'\big(\lambda a + m(1-\lambda)r\big)\big|^q \right. \right.$$

$$
+ mh(1-t)\left|f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right] dt \right\}^{\frac{1}{q}}
$$

$$
= \left[\frac{2k}{\mu+k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1-\frac{1}{q}}\left[\int_0^{\frac{1}{2}}\left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right)\left(h(t) + h(1-t)\right)dt\right]^{\frac{1}{q}}
$$

$$
\times \left[\left|f'\left(\lambda a + m(1-\lambda)r\right)\right|^q + m\left|f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.
$$

This completes the proof. □

Now, we point out some special cases of Theorem 3.1.

Corollary 3.1 *In Theorem* 3.1, *if we choose* $h(t) = t$ *and* $r = b$, *then we derive the following inequality for m-convex functions*:

$$
\left|\mathcal{T}_{k,\mu}(m,\lambda,b)\right|
$$

$$
\leq \frac{2k}{\mu+k}\left[1 - \frac{1}{2^{\frac{\mu}{k}}}\right]\left[\frac{|f'(\lambda a + m(1-\lambda)b)|^q + m|f'(\lambda b + (1-\lambda)\frac{a}{m})|^q}{2}\right]^{\frac{1}{q}}. \tag{3.2}
$$

Especially if we put $k = 1$, *we obtain Theorem* 3.2 *in* [35].

Corollary 3.2 *In Theorem* 3.1, *if we choose* $h(t) = t$, $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *then we derive the following inequality for convex functions*:

$$
\left|-\frac{f(a) + f(r)}{r-a} + \frac{\Gamma_k(\mu+k)}{(r-a)^{\frac{\mu}{k}+1}}\left[{_kJ}_{a^+}^{\mu}f(r) + {_kJ}_{r^-}^{\mu}f(a)\right]\right|
$$

$$
\leq \frac{2k}{\mu+k}\left[1 - \frac{1}{2^{\frac{\mu}{k}}}\right]\left[\frac{|f'(r)|^q + |f'(a)|^q}{2}\right]^{\frac{1}{q}}.
$$

Remark 3.1 In Corollary 3.2,
  (a) if we put $k = 1$, we can obtain Theorem 2.3 in [12],
  (b) if we put $k = 1$ and $r = b$, we can obtain Corollary 2.4 in [12],
  (c) if we put $k = 1 = \mu$ and $r = b$, we can obtain Theorem 1 in [37],
  (d) if we put $\mu = q = k = 1$ and $r = b$, we can obtain Theorem 2.2 in [36].

Corollary 3.3 *In Theorem* 3.1, *if we choose* $h(t) = t^s$, $s \in (0,1]$, *then we have the following inequality for* (s,m)-*Breckner convex functions*:

$$
\left|\mathcal{T}_{k,\mu}(m,\lambda,r)\right| \leq \left[\frac{2k}{\mu+k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1-\frac{1}{q}}
$$

$$
\times \left[\beta\left(\frac{1}{2}, s+1, \frac{\mu}{k}+1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}+1, s+1\right)\right.
$$

$$
\left. + \frac{k}{k(s+1)+\mu} - \frac{k}{k(s+1)+\mu}\left(\frac{1}{2}\right)^{\frac{sk+\mu}{k}}\right]^{\frac{1}{q}}
$$

$$
\times \left[\left|f'\left(\lambda a + m(1-\lambda)r\right)\right|^q + m\left|f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}. \tag{3.3}
$$

Especially if we choose $m = 1 = k$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we can get Theorem* 7 *in* [38].

Corollary 3.4 *In Theorem 3.1, if we put $h(t) = 1$, then we obtain the following inequality for (m, P)-convex functions:*

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \frac{2k}{\mu + k}\left[1 - \frac{1}{2^{\frac{\mu}{k}}}\right]\left[\left|f'(\lambda a + m(1 - \lambda)r)\right|^q + m\left|f'\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Especially if we choose $m = 1$ and $\lambda = 1$ or $\lambda = 0$, we have

$$\left|-\frac{f(a) + f(r)}{2} + \frac{\Gamma_k(\mu + k)}{2(r - a)^{\frac{\mu}{k}}}\left[{}_kJ_{a^+}^{\mu}f(r) + {}_kJ_{r^-}^{\mu}f(a)\right]\right|$$

$$\leq \frac{k(r - a)}{\mu + k}\left[1 - \frac{1}{2^{\frac{\mu}{k}}}\right]\left[\left|f'(r)\right|^q + \left|f'(a)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 3.5 *In Theorem 3.1, if we take $h(t) = t^{-s}$, $s \in (0, 1)$, then we get the following inequality for (m, s)-Godunova-Liven-Dragomir convex functions:*

