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Eigenvalues of the resistance-distance matrix of complete multipartite graphs

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Abstract

Let $G = (V, E)$ be a simple graph. The resistance distance between $i, j \in V$, denoted by r_{ij}, is defined as the net effective resistance between nodes i and j in the corresponding electrical network constructed from G by replacing each edge of G with a resistor of 1 Ohm. The resistance-distance matrix of G, denoted by $R(G)$, is a $|V| \times |V|$ matrix whose diagonal entries are 0 and for $i \neq j$, whose ij-entry is r_{ij}. In this paper, we determine the eigenvalues of the resistance-distance matrix of complete multipartite graphs. Also, we give some lower and upper bounds on the largest eigenvalue of the resistance-distance matrix of complete multipartite graphs. Moreover, we obtain a lower bound on the second largest eigenvalue of the resistance-distance matrix of complete multipartite graphs.

Keywords: resistance distance; resistance-distance matrix; largest resistance-distance eigenvalue; second largest resistance-distance eigenvalue

1 Introduction

Throughout the paper we consider only simple graphs, that is, graphs without loops and multi-edges. Let $G = (V, E)$ be a connected graph with a vertex set $V = \{1, 2, \dots, n\}$ and an edge set $E = E(G)$. The resistance distance [1] between any two vertices i and j, denoted by r_{ij}, is defined as the net effective resistance between nodes i and j in the corresponding electrical network constructed from G by replacing each edge with a resistor of 1 Ohm. The resistance-distance matrix of G, denoted by $R(G)$, is a $|V| \times |V|$ matrix whose diagonal entries are 0 and for $i \neq j$, whose ij-entry is r_{ij}. Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ denote the eigenvalues of $R(G)$. They are usually called the resistance-distance eigenvalues of G. In recent years, much study has been done on resistance distances. For more information, the readers are referred to most recent papers [2–12] and the references therein. In this paper, we study the resistance-distance matrix of complete multipartite graphs. The paper is organized as follows. In Section 2, we compute resistance distances in complete multipartite graphs. In Section 3, we determine the eigenvalues of the resistance distance matrix of complete multipartite graphs. In Section 4, we give some lower and upper bounds on the largest eigenvalue of the resistance-distance matrix of complete multipartite graphs. In Section 5, we obtain a lower bound on the second largest eigenvalue of the resistance-distance matrix of complete multipartite graphs.

2 Resistance distances in complete multipartite graphs

In this section, we compute resistance distances between any pair of vertices in the complete multipartite graph $K_{n_1,n_2,\dots,n_k}$ via electrical network approach. Recall that G is a complete k-partite graph if the vertex set V can be partitioned into k parts $V_1, V_2, \dots, V_k$ such that $uv \in E(G)$ if and only if u and v are in different parts. If $|V_i| = n_i$ ($i = 1, 2, \dots, k$), then G is denoted by $K_{n_1,n_2,\dots,n_k}$. The following two lemmas play essential roles further.

Lemma 2.1 ([13]) *Let i, j be vertices of G satisfying that they have the same neighborset N in $V/\{i,j\}$. Then*

$$r_{ij} = \begin{cases} \frac{2}{|N|+2} & \text{if } ij \in E(G), \\ \frac{2}{|N|} & \text{otherwise.} \end{cases} \tag{1}$$

Lemma 2.2 ([13] (The reduction principle)) *If $S \subset V$ satisfies that all vertices in S have the same neighborset N in $G - S$. Let H be the graph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N. Then the resistance distance between any two vertices of S in G is the same as the resistance distance between them in H.*

Now we are ready to give the main result of this section.

