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Bergman projections on weighted Fock spaces in several complex variables

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Abstract

Let ϕ be a real-valued plurisubharmonic function on $\mathbb{C}^n$ whose complex Hessian has uniformly comparable eigenvalues, and let $\mathcal{F}^p(\phi)$ be the Fock space induced by ϕ. In this paper, we conclude that the Bergman projection is bounded from the pth Lebesgue space $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ for $1 \leq p \leq \infty$. As a remark, we claim that Bergman projections are also well defined and bounded on Fock spaces $\mathcal{F}^p(\phi)$ with $0 < p < 1$. We also obtain the estimates for the distance induced by ϕ and the $L^p(\phi)$-norm of Bergman kernel for $\mathcal{F}^2(\phi)$.

Keywords: Bergman kernel; Bergman projection; reverse-Hölder inequality

1 Introduction

The symbol dv denotes the Lebesgue volume measure on $\mathbb{C}^n$, and

$$B(z,r) = \{w \in \mathbb{C}^n : |w-z| < r\} \quad \text{for } z \in \mathbb{C}^n \text{ and } r > 0.$$

Suppose $\phi : \mathbb{C}^n \to \mathbb{R}$ is a C^2 plurisubharmonic function. We say that ϕ belongs to the weight class $\mathbf{W}$ if ϕ satisfies the following statements:

(I) There exists $c > 0$ such that for $z \in \mathbb{C}^n$

$$\inf_{z\in\mathbb{C}^n} \sup_{w\in B(z,c)} \Delta\phi(w) > 0; \tag{1}$$

(II) $\Delta\phi$ satisfies the reverse-Hölder inequality

$$\|\Delta\phi\|_{L^\infty(B(z,r))} \leq Cr^{-2n} \int_{B(z,r)} \Delta\phi \, dv, \quad \forall z \in \mathbb{C}^n, r > 0 \tag{2}$$

for some $0 < C < +\infty$;

(III) The eigenvalues of H_ϕ are comparable, i.e., there exists $\delta_0 > 0$ such that

$$(H_\phi(z)u, u) \geq \delta_0 \Delta\phi(z)|u|^2, \quad \forall z, u \in \mathbb{C}^n,$$

where

$$H_\phi = \left(\frac{\partial^2 \phi}{\partial z_j \, \partial \bar{z}_k}\right)_{j,k}.$$

Suppose $0 < p < \infty$, $\phi \in \mathbf{W}$. The space $L^p(\phi)$ consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{p,\phi} = \left(\int_{\mathbb{C}^n} \left|f(z)\right|^p e^{-p\phi(z)}\,dv(z)\right)^{\frac{1}{p}} < \infty.$$

$L^\infty(\phi)$ is the set of all Lebesgue measurable functions f on $\mathbb{C}^n$ with

$$\|f\|_{\infty,\phi} = \sup_{z\in\mathbb{C}^n} \left|f(z)\right| e^{-\phi(z)} < \infty.$$

Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. The weighted Fock space is defined as

$$\mathcal{F}^p(\phi) = L^p(\phi) \cap H\left(\mathbb{C}^n\right)$$

with the same norm $\|\cdot\|_{p,\phi}$. It is easy to check that $\mathcal{F}^p(\phi)$ is a Banach space under $\|\cdot\|_{p,\phi}$ if $1 \le p < \infty$, and $\mathcal{F}^p(\phi)$ is a Fréchet space with the metric $\varrho(f,g) = \|f-g\|_{p,\phi}^p$ whenever $0 < p < 1$. Taking $\phi(z) = \frac{1}{2}|z|^2$, $\mathcal{F}^p(\phi)$ is the classical Fock space which has been studied by many authors, see [1–3] and the references therein. Notice that the weight function φ on $\mathbb{C}^n$ with the restriction that $dd^c\varphi \simeq dd^c|z|^2$ in [4] and [5] belongs to $\mathbf{W}$.

In the one-dimensional case, an important contribution to weighted Fock spaces was given by Christ [6] (but see also [7, 8]). They work under the assumption that ϕ is subharmonic and that $\Delta\phi\,dA$ is a doubling measure, where dA is the area measure on $\mathbb{C}$. Notice that the hypotheses on $\Delta\phi\,dA$ are a sort of finite-type assumption and are automatically verified when ϕ is a subharmonic non-harmonic polynomial.

The result of Christ was extended by Delin to several complex variables under the assumption of strict plurisubharmonicity of the weight in [9]. Dall'Ara [10] tried to extend Christ's approach to $n \ge 2$. Given $\phi \in \mathbf{W}$, let $K(\cdot,\cdot)$ be the weighted Bergman kernel for $\mathcal{F}^2(\phi)$. In particular, Theorem 20 of [10] proves that there is a constant $C, \epsilon > 0$ such that

$$\left|K(z,w)\right| \le Ce^{\phi(z)+\phi(w)} \frac{e^{-\epsilon d(z,w)}}{\rho_\phi(z)^n \rho_\phi(w)^n} \tag{3}$$

for $z, w \in \mathbb{C}^n$, where $d(\cdot,\cdot)$, $\rho_\phi(\cdot)$ described in Section 2.