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \left[\frac{2k}{\mu + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1 - \frac{1}{q}}$$

$$\times \left[\beta\left(\frac{1}{2}, 1 - s, \frac{\mu}{k} + 1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, 1 - s\right) + \frac{k}{\mu + (1 - s)k}\left(1 - 2^{\frac{sk - \mu}{k}}\right)\right]^{\frac{1}{q}}$$

$$\times \left[\left|f'(\lambda a + m(1 - \lambda)r)\right|^q + m\left|f'\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Especially if we put $m = 1$ and $\lambda = 1$ or $\lambda = 0$, we get

$$\left|-\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}}\left[{}_kJ_{a^+}^{\mu}f(r) + {}_kJ_{r^-}^{\mu}f(a)\right]\right|$$

$$\leq \left[\frac{2k}{\mu + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1 - \frac{1}{q}}$$

$$\times \left[\beta\left(\frac{1}{2}, 1 - s, \frac{\mu}{k} + 1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, 1 - s\right) + \frac{k}{\mu + (1 - s)k}\left(1 - 2^{\frac{sk - \mu}{k}}\right)\right]^{\frac{1}{q}}$$

$$\times \left[\left|f'(a)\right|^q + \left|f'(r)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 3.6 *In Theorem 3.1, if we choose $h(t) = t(1 - t)$, then we obtain the following inequality for (m, tgs)-convex functions:*

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right| \leq \left[\frac{2k}{\mu + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1 - \frac{1}{q}}\left[\frac{4k^2 - 2^{-\frac{\mu}{k}}(k\mu + 4k^2)}{2(\mu + 2k)(\mu + 3k)}\right]^{\frac{1}{q}}$$

$$\times \left[\left|f'(\lambda a + m(1 - \lambda)r)\right|^q + m\left|f'\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Especially if we put m = 1 and λ = 1 or λ = 0, we get

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\leq \left[\frac{2k}{\mu + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}}} \right) \right]^{1 - \frac{1}{q}} \left[\frac{4k^2 - 2^{-\frac{\mu}{k}} (k\mu + 4k^2)}{(\mu + 2k)(\mu + 3k)} \right]^{\frac{1}{q}} \left[\frac{|f'(a)|^q + |f'(r)|^q}{2} \right]^{\frac{1}{q}}.$$

Corollary 3.7 *In Theorem* 3.1, *if we choose* $h(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, *then we obtain the following inequality for m-MT-convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right| \leq \left[\frac{2k}{\mu + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}}} \right) \right]^{1 - \frac{1}{q}} \left[\frac{1}{2} \left(\beta \left(\frac{1}{2}, \frac{1}{2}, \frac{\mu}{k} + \frac{1}{2} \right) - \beta \left(\frac{1}{2}, \frac{\mu}{k} + \frac{1}{2}, \frac{1}{2} \right) \right) \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + m(1 - \lambda)r) \right|^q + m \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.$$

Especially if we put m = 1 and λ = 1 or λ = 0, we get

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\leq \left[\frac{2k}{\mu + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}}} \right) \right]^{1 - \frac{1}{q}} \left[\beta \left(\frac{1}{2}, \frac{1}{2}, \frac{\mu}{k} + \frac{1}{2} \right) - \beta \left(\frac{1}{2}, \frac{\mu}{k} + \frac{1}{2}, \frac{1}{2} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\frac{|f'(a)|^q + |f'(r)|^q}{2} \right]^{\frac{1}{q}}.$$

Now, we prepare to introduce the second theorem as follows.

Theorem 3.2 *Under the assumptions of Theorem* 3.1, *the resulting expression exists*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right| \leq \left[\int_0^{\frac{1}{2}} \left((1 - t)^{\frac{\mu}{k}q} - t^{\frac{\mu}{k}q} \right) \left(h(t) + h(1 - t) \right) dt \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + m(1 - \lambda)r) \right|^q + m \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}, \qquad (3.4)$$

where $\lambda \in [0, 1] \setminus \frac{1}{2}$, *k* > 0 *and* $\mu > 0$.

Proof Using Lemma 2.1, Hölder's inequality and the (h, m)-convexity of $|f'|^q$, we have

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\leq \int_0^1 \left| (1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right| \left| f'\left(t(\lambda a + m(1 - \lambda)r) + m(1 - t)\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right) \right| dt$$

$$\leq \left(\int_0^1 1^p \, dt \right)^{\frac{1}{p}} \left[\int_0^1 \left| (1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right|^q \right.$$

$$\times \left. \left| f'\left(t(\lambda a + m(1 - \lambda)r) + m(1 - t)\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right) \right|^q dt \right]^{\frac{1}{q}}$$

$$= \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right)^q \right.$$

$$\times \left| f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) \right|^q \, dt$$

$$+ \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}} \right)^q$$

$$\left. \times \left| f'\left(t(\lambda a + m(1-\lambda)r) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right) \right|^q \, dt \right]^{\frac{1}{q}}$$

$$\leq \left\{ \int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}q} - t^{\frac{\mu}{k}q} \right) \right.$$

$$\times \left[h(t) \left| f'(\lambda a + m(1-\lambda)r) \right|^q + mh(1-t) \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \, dt \right]$$

$$+ \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}q} - (1-t)^{\frac{\mu}{k}q} \right)$$

$$\left. \times \left[h(t) \left| f'(\lambda a + m(1-\lambda)r) \right|^q + mh(1-t) \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \right] \, dt \right\}^{\frac{1}{q}}$$

$$= \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}q} - t^{\frac{\mu}{k}q} \right) \left(h(t) + h(1-t) \right) dt \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + m(1-\lambda)r) \right|^q + m \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.$$

Here, we use $(A - B)^q \leq A^q - B^q$ for any $A \geq B \geq 0$ and $q \geq 1$.