Theorem 2.3 *Resistance distances in $K_{n_1,n_2,\dots,n_k}$ can be computed as follows:*

$$r_{uv} = \begin{cases} \frac{2}{n-n_i} & \text{if } u, v \in V_i, \\ \frac{(n-1)(2n-n_i-n_j)}{n[n^2-(n_i+n_j)n+n_in_j]} & \text{if } u \in V_i, v \in V_j, \text{ and } i \neq j. \end{cases} \tag{2}$$

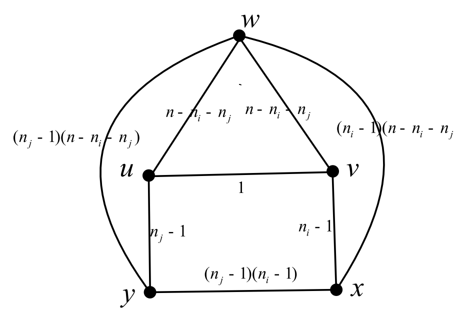
Proof For $u, v \in V_i$, it is easily seen that u and v have the same neighborset N with $|N| = n - n_i$. Hence, by Lemma 2.1, we have

$$r_{uv} = \frac{2}{n - n_i},$$

as required.

Now suppose that $u \in V_i$ and $v \in V_j$. Let $S = V_i \cup V_j$ and $N = V - S$. Let G^* be the graph obtained from $K_{n_1,n_2,\dots,n_k}$ by deleting all the edges between vertices in N. Then, by the reduction principle, the resistance distance between u and v in $K_{n_1,n_2,\dots,n_k}$ is equal to the resistance distance between u and v in G^*. In what follows, we compute the resistance distance between u and v in G^*. If we apply a unit potential across u and v, then by symmetry, all the vertices in $V_i \setminus \{u\}$ have the same potential, all the vertices in $V_j \setminus \{v\}$ have the same potential, and all the vertices in N have the same potential. In an electrical point of view, vertices that have the same potential can be regarded as identical so that they can be shortened together. Consequently, we shorten all the vertices in $V_i \setminus \{u\}$ together to get a new vertex x, shorten all the vertices in $V_j \setminus \{v\}$ together to get a new vertex y, and shorten all the vertices in N together to get a new vertex w. Then G^* can be simplified to the network $\mathcal{N}$ as shown in Figure 1, where the weight c_{ab} on each edge ab denotes the edge conductance (i.e., reciprocal of edge resistance).

Figure 1 The simplified graph G^*.



Now apply a unit potential across u and v in $\mathcal{N}$ and suppose the absolute potentials of u, v, x, y, w are $V_u = 1$, $V_v = 0$, V_x, V_y, V_w, respectively. Then, by Kirchhoff's laws, we have

$$(V_x - V_y)c_{xy} + (V_x - V_v)c_{xv} + (V_x - V_w)c_{xw} = 0,$$
$$(V_y - V_x)c_{xy} + (V_y - V_u)c_{yu} + (V_y - V_w)c_{yw} = 0,$$
$$(V_w - V_u)c_{wu} + (V_w - V_x)c_{xw} + (V_w - V_y)c_{yw} + (V_w - V_v)c_{wv} = 0.$$

Then simple calculation shows that

$$V_x = \frac{(n_i-1)(n_j-1)V_y + (n_i-1)(n-n_i-n_j)V_w}{n_i - 1 + (n_i-1)(n_j-1) + (n_i-1)(n-n_i-n_j)}, \tag{3}$$
$$V_y = \frac{n_j - 1 + (n_i-1)(n_j-1)V_x + (n_j-1)(n-n_i-n_j)V_w}{n_j - 1 + (n_i-1)(n_j-1) + (n_j-1)(n-n_i-n_j)}, \tag{4}$$
$$V_w = \frac{n - n_i - n_j + (n_i-1)(n-n_i-n_j)V_x + (n_j-1)(n-n_i-n_j)V_y}{2(n-n_i-n_j) + (n_i-1)(n-n_i-n_j) + (n_j-1)(n-n_i-n_j)}. \tag{5}$$

Solving the above linear system, we get

$$V_x = \frac{n(n_j-1) + (n-1)(n-n_i-n_j)}{(2n-n_i-n_j)(n-1)},$$
$$V_y = \frac{n(n-n_i)}{(2n-n_i-n_j)(n-1)},$$
$$V_w = \frac{n-n_i}{2n-n_i-n_j}.$$