In the setting of Bergman spaces, the Bergman projection is bounded on p-Bergman spaces for $1 < p < \infty$, it also maps L^∞ into Bloch spaces, see [11] for details. With the Bergman kernel $K(\cdot,\cdot)$ for $\mathcal{F}^2(\phi)$, the Bergman projection P can be represented as

$$Pf(z) = \int_{\mathbb{C}^n} K(z,w) f(w) e^{-2\phi(w)}\,dv(w), \quad z \in \mathbb{C}^n.$$

It is well known that $P(f) = f$ for $f \in \mathcal{F}^2(\phi)$. The purpose of this work is to discuss the boundedness of Bergman projection acting on $\mathcal{F}^p(\phi)$ for general p. Section 2 is devoted to some basic estimates, including the distance $d(\cdot,\cdot)$ and the $L^p(\phi)$-norm of the Bergman kernel. In Section 3, we will discuss the boundedness of Bergman projections from $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ with $1 \le p \le \infty$. We also show that the Bergman projection is well defined and bounded on $\mathcal{F}^p(\phi)$ for $p < 1$.

In what follows, we always suppose $\phi \in \mathbf{W}$ and use C to denote positive constants whose values may change from line to line but do not depend on the functions being considered. Two quantities A and B are called equivalent, denoted by '$A \simeq B$', if there exists some C such that $C^{-1}A \le B \le CA$.

2 Some basic estimates

In this section, we are going to give some estimates, which will be useful in the following section. At the beginning, we will give some notations.

For $z \in \mathbb{C}^n$, set

$$\rho_\phi(z) = \sup\Bigl\{r > 0 : \sup_{w\in B(z,r)} \Delta\phi(w) \le r^{-2}\Bigr\}. \tag{4}$$

By (1), there exist $c, s > 0$ such that for $z \in \mathbb{C}^n$

$$\sup_{w\in B(z,c)} \Delta\phi(w) \ge s.$$

We then have some $M > 0$ such that

$$\sup_{z\in\mathbb{C}^n} \rho_\phi(z) \le M.$$

Moreover, there are some positive constants C, M_1 and M_2 such that for all $z, w \in \mathbb{C}^n$, we have

$$C^{-1}\theta^{-M_1}\rho_\phi(w) \le \rho_\phi(z) \le C\theta^{M_2}\rho_\phi(w), \tag{5}$$

where $\theta = \max(1, \frac{|z-w|}{\rho_\phi(w)})$. We can see this in Proposition 10 of [10].

Given $r > 0$, write

$$B^r(z) = B\bigl(z, r\rho_\phi(z)\bigr) \quad \text{and} \quad B(z) = B^1(z).$$

Then (5) implies that there is some C such that for $z \in \mathbb{C}^n$

$$C^{-1}\rho_\phi(w) \le \rho_\phi(z) \le C\rho_\phi(w) \quad \text{for } w \in B(z). \tag{6}$$

By (6) and the triangle inequality, we have $m_1, m_2 > 0$ such that

$$B(z) \subseteq B^{m_1}(w), \qquad B(w) \subseteq B^{m_2}(z) \quad \text{whenever } w \in B(z). \tag{7}$$

Given a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}^n$, we say that $\{a_k\}_{k=1}^\infty$ is a lattice if $\{B(a_k)\}_{k=1}^\infty$ covers $\mathbb{C}^n$ and the balls of $\{B^{\frac{1}{5}}(a_k)\}_{k=1}^\infty$ are pairwise disjoint. This lattice exists by a standard covering lemma, see Theorem 2.1 in [12], or Proposition 7 in [10] as well. Moreover, for the lattice $\{a_k\}_k$ and any $m > 0$, there exists some integer N such that each $z \in \mathbb{C}^n$ can be in at most N balls of $\{B^m(a_k)\}_k$. Equivalently,

$$\sum_{k=1}^\infty \chi_{B^m(a_k)}(z) \le N \quad \text{for } z \in \mathbb{C}^n. \tag{8}$$

To the radius function ρ_ϕ defined as (4), we associate the Riemannian metric $\rho_\phi(z)^{-2}\,dz \otimes d\bar{z}$. In fact, we are interested only in the associated Riemannian distance, which we describe explicitly. If $\gamma : [0,1] \to \mathbb{C}^n$ is piecewise C^1 curves, we define

$$L_{\rho_\phi}(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))}\,dt.$$

Given $z, w \in \mathbb{C}^n$, we put

$$d(z,w) = \inf_\gamma L_{\rho_\phi}(\gamma),$$

where the inf is taken as γ varies over the collection of curves with $\gamma(0) = z$ and $\gamma(1) = w$. We then have the estimate for this distance as follows.