Let us point out some special cases of Theorem 3.2.

Corollary 3.8 *In Theorem* 3.2, *if we put* $h(t) = t^s$, $s \in (0, 1]$, *then we get the following inequality for* (s, m)-*Breckner convex functions:*

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\leq \left[\beta\left(\frac{1}{2}, s+1, \frac{\mu}{k}q+1 \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}q+1, s+1 \right) + \frac{k}{\mu q + (s+1)k}\left(1 - 2^{-\frac{\mu q + sk}{k}} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(\lambda a + (1-\lambda)r) \right|^q + m \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.$$

Especially if we put $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we have*

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r-a)^{\frac{\mu}{k}+1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\leq \left[\beta\left(\frac{1}{2}, s+1, \frac{\mu}{k}q+1 \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}q+1, s+1 \right) + \frac{k}{\mu q + (s+1)k}\left(1 - 2^{-\frac{\mu q + sk}{k}} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(r) \right|^q + \left| f'(a) \right|^q \right]^{\frac{1}{q}}.$$

Corollary 3.9 *In Theorem* 3.2, *if we take* $h(t) = t^{-s}$, $s \in (0, 1]$, *then we get the following inequality for* (m, s)-*Godunova-Levin-Dragomir convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\leq \left[\beta\left(\frac{1}{2}, 1 - s, \frac{\mu}{k} + 1 \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, 1 - s \right) + \frac{k}{k(1 - s) + \mu} \left(1 - 2^{\frac{sk - \mu}{k}} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'\left(\lambda a + m(1 - \lambda)r \right) \right|^q + m\left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.$$

Especially if we take $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we have*

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\leq \left[\beta\left(\frac{1}{2}, 1 - s, \frac{\mu}{k} + 1 \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, 1 - s \right) + \frac{k}{k(1 - s) + \mu} \left(1 - 2^{\frac{sk - \mu}{k}} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'(r) \right|^q + \left| f'(a) \right|^q \right]^{\frac{1}{q}}.$$

Corollary 3.10 *In Theorem* 3.2, *if we put* $h(t) = t(1 - t)$, *then we get the following inequality for* (m, tgs)-*convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\leq \left[\frac{2k^2 - (\frac{1}{2})^{\frac{\mu q + k}{k}}(4k^2 + kuq)}{(\mu q + 2k)(\mu q + 3k)} \right]^{\frac{1}{q}}$$

$$\times \left[\left| f'\left(\lambda a + m(1 - \lambda)r \right) \right|^q + m\left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.$$

Especially if we put $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we have*

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\leq \left[\frac{2k^2 - (\frac{1}{2})^{\frac{\mu q + k}{k}}(4k^2 + kuq)}{(\mu q + 2k)(\mu q + 3k)} \right]^{\frac{1}{q}} \left[\left| f'(r) \right|^q + \left| f'(a) \right|^q \right]^{\frac{1}{q}}.$$

Corollary 3.11 *In Theorem* 3.2, *if we put* $h(t) = \frac{\sqrt{1 - t}}{2\sqrt{t}}$, *then we get the following inequality for* m-MT-*convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\leq \left[\beta\left(\frac{1}{2}, \frac{1}{2}, \frac{\mu}{k}q + \frac{1}{2} \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}q + \frac{1}{2}, \frac{1}{2} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\frac{|f'(\lambda a + m(1 - \lambda)r)|^q + m|f'(\lambda r + (1 - \lambda)\frac{a}{m})|^q}{2} \right]^{\frac{1}{q}}.$$

Especially if we put m = 1 and λ = 0 or λ = 1, we have

$$
\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}} \left[{}_k J^{\mu}_{a^+} f(r) + {}_k J^{\mu}_{r^-} f(a) \right] \right|
$$

$$
\leq \left[\beta\left(\frac{1}{2}, \frac{1}{2}, \frac{\mu}{k} q + \frac{1}{2} \right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} q + \frac{1}{2}, \frac{1}{2} \right) \right]^{\frac{1}{q}} \left[\frac{|f'(r)|^q + |f'(a)|^q}{2} \right]^{\frac{1}{q}}.
$$

Now, we are ready to state the third theorem in this section.

Theorem 3.3 *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0, 1] \subseteq J$) be a non-negative function, and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o along with $a, r \in I$, $0 \leq a < mr$, for some fixed $m \in (0, 1]$. If $f' \in L[a, mr]$ and $|f'|^q$ for $q > 1$ is (h, m)-convex on $[a, mr]$, then the following inequality holds:*

$$
\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|
$$

$$
\leq \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k} p}} \right) \right]^{\frac{1}{p}}
$$

$$
\times \left\{ \int_0^1 \left[h(t) \left| f'(\lambda a + m(1 - \lambda)r) \right|^q \right. \right.
$$

$$
\left. \left. + m h(1 - t) \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right] dt \right\}^{\frac{1}{q}},
\tag{3.5}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\mu > 0$, $k > 0$ and $\lambda \in [0, 1] \setminus \frac{1}{2}$.