Denote the total current flows from u to v by I. Then, by Ohm's law, we have

$$\begin{aligned} r_{uv} &= \frac{V_u - V_v}{I} = \frac{1-0}{I} \\ &= \frac{1}{(V_u - V_w)(n-n_i-n_j) + (V_u - V_y)(n_j - 1) + (V_u - V_v) \times 1} \\ &= \frac{1}{(1 - \frac{n-n_i}{2n-n_i-n_j})(n-n_i-n_j) + [1 - \frac{n(n-n_i)}{(2n-n_i-n_j)(n-1)}](n_j-1) + 1} \\ &= \frac{(n-1)(2n-n_i-n_j)}{n[n^2 - (n_i+n_j)n + n_in_j]}, \end{aligned}$$

as required. □

Remark It should be mentioned that resistance distances in complete multipartite graphs have also been determined in [10] via an alternative method.

By Theorem 2.3, for simplicity, in the following, we use r_i to denote the resistance distance between any two vertices in V_i, and use r_{ij} to denote the resistance distance between any $u \in V_i$ and $v \in V_j$. Thus the resistance-distance matrix $R(K_{n_1,n_2,\ldots,n_k})$ of $K_{n_1,n_2,\ldots,n_k}$ is

$$\begin{pmatrix} r_1J_{n_1,n_1} - r_1I_{n_1} & r_{12}J_{n_1,n_2} & r_{13}J_{n_1,n_3} & \cdots & r_{1k}J_{n_1,n_k} \\ r_{12}J_{n_2,n_1} & r_2J_{n_2,n_2} - r_2I_{n_2} & r_{23}J_{n_2,n_3} & \cdots & r_{2k}J_{n_2,n_k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{1k}J_{n_k,n_1} & r_{2k}J_{n_k,n_2} & r_{3k}J_{n_k,n_3} & \cdots & r_kJ_{n_k,n_k} - r_kI_{n_k} \end{pmatrix},$$

where J_{st} denotes the $s \times t$ matrix of all ones, I_l denotes the identity matrix of order l. In what follows, we always write $R(K_{n_1,n_2,\ldots,n_k})$ to R for short.

3 The eigenvalues of the resistance-distance matrix of complete multipartite graphs

In this section we obtain the eigenvalues of the resistance-distance matrix of complete k-partite graphs $K_{n_1,n_2,\ldots,n_k}$.

Theorem 3.1 *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ on n vertices. Then the characteristic polynomial of R is*

$$R_G(x) = \det(xI_n - R) = \prod_{i=1}^{k}(x + r_i)^{n_i-1}|xI_k - D_2|,$$

where

$$D_2 = \begin{pmatrix} (n_1-1)r_1 & n_2r_{12} & n_3r_{13} & \cdots & n_kr_{1k} \\ n_1r_{12} & (n_2-1)r_2 & n_3r_{23} & \cdots & n_kr_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_1r_{1k} & n_2r_{2k} & n_3r_{3k} & \cdots & (n_k-1)r_k \end{pmatrix}. \tag{6}$$

Proof As given above, the resistance-distance matrix of $K_{n_1,n_2,\ldots,n_k}$ is

$$R = \begin{pmatrix} r_1J_{n_1,n_1} - r_1I_{n_1} & r_{12}J_{n_1,n_2} & r_{13}J_{n_1,n_3} & \cdots & r_{1k}J_{n_1,n_k} \\ r_{12}J_{n_2,n_1} & r_2J_{n_2,n_2} - r_2I_{n_2} & r_{23}J_{n_2,n_3} & \cdots & r_{2k}J_{n_2,n_k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{1k}J_{n_k,n_1} & r_{2k}J_{n_k,n_2} & r_{3k}J_{n_k,n_3} & \cdots & r_kJ_{n_k,n_k} - r_kI_{n_k} \end{pmatrix}.$$