Lemma 1 *There exist $\alpha, \beta, C > 0$ such that for $z, w \in \mathbb{C}^n$*

$$\frac{1}{C}\left(\frac{|z-w|}{\rho_\phi(z)}\right)^\alpha \le d(z,w) \le C\left(\frac{|z-w|}{\rho_\phi(z)}\right)^\beta.$$

Proof First, we claim that there is some $C > 0$ such that

$$d(z,w) \ge C\left(\frac{|z-w|}{\rho_\phi(z)}\right)^\alpha. \tag{9}$$

In fact, set μ to be

$$\mu\big(B(z,r)\big) = r^2 \|\Delta\phi\|_{L^\infty(B(z,r))}, \quad z \in \mathbb{C}^n, r > 0. \tag{10}$$

By (2), it is easy to check that there is some $M > 2$ such that

$$\mu\big(B(z,2r)\big) \le M\mu\big(B(z,r)\big). \tag{11}$$

Moreover,

$$\mu\big(B\big(z,\rho_\phi(z)\big)\big) = 1 \tag{12}$$

because of (4). Given any $r \le R$, it is easy to check that

$$\mu\big(B(z,r)\big) \le \left(\frac{r}{R}\right)^2 \mu\big(B(z,R)\big) \le \mu\big(B(z,R)\big) \tag{13}$$

for $z \in \mathbb{C}^n$ because of (10). Also, there is a positive integer m such that $2^{m-1}r < R \le 2^m r$. Hence, (11) and (12) tell us

$$\mu\big(B(z,R)\big) \le \mu\big(B\big(z,2^m r\big)\big) \le M\mu\big(B\big(z,2^{m-1}r\big)\big) \le \cdots \le M^m\mu\big(B(z,r)\big).$$

Since $M^{m-1} = 2^{(m-1)\log_2 M} \le (\frac{R}{r})^{\log_2 M}$, we get

$$\mu\big(B(z,R)\big) \le M\left(\frac{R}{r}\right)^{\log_2 M} \mu\big(B(z,r)\big). \tag{14}$$

For $z, w \in \mathbb{C}^n$, notice that $B(w, |w-z|) \subset B(z, 2|w-z|)$. If $|w-z| < \rho_\phi(z)$, take any piecewise C^1 curve $\gamma : [0,1] \to \mathbb{C}^n$ connecting z and w, and let T_0 be the minimum time such that $|z - \gamma(T_0)| = \rho_\phi(z)$. By (6), $\rho_\phi(\gamma(t)) \simeq \rho_\phi(z)$ for $t \in [0, T_0)$. This implies

$$\int_0^1 \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))}\,dt \geq \frac{C}{\rho_\phi(z)} \int_0^{T_0} \left|\gamma'(t)\right| dt \geq C\frac{|z-w|}{\rho_\phi(z)}.$$

If $|z-w| \geq \rho_\phi(w)$, then (11), (10), (13) and (12) give

$$\begin{aligned}
\mu\left(B\left(z, \frac{1}{4}|z-w|\right)\right) &\geq C\mu\big(B\big(z, 2|z-w|\big)\big) \geq C\mu\big(B\big(w, |z-w|\big)\big) \\
&\geq C\left(\frac{|z-w|}{\rho_\phi(w)}\right)^2 \mu\big(B\big(w, \rho_\phi(w)\big)\big) \\
&= C\left(\frac{|z-w|}{\rho_\phi(w)}\right)^2.
\end{aligned}$$

On the other hand, for $\zeta \in \overline{B(z, \frac{1}{4}|z-w|)}$, there are

$$B\left(\zeta, \frac{1}{4}|z-w|\right) \subset B\left(z, \frac{1}{2}|z-w|\right)$$

and

$$B\left(z, \frac{1}{4}|z-w|\right) \subset B\left(\zeta, \frac{1}{2}|z-w|\right).$$

Combining the above with (11), we know

$$\mu\left(B\left(z, \frac{1}{4}|z-w|\right)\right) \simeq \mu\left(B\left(\zeta, \frac{1}{2}|z-w|\right)\right).$$

By the fact $\log_2 M > 0$, (13), (14) and (12), there exists $t > 0$ such that

$$\begin{aligned}
\mu\left(B\left(z, \frac{1}{4}|z-w|\right)\right) &\simeq \mu\left(B\left(\zeta, \frac{1}{2}|z-w|\right)\right) \\
&\leq C\left(\frac{|z-w|}{\rho_\phi(\zeta)}\right)^t \mu\big(B\big(\zeta, \rho_\phi(\zeta)\big)\big) \\
&\simeq \left(\frac{|z-w|}{\rho_\phi(\zeta)}\right)^t.
\end{aligned}$$