Proof Applying Lemma 2.1, Hölder's inequality and the (h, m)-convexity of $|f'|^q$, we have

$$
\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|
$$

$$
\leq \int_0^1 \left| (1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right| \left| f'\left(t(\lambda a + m(1 - \lambda)r) + m(1 - t)\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right) \right| dt
$$

$$
\leq \left[\int_0^1 \left| (1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right|^p \right]^{\frac{1}{p}}
$$

$$
\times \left[\left| f'\left(t(\lambda a + m(1 - \lambda)r) + m(1 - t)\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right) \right|^q dt \right]^{\frac{1}{q}}
$$

$$
= \left[\int_0^{\frac{1}{2}} \left((1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}} \right)^p dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1 - t)^{\frac{\mu}{k}} \right)^p dt \right]^{\frac{1}{p}}
$$

$$
\times \left\{ \int_0^1 \left[h(t) \left| f'(\lambda a + m(1 - \lambda)r) \right|^q \right. \right.
$$

$$
\left. \left. + m h(1 - t) \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right] dt \right\}^{\frac{1}{q}}
$$

$$
\leq \left[\int_0^{\frac{1}{2}} \left((1 - t)^{\frac{\mu}{k} p} - t^{\frac{\mu}{k} p} \right) dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k} p} - (1 - t)^{\frac{\mu}{k} p} \right) dt \right]^{\frac{1}{p}}
$$

$$
\times \left\{ \int_0^1 \left[h(t) \left| f'(\lambda a + m(1 - \lambda)r) \right|^q + m h(1 - t) \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right] dt \right\}^{\frac{1}{q}}
$$

$$= \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}}$$

$$\times \left\{ \int_0^1 \left[h(t) \left| f'(\lambda a + m(1-\lambda)r) \right|^q + mh(1-t) \left| f'\left(\lambda r + (1-\lambda)\frac{a}{m} \right) \right|^q \right] dt \right\}^{\frac{1}{q}}.$$

Here, we use the fact that $(A - B)^q \le A^q - B^q$ for any $A \ge B \ge 0$ and $q \ge 1$, which completes the proof. □

Now, we point out some special cases of Theorem 3.3.

Corollary 3.12 *In Theorem* 3.3, *if we choose* $h(t) = t$ *and* $r = b$, *then we obtain the following inequality for* m-*convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, b) \right|$$

$$\le \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(\lambda a + (1-\lambda)b)|^q + m|f'(\lambda b + (1-\lambda)\frac{a}{m})|^q}{2} \right]^{\frac{1}{q}}.$$

Especially if we put $k = 1$, *we obtain Theorem* 3.3 *in* [35]. *Further, if we put* $m = 1$, *we obtain Theorem* 2.6 *in* [12].

Corollary 3.13 *In Theorem* 3.3, *if we choose* $h(t) = t$, $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *then we obtain the following inequality for convex functions*:

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r-a)^{\frac{\mu}{k}+1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\le \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(r)|^q + |f'(a)|^q}{2} \right]^{\frac{1}{q}}.$$

Remark 3.2 In Corollary 3.13,
   (a) if we take $k = 1$ and $r = b$, we can get Corollary 2.7 in [12],
   (b) if we take $k = 1 = \mu$ and $r = b$, we can get Corollary 2.8 in [12].

Corollary 3.14 *In Theorem* 3.3, *if we choose* $h(t) = t^s, s \in (0, 1]$, *then we obtain the following inequality for* (s, m)-*Breckner convex functions*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|$$

$$\le \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(\lambda a + (1-\lambda)r)|^q + m|f'(\lambda r + (1-\lambda)\frac{a}{m})|^q}{s + 1} \right]^{\frac{1}{q}}.$$

Especially if we put $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *then we have*

$$\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r-a)^{\frac{\mu}{k}+1}} \left[{}_kJ_{a^+}^\mu f(r) + {}_kJ_{r^-}^\mu f(a) \right] \right|$$

$$\le \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(a)|^q + |f'(r)|^q}{s + 1} \right]^{\frac{1}{q}}.$$

Corollary 3.15 *In Theorem* 3.3, *if we put* $h(t) = 1$, *then we obtain the following inequality for* (m, P)-*convex functions*:

$$
\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|
$$
$$
\leq \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\left| f'(\lambda a + (1 - \lambda)r) \right|^q + m \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.
$$

Especially if we put $m = 1$ *and* $\lambda = 1$ *or* $\lambda = 0$, *we have*

$$
\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k}+1}} \left[{}_k J_{a^+}^{\mu} f(r) + {}_k J_{r^-}^{\mu} f(a) \right] \right|
$$
$$
\leq \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\left| f'(r) \right|^q + \left| f'(a) \right|^q \right]^{\frac{1}{q}}.
$$

Corollary 3.16 *In Theorem* 3.3, *if we take* $h(t) = t^{-s}$, $s \in (0, 1)$, *then we obtain the following inequality for* (s, m)-*Godunova-Levin-Dragomir convex functions*:

$$
\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|
$$
$$
\leq \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(\lambda a + (1 - \lambda)r)|^q + m|f'(\lambda r + (1 - \lambda)\frac{a}{m})|^q}{1 - s} \right]^{\frac{1}{q}}.
$$