Hence, by linear algebra knowledge,

$$\begin{aligned} R_G(x) &= \det(xI_n - R) \\ &= \begin{vmatrix} (x+r_1)I_{n_1} - r_1J_{n_1,n_1} & -r_{12}J_{n_1,n_2} & -r_{13}J_{n_1,n_3} & \cdots & -r_{1k}J_{n_1,n_k} \\ -r_{12}J_{n_2,n_1} & (x+r_2)I_{n_2} - r_2J_{n_2,n_2} & -r_{23}J_{n_2,n_3} & \cdots & -r_{2k}J_{n_2,n_k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -r_{1k}J_{n_k,n_1} & -r_{2k}J_{n_k,n_2} & -r_{3k}J_{n_k,n_3} & \cdots & (x+r_k)I_{n_k} - r_kJ_{n_k,n_k} \end{vmatrix} \end{aligned}$$

$$= \prod_{i=1}^{k}(x+r_i)^{n_i-1}\begin{vmatrix} x-(n_1-1)r_1 & -n_2r_{12} & -n_3r_{13} & \cdots & -n_kr_{1k} \\ -n_1r_{12} & x-(n_2-1)r_2 & -n_3r_{23} & \cdots & -n_kr_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_1r_{1k} & -n_2r_{2k} & -n_3r_{3k} & \cdots & x-(n_k-1)r_k \end{vmatrix}.$$

This completes the proof. □

Corollary 3.2 *Let $G = K_{n_1,n_2,\ldots,n_k}$. Then the eigenvalues for the resistance-distance matrix of G are $-r_i$ of multiplicity $n_i - 1$ ($i = 1, 2, \ldots, k$) and the remaining eigenvalues satisfy the following*:

$$|xI_k - D_2| = 0,$$

where D_2 is given by (6).

Corollary 3.3 *The largest eigenvalue of the resistance-distance matrix of complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is given by*

$$|xI_k - D_2| = 0.$$

4 Lower and upper bounds on the largest eigenvalue of the resistance-distance matrix of complete multipartite graphs

In this section we give some lower and upper bounds on $\rho_1(G)$ of complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$. For this we need the following two results.

Lemma 4.1 ([14]) *If $q_1, q_2, \ldots, q_n$ are positive numbers, then*

$$\min_i \frac{p_i}{q_i} \le \frac{p_1 + p_2 + \cdots + p_n}{q_1 + q_2 + \cdots + q_n} \le \max_i \frac{p_i}{q_i}$$

for any real numbers $p_1, p_2, \ldots, p_n$. Equality holds on both sides if and only if all the ratios $\frac{p_i}{q_i}$ are equal.

Now we obtain the spectral radius for the resistance-distance matrix of $K_{n_1,n_1,\ldots,n_1}$.

Lemma 4.2 *Let G be a complete k-partite graph $K_{n_1,n_1,\ldots,n_1}$ of order n. Then*

$$\rho_1(G) = \frac{2}{n-n_1}\left[n_1 - 1 + \frac{n_1(k-1)(n-1)}{n}\right].$$

Furthermore, all the remaining eigenvalues are $-\frac{2}{n-n_1}$ and $-\frac{2}{n}$, with multiplicities $n-k$ and $k-1$, respectively.

Proof For $G = K_{n_1,n_1,\ldots,n_1}$, by Theorem 2.3, we have $r_i = \frac{2}{n-n_1}$ ($i = 1, 2, \ldots, k$) and $r_{ij} = \frac{n-1}{n}(\frac{1}{n-n_1} + \frac{1}{n-n_1})$ for all i and j, $i \neq j$. Since

$$D_2 = \begin{pmatrix} (n_1-1)r_1 & n_1r_{12} & n_1r_{12} & \cdots & n_1r_{12} \\ n_1r_{12} & (n_1-1)r_1 & n_1r_{12} & \cdots & n_1r_{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_1r_{12} & n_1r_{12} & n_1r_{12} & \cdots & (n_1-1)r_1 \end{pmatrix},$$

it is easily obtained that the eigenvalues of D_2 are

$$(n_1-1)r_1+(k-1)r_{12},\underbrace{(n_1-1)r_1-n_1r_{12},\ldots,(n_1-1)r_1-n_1r_{12}}_{k-1}.$$