Especially if we take $m = 1$ *and* $\lambda = 1$ *or* $\lambda = 0$, *we have*

$$
\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k}+1}} \left[{}_k J_{a^+}^{\mu} f(r) + {}_k J_{r^-}^{\mu} f(a) \right] \right|
$$
$$
\leq \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\frac{|f'(r)|^q + |f'(a)|^q}{1 - s} \right]^{\frac{1}{q}}.
$$

Corollary 3.17 *In Theorem* 3.3, *if we put* $h(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, *then we obtain the following inequality for* m-*MT-convex functions*:

$$
\left| \mathcal{T}_{k,\mu}(m, \lambda, r) \right|
$$
$$
\leq \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}}
$$
$$
\times \left[\left| f'(\lambda a + (1 - \lambda)r) \right|^q + m \left| f'\left(\lambda r + (1 - \lambda)\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}}.
$$

Especially if we put $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we have*

$$
\left| -\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k}+1}} \left[{}_k J_{a^+}^{\mu} f(r) + {}_k J_{r^-}^{\mu} f(a) \right] \right|
$$
$$
\leq \left(\frac{\pi}{4} \right)^{\frac{1}{q}} \left[\frac{2k}{\mu p + k} \left(1 - \frac{1}{2^{\frac{\mu}{k}p}} \right) \right]^{\frac{1}{p}} \left[\left| f'(r) \right| + \left| f'(a) \right| \right]^{\frac{1}{q}}.
$$

Corollary 3.18 *In Theorem* 3.3, *if we choose* $h(t) = t(1 - t)$, *then we obtain the following inequality for* (m, tgs)-*convex functions*:

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \left(\frac{1}{6}\right)^{\frac{1}{q}} \left[\frac{2k}{\mu p + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}p}}\right)\right]^{\frac{1}{p}} \left[\left|f'(\lambda a + (1 - \lambda)r)\right|^q + m\left|f'\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Especially if we choose $m = 1$ *and* $\lambda = 0$ *or* $\lambda = 1$, *we have*

$$\left|-\frac{f(a) + f(r)}{r - a} + \frac{\Gamma_k(\mu + k)}{(r - a)^{\frac{\mu}{k} + 1}}\left[{_kJ_{a^+}^\mu f(r)} + {_kJ_{r^-}^\mu f(a)}\right]\right|$$

$$\leq \left(\frac{1}{6}\right)^{\frac{1}{q}} \left[\frac{2k}{\mu p + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}p}}\right)\right]^{\frac{1}{p}} \left[\left|f'(r)\right|^q + \left|f'(a)\right|^q\right]^{\frac{1}{q}}.$$

4 k-fractional inequalities for (α, m)-convex functions

Using Lemma 2.1 again, we state the following theorems.

Theorem 4.1 *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I^o *along with* $a, r \in I$ *and* $0 \leq a < mr$. *If* $|f'|^q$ *for* $q \geq 1$ *is* (α, m)-*convex on* $[a, mr]$ *and* $f' \in L^1[a, mr]$. *Then the following inequality for* k-*fractional integrals holds*:

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \left[\frac{2k}{\mu + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1 - \frac{1}{q}} \left\{\left[\left[\beta\left(\frac{1}{2}, \alpha + 1, \frac{\mu}{k} + 1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, \alpha + 1\right)\right.\right.\right.$$

$$\left. + \frac{k}{\mu + (\alpha + 1)k} - \frac{k}{\mu + (\alpha + 1)k}\left(\frac{1}{2}\right)^{\frac{\mu + \alpha k}{k}}\right]\left|f'(\lambda a + m(1 - \lambda)r)\right|^q$$

$$- \left[\beta\left(\frac{1}{2}, \alpha + 1, \frac{\mu}{k} + 1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k} + 1, \alpha + 1\right) + \frac{k}{\mu + (\alpha + 1)k}\right.$$

$$- \frac{k}{\mu + (\alpha + 1)k}\left(\frac{1}{2}\right)^{\frac{\mu + \alpha k}{k}}$$

$$\left.\left. + \frac{2k}{\mu + k}\left(\frac{1}{2}\right)^{\frac{\mu}{k}} - \frac{2k}{\mu + k}\right]m\left|f'\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}}, \tag{4.1}$$

where $\lambda \in (0, 1]\backslash\frac{1}{2}$, $k > 0$ *and* $\mu > 0$.

Proof Using Lemma 2.1, the power mean inequality and the (α, m)-convexity of $|f'|^q$, we have

$$\left|\mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \int_0^1 \left|(1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right|\left|f'\left(t(\lambda a + m(1 - \lambda)r) + m(1 - t)\left(\lambda r + (1 - \lambda)\frac{a}{m}\right)\right)\right| dt$$

$$\leq \left[\int_0^1 \left|(1 - t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| dt\right]^{1 - \frac{1}{q}}$$

$$\times \left[\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left|f'\left(t\big(\lambda a + m(1-\lambda)r\big) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right]^{\frac{1}{q}}$$

$$\le \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right) dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}}\right) dt\right]^{1-\frac{1}{q}}$$

$$\times \left\{\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right|\right.$$

$$\times \left.\left[t^\alpha \left|f'\big(\lambda a + m(1-\lambda)r\big)\right|^q + m\big(1-t^\alpha\big)\left|f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q\right] dt\right\}^{\frac{1}{q}}$$

$$= \left[\frac{2k}{\mu+k}\left(1-\frac{1}{2^{\frac{\mu}{k}}}\right)\right]^{1-\frac{1}{q}} \left\{\left[\beta\left(\frac{1}{2},\alpha+1,\frac{\mu}{k}+1\right) - \beta\left(\frac{1}{2},\frac{\mu}{k}+1,\alpha+1\right)\right.\right.$$

$$\left.+ \frac{k}{\mu+(\alpha+1)k} - \frac{k}{\mu+(\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu+\alpha k}{k}}\right]\left|f'\big(\lambda a + m(1-\lambda)r\big)\right|^q$$

$$- \left[\beta\left(\frac{1}{2},\alpha+1,\frac{\mu}{k}+1\right) - \beta\left(\frac{1}{2},\frac{\mu}{k}+1,\alpha+1\right) + \frac{k}{\mu+(\alpha+1)k}\right.$$

$$\left.\left.- \frac{k}{\mu+(\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu+\alpha k}{k}} + \frac{2k}{\mu+k}\left(\frac{1}{2}\right)^{\frac{\mu}{k}} - \frac{2k}{\mu+k}\right]m\left|f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}},$$

which completes the proof. □

Theorem 4.2 *Under the assumptions of Theorem* 4.1, *the following inequality for k-fractional integrals holds*:

$$\left|\mathcal{T}_{k,\mu}(m,\lambda,r)\right|$$

$$\le \left\{\left[\beta\left(\frac{1}{2},\alpha+1,\frac{\mu}{k}q+1\right) - \beta\left(\frac{1}{2},\frac{\mu}{k}q+1,\alpha+1\right)\right.\right.$$

$$\left.+ \frac{k}{\mu q+(\alpha+1)k} - \frac{k}{\mu q+(\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu q+\alpha k}{k}}\right]\left|f'\big(\lambda a + m(1-\lambda)r\big)\right|^q$$

$$- \left[\beta\left(\frac{1}{2},\alpha+1,\frac{\mu}{k}q+1\right) - \beta\left(\frac{1}{2},\frac{\mu}{k}q+1,\alpha+1\right) - \frac{k}{\mu q+(\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu q+\alpha k}{k}}\right.$$

$$\left.\left.+ \frac{k}{\mu q+(\alpha+1)k} - \frac{2k}{\mu q+k} + \frac{2k}{\mu q+k}\left(\frac{1}{2}\right)^{\frac{\mu q}{k}}\right]m\left|f'\left(\lambda r + m(1-\lambda)\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}}, \quad (4.2)$$

where $\lambda \in (0,1]\backslash\frac{1}{2}$, $k > 0$ *and* $\mu > 0$.

Proof By making use of Lemma 2.1, Hölder's inequality and the (α,m)-convexity of $|f'|^q$, we get

$$\left|\mathcal{T}_{k,\mu}(m,\lambda,r)\right|$$

$$\le \int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right| \left|f'\left(t\big(\lambda a + m(1-\lambda)r\big) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right)\right| dt$$

$$\le \left(\int_0^1 1^p\, dt\right)^{\frac{1}{p}}\left[\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right|^q\right.$$

$$\times \left| f'\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right) \right) \right|^q dt \right]^{\frac{1}{q}}$$

$$= \left[\int_0^{\frac{1}{2}} \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right|^q \left| f'\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right) \right) \right|^q dt \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \left| t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}}\right|^q \left| f'\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right) \right) \right|^q dt \right]^{\frac{1}{q}}$$

$$\leq \left\{ \int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}q} - t^{\frac{\mu}{k}q}\right) \right.$$

$$\times \left[t^\alpha \left| f'(\lambda a + m(1-\lambda)r)\right|^q + m(1-t^\alpha)\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q dt \right]$$

$$+ \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}q} - (1-t)^{\frac{\mu}{k}q}\right)$$

$$\times \left[t^\alpha \left| f'(\lambda a + m(1-\lambda)r)\right|^q + m(1-t^\alpha)\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right] dt \right\}^{\frac{1}{q}}$$

$$= \left\{ \left[\beta\left(\frac{1}{2}, \alpha+1, \frac{\mu}{k}q+1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}q+1, \alpha+1\right) \right.\right.$$

$$\left. + \frac{k}{\mu q + (\alpha+1)k} - \frac{k}{\mu q + (\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu q + \alpha k}{k}} \right] \left| f'(\lambda a + m(1-\lambda)r)\right|^q$$

$$- \left[\beta\left(\frac{1}{2}, \alpha+1, \frac{\mu}{k}q+1\right) - \beta\left(\frac{1}{2}, \frac{\mu}{k}q+1, \alpha+1\right) - \frac{k}{\mu q + (\alpha+1)k}\left(\frac{1}{2}\right)^{\frac{\mu q + \alpha k}{k}} \right.$$

$$\left.\left. + \frac{k}{\mu q + (\alpha+1)k} - \frac{2k}{\mu q + k} + \frac{2k}{\mu q + k}\left(\frac{1}{2}\right)^{\frac{\mu q}{k}} \right] m \left| f'\left(\lambda r + m(1-\lambda)\frac{a}{m}\right)\right|^q \right\}^{\frac{1}{q}}.$$