Simple calculation shows that

$$(n_1-1)r_1+(k-1)r_{12}=\frac{2}{n-n_1}\left[n_1-1+\frac{n_1(k-1)(n-1)}{n}\right]\quad\text{and}\quad(n_1-1)r_1-n_1r_{12}=-\frac{2}{n}.$$

Thus it follows that

$$\rho_1(G)=\frac{2}{n-n_1}\left[n_1-1+\frac{n_1(k-1)(n-1)}{n}\right].$$

Together with the result in Theorem 2.3, we conclude that eigenvalues other than $\rho_1(G)$ are

$$\underbrace{-\frac{2}{n},\ldots,-\frac{2}{n}}_{k-1},\underbrace{-\frac{2}{n-n_1},\ldots,-\frac{2}{n-n_1}}_{n-k}.$$

□

Theorem 4.3 *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n with $n_1\geq n_2\geq\cdots\geq n_k$. Then*

$$\frac{2}{n-n_k}\left[n_k-1+\frac{n_k(k-1)(n-1)}{n}\right]\leq\rho_1(G)\leq\frac{2}{n-n_1}\left[n_1-1+\frac{n_1(k-1)(n-1)}{n}\right].\tag{7}$$

Moreover, the equality holds in both sides if and only if $n_1=n_2=\cdots=n_k$.

Proof Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ be an eigenvector corresponding to the eigenvalue $\rho_1(G)$ of D_2. Then we have

$$D_2\mathbf{x}=\rho_1(G)\mathbf{x}.\tag{8}$$

From the ith equation of (8), we get

$$\begin{aligned}\rho_1(G)x_i&=(n_i-1)r_ix_i+\sum_{j=1,j\neq i}^{k}n_jr_{ij}x_j\\&=\frac{2(n_i-1)}{n-n_i}x_i+\frac{n-1}{n}\sum_{j=1,j\neq i}^{k}\left[\frac{1}{n-n_i}+\frac{1}{n-n_j}\right]n_jx_j\\&\geq\frac{2(n_k-1)}{n-n_k}x_i+\frac{2(n-1)}{n(n-n_k)}\sum_{j=1,j\neq i}^{k}n_jx_j\quad\text{as } n_i,n_j\geq n_k.\end{aligned}\tag{9}$$

Taking summation on both sides from $i=1$ to k, we get

$$\rho_1(G)\sum_{i=1}^{k}x_i\geq\frac{2(n_k-1)}{n-n_k}\sum_{i=1}^{k}x_i+\frac{2(n-1)(k-1)}{n(n-n_k)}\sum_{i=1}^{k}n_ix_i.$$

By Lemma 4.1, we get

$$\frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} x_i} \geq n_k. \tag{10}$$

From the above two results, we get

$$\rho_1(G) \geq \frac{2(n_k-1)}{n-n_k} + \frac{2(n-1)(k-1)}{n(n-n_k)} n_k = \frac{2}{n-n_k}\left[n_k - 1 + \frac{n_k(k-1)(n-1)}{n}\right].$$

Since $n_i, n_j \leq n_1$ and

$$\frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} x_i} \leq n_1,$$

similarly, from the above, we get

$$\rho_1(G) \leq \frac{2}{n-n_1}\left[n_1 - 1 + \frac{n_1(k-1)(n-1)}{n}\right].$$

First part of the proof is done.

Now suppose that the left-hand side equality holds in (7). Then all the inequalities above must be equalities. From the equality in (9), we get $n_1 = n_2 = \cdots = n_k$. From the equality in (10), we get $n_1 = n_2 = \cdots = n_k$. Similarly, if the right-hand side equality holds in (7), then we have $n_1 = n_2 = \cdots = n_k$.