Here, we use $(A-B)^q \leq A^q - B^q$ for any $A \geq B \geq 0$ and $q \geq 1$. This ends the proof. □

Theorem 4.3 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o along with $a, r \in I$ and $0 \leq a < mr$. If $|f'|^q$ for $q > 1$ is (α, m)-convex on $[a, mr]$ and $f' \in L^1[a, mr]$, then the following inequality for k-Riemann-Liouville fractional integral holds*:

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \left[\frac{2k}{\mu p + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}p}}\right) \right]^{\frac{1}{p}}$$

$$\times \left[\frac{1}{\alpha+1}\left| f'(\lambda a + m(1-\lambda)r)\right|^q + \frac{\alpha m}{\alpha+1}\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right]^{\frac{1}{q}}, \tag{4.3}$$

where $\lambda \in (0, 1]\backslash\frac{1}{2}$, $k > 0$, $\mu > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Employing Lemma 2.1, Hölder's inequality and the (α, m)-convexity of $|f'|^q$, we have

$$\left| \mathcal{T}_{k,\mu}(m, \lambda, r)\right|$$

$$\leq \left[\int_0^1 \left|(1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right|^p \right]^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left| f'\left(t\left(\lambda a + m(1-\lambda)r\right) + m(1-t)\left(\lambda r + (1-\lambda)\frac{a}{m}\right) \right) \right|^q dt \right]^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}} - t^{\frac{\mu}{k}}\right)^p dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}} - (1-t)^{\frac{\mu}{k}}\right)^p dt \right]^{\frac{1}{p}}$$

$$\times \left\{ \int_0^1 \left[t^\alpha \left| f'\left(\lambda a + m(1-\lambda)r\right)\right|^q + m\left(1-t^\alpha\right)\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right] dt \right\}^{\frac{1}{q}}$$

$$\leq \left[\int_0^{\frac{1}{2}} \left((1-t)^{\frac{\mu}{k}p} - t^{\frac{\mu}{k}p}\right) dt + \int_{\frac{1}{2}}^1 \left(t^{\frac{\mu}{k}p} - (1-t)^{\frac{\mu}{k}p}\right) dt \right]^{\frac{1}{p}}$$

$$\times \left\{ \int_0^1 \left[t^\alpha \left| f'\left(\lambda a + m(1-\lambda)r\right)\right|^q + m\left(1-t^\alpha\right)\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right] dt \right\}^{\frac{1}{q}}$$

$$= \left[\frac{2k}{\mu p + k}\left(1 - \frac{1}{2^{\frac{\mu}{k}p}}\right) \right]^{\frac{1}{p}}$$

$$\times \left[\frac{1}{\alpha+1}\left| f'\left(\lambda a + m(1-\lambda)r\right)\right|^q + \frac{\alpha m}{\alpha+1}\left| f'\left(\lambda r + (1-\lambda)\frac{a}{m}\right)\right|^q \right]^{\frac{1}{q}},$$

where we use the fact that $(A - B)^q \leq A^q - B^q$ for any $A \geq B \geq 0$ and $q \geq 1$. This completes the proof. □

Remark 4.1 If we take $\lambda = 0$ or $\lambda = 1$, we can deduce some new k-fractional integral trapezium-like inequalities from the results of Theorems 4.1, 4.2 and 4.3 and their related inequalities.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Du, TS, Liao, JG, Li, YJ: Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions. J. Nonlinear Sci. Appl. **9**, 3112-3126 (2016)
2. Set, E, Sardari, M, Ozdemir, ME, Rooin, J: On generalizations of the Hadamard inequality for (α, m)-convex functions. Kyungpook Math. J. **52**, 307-317 (2012)
3. Du, TS, Li, YJ, Yang, ZQ: A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions. Appl. Math. Comput. **293**, 358-369 (2017)
4. Latif, MA: On some new inequalities of Hermite-Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value. Ukr. Math. J. **67**(10) 1552-1571 (2016)
5. Matłoka, M: Inequalities for h-preinvex functions. Appl. Math. Comput. **234**, 52-57 (2014)
6. Özdemir, ME, Gürbüz, M, Yildiz, Ç: Inequalities for mappings whose second derivatives are quasi-convex or h-convex functions. Miskolc Math. Notes **15**(2), 635-649 (2014)
7. Xi, BY, Qi, F, Zhang, TY: Some inequalities of Hermite-Hadamard type for m-harmonic-arithmetically convex functions. Science Asia **41**, 357-361 (2015)
8. Sarikaya, MZ, Set, E, Yaldiz, H, Başak, N: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403-2407 (2013)
9. Dragomir, SS, Bhatti, MI, Iqbal, M, Muddassar, M: Some new Hermite-Hadamard's type fractional integral inequalities. J. Comput. Anal. Appl. **18**(4), 655-661 (2015)

10. Iqbal, M, Bhatti, MI, Nazeer, K: Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals. Bull. Korean Math. Soc. **52**(3), 707-716 (2015)

11. Özdemir, ME, Dragomir, SS, Yildiz, Ç: The Hadamard inequality for convex function via fractional integrals. Acta Math. Sci. Ser. B Engl. Ed. **33B**(5), 1293-1299 (2013)

12. Sarikaya, MZ, Budak, H: Generalized Hermite-Hadamard type integral inequalities for fractional integrals. Filomat **30**(5), 1315-1326 (2016)

13. İşcan, İ: Generalization of different type integral inequalities for *s*-convex functions via fractional integrals. Appl. Anal. **93**, 1846-1862 (2014)

14. Anastassiou, GA: Generalised fractional Hermite-Hadamard inequalities involving *m*-convexity and (*s*, *m*)-convexity. Facta Univ., Ser. Math. Inform. **28**(2), 107-126 (2013)

15. Wang, J, Deng, J, Fečkan, M: Hermite-Hadamard-type inequalities for *r*-convex functions based on the use of Riemann-Liouville fractional integrals. Ukr. Math. J. **65**(2), 193-211 (2013)

16. Chen, FX: Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals. Appl. Math. Comput. **268**, 121-128 (2015)

17. Awan, MU, Noor, MA, Mihai, MV, Noor, KI: Fractional Hermite-Hadamard inequalities for differentiable *s*-Godunova-Levin functions. Filomat **30**(12), 3235-3241 (2016)

18. Noor, MA, Noor, KI, Mihai, MV, Awan, MU: Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions. Sci. Bull. 'Politeh.' Univ. Buchar., Ser. A, Appl. Math. Phys. **78**(3), 163-174 (2016)

19. Qaisar, S, Iqbal, M, Muddassar, M: New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals. J. Comput. Anal. Appl. **20**, 1318-1328 (2016)

20. Kashuri, A, Liko, R: Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT$_m$-preinvex functions. Proyecciones **36**(1), 45-80 (2017)

21. Matłoka, M: Some inequalities of Hadamard type for mappings whose second derivatives are *h*-convex via fractional integrals. J. Fract. Calc. Appl. **6**(1), 110-119 (2015)

22. Mubeen, S, Habibullah, GM: *k*-fractional integrals and applications. Int. J. Contemp. Math. Sci. **7**(2), 89-94 (2012)

23. Agarwal, P, Tariboon, J, Ntouyas, SK: Some generalized Riemann-Liouville *k*-fractional integral inequalities. J. Inequal. Appl. **2016**, Article ID 122 (2016)

24. Romero, LG, Luque, LL, Dorrego, GA, Cerutti, RA: On the *k*-Riemann-Liouville fractional derivative. Int. J. Contemp. Math. Sci. **8**(1), 41-51 (2013)

25. Sarikaya, MZ, Karaca, A: On the *k*-Riemann-Liouville fractional integral and applications. Int. J. Stat. Math. **1**(3), 33-43 (2014)

26. Sarikaya, MZ, Dahmani, Z, Kiris, ME, Ahmad, F: (*k*, *s*)-Riemann-Liouville fractional integral and applications. Hacet. J. Math. Stat. **45**(1), 77-89 (2016)

27. Ali, A, Gulshan, G, Hussain, R, Latif, A, Muddassar, M: Generalized inequalities of the type of Hermite-Hadamard-Fejer with quasi-convex functions by way of *k*-fractional derivatives. J. Comput. Anal. Appl. **22**(7), 1208-1219 (2017)

28. Awan, MU, Noor, MA, Mihai, MV, Noor, KI: On bounds involving *k*-Appell's hypergeometric functions. J. Inequal. Appl. **2017**, Article ID 118 (2017)

29. Set, E, Tomar, M, Sarikaya, MZ: On generalized Grüss type inequalities for *k*-fractional integrals. Appl. Math. Comput. **269**, 29-34 (2015)

30. Miheşan, VG: A generalization of the convexity. In: Seminar on Functial Equations, Approx, Convex, Cluj-Napoca (Romania) (1993)

31. Omotoyinbo, O, Mogbademu, A: Some new Hermite-Hadamard integral inequalities for convex functions. Int. J. Sci. Innov. Technol. **1**(1), 1-12 (2014)

32. Varošanec, S: On *h*-convexity. J. Math. Anal. Appl. **326**, 303-311 (2007)

33. Tunç, M, Göv, E, Şanal, Ü: On *tgs*-convex function and their inequalities. Facta Univ., Ser. Math. Inform. **30**(5), 679-691 (2015)

34. Özdemir, ME, Akdemir, AO, Set, E: On (*h*, *m*)-convexity and Hadamard-type inequalities. Available at arXiv:1103.6163

35. Faird, G, Tariq, B: Some integral inequalities for *m*-convex functions via fractional integrals. J. Inequal. Spec. Funct. **8**(1), 170-185 (2017)

36. Dragomir, SS, Agarwal, RP: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91-95 (1998)

37. Pearce, CEM, Pečarić, J: Inequalities for differentiable mappings with application to special means and quadrature formula. Appl. Math. Lett. **13**(2), 51-55 (2000)

38. Yildiz, Ç, Özdemir, ME, Önalan, HK: Fractional integral inequalities via *s*-convex functions. Turk. J. Anal. Number Theory **5**(1), 18-22 (2017)