Conversely, let $G \cong K_{n_1,n_1,\ldots,n_1}$. By Lemma 4.2, equalities on both sides hold in (7). □

Now we give another upper bound on $\rho_1(G)$ of complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$.

Theorem 4.4 *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n with $n_1 \geq n_2 \geq \cdots \geq n_k$. Then*

$$\begin{aligned} \rho_1(G) \leq{} & \frac{2(n_1-1)}{n-n_1} + \frac{n_1(n-1)^2(k-1)}{n(n-n_1)(n_k-1)} - \frac{(n-1)(n-n_1)}{n(n_1-1)} \\ & + \frac{(n-1)(n-n_k)(k-1)}{n(n_k-1)}\left[1 - \frac{(2n-1)}{n-n_1} + \frac{n(n-1)}{(n-n_1)^2}\right] \end{aligned} \tag{11}$$

with equality holding if and only if $n_1 = n_2 = \cdots = n_k$.

Proof Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\rho_1(G)$ of $C^{-1}D_2C$, where $C = \mathrm{diag}((n_1-1)r_1, (n_2-1)r_2, \ldots, (n_k-1)r_k)$. Then we have

$$C^{-1}D_2C\mathbf{X} = \rho_1(G)\mathbf{X}. \tag{12}$$

We can assume that $x_i = 1$ and $x_k \leq 1$ for all k. From the ith equation of (12), we get

$$\rho_1(G)x_i = (n_i-1)r_i x_i + \sum_{j=1, j\neq i}^{k} \frac{(n_j-1)r_j n_j r_{ij}}{(n_i-1)r_i} x_j,$$

that is,

$$\rho_1(G) \leq (n_i-1)r_i + \sum_{j=1,j\neq i}^{k} \frac{(n_j-1)r_jn_jr_{ij}}{(n_i-1)r_i} \tag{13}$$

$$= \frac{2(n_i-1)}{n-n_i} + \sum_{j=1,j\neq i}^{k} \frac{(n-1)(n_j-1)n_j}{n(n_i-1)}\left[\frac{1}{n-n_j} + \frac{n-n_i}{(n-n_j)^2}\right]. \tag{14}$$

Since

$$\frac{n_j(n_j-1)}{n-n_j} = 1-n-n_j + \frac{n(n-1)}{n-n_j}$$

and

$$\frac{n_j(n_j-1)}{(n-n_j)^2} = 1 - \frac{2n-1}{n-n_j} + \frac{n(n-1)}{(n-n_j)^2},$$

from (14), we get

$$\rho_1(G) \leq \frac{2(n_i-1)}{n-n_i} + \frac{n-1}{n(n_i-1)}\sum_{j=1,j\neq i}^{k}\left[1-n-n_j+\frac{n(n-1)}{n-n_j}\right] + \frac{(n-1)(n-n_i)}{n(n_i-1)} \times \sum_{j=1,j\neq i}^{k}\left[1-\frac{2n-1}{n-n_j}+\frac{n(n-1)}{(n-n_j)^2}\right].$$

Since

$$f(x) = \frac{n(n-1)}{(n-x)^2} - \frac{2n-1}{(n-x)}$$

is an increasing function on x, from the above, we get

$$\begin{aligned}\rho_1(G) &\leq \frac{2(n_i-1)}{n-n_i} + \frac{n-1}{n(n_i-1)}\left[-(n-1)(k-1)-(n-n_i)+\frac{n(n-1)(k-1)}{n-n_1}\right] \\ &\quad + \frac{(n-1)(n-n_i)}{n(n_i-1)}\left[1-\frac{2n-1}{n-n_1}+\frac{n(n-1)}{(n-n_1)^2}\right](k-1) \\ &= \frac{2(n_i-1)}{n-n_i} + \frac{n_1(n-1)^2(k-1)}{n(n-n_1)(n_i-1)} - \frac{(n-1)(n-n_i)}{n(n_i-1)} \\ &\quad + \frac{(n-1)(n-n_i)(k-1)}{n(n_i-1)}\left[1-\frac{(2n-1)}{n-n_1}+\frac{n(n-1)}{(n-n_1)^2}\right].\end{aligned} \tag{15}$$

Let us consider a function

$$g(x) = 1 - \frac{(2n-1)}{n-x} + \frac{n(n-1)}{(n-x)^2}, \quad 1 \leq x \leq n-1.$$

Then

$$g'(x) = \frac{1}{(n-x)^3}\left[(2n-1)x-n\right] > 0.$$

Therefore $g(x) \geq g(1) = 0$. Since

$$\frac{n-n_1}{n_1-1} \leq \frac{n-n_i}{n_i-1} \leq \frac{n-n_k}{n_k-1}$$

and $n_k \leq n_i \leq n_1$ $(1 \leq i \leq k)$, from (15), we get

$$\begin{aligned}\rho_1(G) \leq{}& \frac{2(n_1-1)}{n-n_1} + \frac{n_1(n-1)^2(k-1)}{n(n-n_1)(n_k-1)} - \frac{(n-1)(n-n_1)}{n(n_1-1)} \\ &+ \frac{(n-1)(n-n_k)(k-1)}{n(n_k-1)}\left[1 - \frac{(2n-1)}{n-n_1} + \frac{n(n-1)}{(n-n_1)^2}\right],\end{aligned} \tag{16}$$

which gives the required result in (11). First part of the proof is done.

Now suppose that equality holds in (11). Then all the inequalities above must be equalities. From equality in (13), we get $x_1 = x_2 = \cdots = x_k$. From equality in (15), we get $n_1 = n_2 = \cdots = n_{i-1} = n_{i+1} = \cdots = n_k$. From equality in (16), we get $n_i = n_1 = n_k$. From these results we conclude that $n_1 = n_2 = \cdots = n_k$.

Conversely, let $n_1 = n_2 = \cdots = n_k$. Then $n = n_1 k$. Now,

$$\begin{aligned}&\frac{2(n_1-1)}{n-n_1} + \frac{n_1(n-1)^2(k-1)}{n(n-n_1)(n_1-1)} - \frac{(n-1)(n-n_1)}{n(n_1-1)} + \frac{(n-1)(n-n_1)(k-1)}{n(n_1-1)} \\ &\quad\times\left[1 - \frac{(2n-1)}{n-n_1} + \frac{n(n-1)}{(n-n_1)^2}\right] \\ &= \frac{2(n_1-1)}{n-n_1} + \frac{(n-1)^2}{n(n_1-1)} - \frac{(n-1)(n-n_1)}{n(n_1-1)} + \frac{(n-1)(k-1)^2}{k(n_1-1)} \times \frac{n_1(n_1-1)}{(n-n_1)^2} \\ &= \frac{2(n_1-1)}{n-n_1} + \frac{2(n-1)}{n} = \rho_1(G), \quad \text{by Lemma 4.2.}\end{aligned}$$

□

5 Lower bound on the second largest eigenvalue of the resistance-distance matrix of complete multipartite graphs

In this section we find a lower bound on the second largest eigenvalue of the resistance-distance matrix of complete multipartite graphs. For this we need the following result.

Lemma 5.1 ([15]) *Let A be a $p \times p$ symmetric matrix, and let A_k be its leading $k \times k$ submatrix; that is, A_k is the matrix obtained from A by deleting its last $p-k$ rows and columns. Then, for $i = 1, 2, \ldots, k$,*

$$\lambda_{p-i+1}(A) \leq \lambda_{k-i+1}(A_k) \leq \lambda_{k-i+1}(A), \tag{17}$$

where $\lambda_i(A)$ is the ith largest eigenvalue of A.

Theorem 5.2 *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n with $n_1 \geq n_2 \geq \cdots \geq n_k$. Then*

$$\begin{aligned}\rho_2(G) \geq \max_{1\leq i<j\leq k}\Bigg[&\frac{(n+1)(n_i+n_j) - 2(n+n_in_j)}{(n-n_i)(n-n_j)} - \frac{(n-1)(n_i+n_j)}{(n-n_i)(n-n_j)} \\ &\times\sqrt{\left(1 + \frac{n_in_j}{n^2}\right) - \frac{4n_in_j}{n(n_i+n_j)}}\Bigg].\end{aligned} \tag{18}$$

Proof By Lemma 5.1, we have $\rho_1(G) \geq \max_{1\leq i<j\leq k} \rho_1'$ and $\rho_2(G) \geq \max_{1\leq i<j\leq k} \rho_2'$, where ρ_1' and ρ_2' are given by

$$\begin{vmatrix} (n_i-1)r_i-\rho & n_jr_{ij} \\ n_ir_{ij} & (n_j-1)r_j-\rho \end{vmatrix} = 0,$$

that is,

$$\rho^2 - \big[(n_i-1)r_i + (n_j-1)r_j\big]\rho + (n_i-1)(n_j-1)r_ir_j - n_in_jr_{ij}^2 = 0.$$

So,

$$\rho_1' = \frac{(n_i-1)r_i + (n_j-1)r_j + \sqrt{[(n_i-1)r_i-(n_j-1)r_j]^2 + 4n_in_jr_{ij}^2}}{2}$$

and

$$\rho_2' = \frac{(n_i-1)r_i + (n_j-1)r_j - \sqrt{[(n_i-1)r_i-(n_j-1)r_j]^2 + 4n_in_jr_{ij}^2}}{2}. \tag{19}$$

Now,

$$\begin{aligned}(n_i-1)r_i-(n_j-1)r_j &= \frac{2(n_i-1)}{n-n_i} - \frac{2(n_j-1)}{n-n_j} \\ &= \frac{2(n-1)(n_i-n_j)}{(n-n_i)(n-n_j)}.\end{aligned}$$

Using the above result, we get

$$\begin{aligned}&\big[(n_i-1)r_i-(n_j-1)r_j\big]^2 + 4n_in_jr_{ij}^2 \\ &\quad= \frac{4(n-1)^2(n_i-n_j)^2}{(n-n_i)^2(n-n_j)^2} + \frac{4n_in_j(n-1)^2}{n^2}\left(\frac{1}{n-n_i} + \frac{1}{n-n_j}\right)^2 \\ &\quad= \frac{4(n-1)^2(n_i+n_j)^2}{(n-n_i)^2(n-n_j)^2}\left[\left(1+\frac{n_in_j}{n^2}\right) - \frac{4n_in_j}{n(n_i+n_j)}\right].\end{aligned} \tag{20}$$

Moreover,

$$\begin{aligned}(n_i-1)r_i+(n_j-1)r_j &= \frac{2(n_i-1)}{n-n_i} + \frac{2(n_j-1)}{n-n_j} \\ &= \frac{2[(n+1)(n_i+n_j) - 2(n+n_in_j)]}{(n-n_i)(n-n_j)}.\end{aligned} \tag{21}$$

Using (20) and (21) in (19), we get the required result in (18). □

Corollary 5.3 *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n with $n_1 \geq n_2 \geq \cdots \geq n_k$. Then*

$$\rho_2(G) \geq \frac{(n+1)(n_1+n_2) - 2(n+n_1n_2)}{(n-n_1)(n-n_2)} - \frac{(n-1)(n_1+n_2)}{(n-n_1)(n-n_2)}\sqrt{\left(1+\frac{n_1n_2}{n^2}\right) - \frac{4n_1n_2}{n(n_1+n_2)}}.$$

6 Conclusion

In this paper, resistance distances in complete multipartite graphs are given via the standard electrical approach. Then eigenvalues of the resistance-distance matrix of complete multipartite graphs are studied, with emphasis being placed on bounds for the largest and second largest eigenvalues. However, up to now, the study on the eigenvalues of a resistance-distance matrix has still been in its infancy. Further study in this field is greatly anticipated.

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