Hence, $(\frac{|z-w|}{\rho_\phi(w)})^2 \leq C(\frac{|z-w|}{\rho_\phi(\zeta)})^t$. This implies

$$\rho_\phi(\zeta) \leq C|z-w|\left(\frac{|z-w|}{\rho_\phi(w)}\right)^{-\alpha}, \qquad \zeta \in \overline{B\left(z, \frac{1}{4}|z-w|\right)},$$

where $\alpha = \frac{2}{t} > 0$. For any piecewise C^1 curves Γ, defined as $\gamma : [0,1] \to \mathbb{C}^n$ with $\gamma(0) = z$ and $\gamma(1) = w$, we have

$$\begin{aligned}\int_\Gamma \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))}\,dt &\geq \int_{\Gamma\cap\overline{B(z,\frac{1}{4}|z-w|)}} \frac{|\gamma'(t)|}{\rho_\phi(\gamma(t))}\,dt \\ &\geq \frac{1}{|z-w|(\frac{|z-w|}{\rho_\phi(w)})^{-\alpha}} \int_{\Gamma\cap\overline{B(z,\frac{1}{4}|z-w|)}} \left|\gamma'(t)\right| dt \\ &\geq C\left(\frac{|z-w|}{\rho_\phi(w)}\right)^\alpha.\end{aligned}$$

This yields (9) is true. Now, we are going to prove the other direction. For $z, w \in \mathbb{C}^n$, take $\gamma(t) = z + t(w-z)$ and $\gamma(t_0) \in \partial B(z)$ (set $t_0 = 1$ if $w \in B(z)$). Then (5) gives

$$\begin{aligned}d(z,w) &\leq |w-z| \int_0^1 \frac{dt}{\rho_\phi(\gamma(t))} \\ &\leq C|w-z|\left(\int_0^{t_0} + \int_{t_0}^1\right) \frac{dt}{\rho_\phi(\gamma(t))} \\ &\leq C\frac{|w-z|}{\rho_\phi(z)} \int_0^1 dt + C\left(\frac{|w-z|}{\rho_\phi(z)}\right)^{1+M_1} \int_0^1 t^{M_1}\,dt \\ &\leq C\left(\frac{|w-z|}{\rho_\phi(z)}\right)^\beta,\end{aligned}$$

where $\beta > 0$. The proof is completed. □

Now, we can estimate the following integral.

Lemma 2 *Given $p > 0$ and $k \in \mathbb{R}$, we have*

$$\int_{\mathbb{C}^n} \rho_\phi(\zeta)^k e^{-pd(z,\zeta)}\,dv(\zeta) \leq C\rho_\phi(z)^{k+2n},$$

where $C > 0$ is a constant depending only on n, p and k.

Proof By (6), it is easy to check that

$$\int_{B(z)} \rho_\phi(\zeta)^k e^{-pd(z,\zeta)}\,dv(\zeta) \leq \int_{B(z)} \rho_\phi(\zeta)^k\,dv(\zeta) \leq C\rho_\phi(z)^{k+2n}.$$

Estimate (9) gives

$$\begin{aligned}\int_{\mathbb{C}^n\setminus B(z)} \rho_\phi(\zeta)^k e^{-pd(z,\zeta)}\,dv(\zeta) &\leq \int_{\mathbb{C}^n\setminus B(z)} \rho_\phi(\zeta)^k e^{-pC_1(\frac{|z-\zeta|}{\rho_\phi(z)})^\alpha}\,dv(\zeta) \\ &\leq \int_{\mathbb{C}^n\setminus B(z)} \rho_\phi(\zeta)^k \int_{pC_1(\frac{|z-\zeta|}{\rho_\phi(z)})^\alpha}^\infty e^{-s}\,ds\,dv(\zeta) \\ &\leq \int_{pC_1}^\infty e^{-s} \int_{B^{(\frac{s}{pC_1})^{\frac{1}{\alpha}}}(z)} \rho_\phi(\zeta)^k\,dv(\zeta)\,ds.\end{aligned}$$

By (5), the inequality above is no more than

$$\begin{aligned}&\int_{pC_1}^{\infty}\sup_{\zeta\in B^{(\frac{s}{pC_1})^{\frac{1}{\alpha}}}(z)}\rho_\phi(\zeta)^k v\big(B^{(\frac{s}{pC_1})^{\frac{1}{\alpha}}}(z)\big)e^{-s}\,ds\\&\quad\le C\rho_\phi(z)^{k+2n}\int_{pC_1}^{\infty}s^{\frac{2n+\max\{kM_2,-kM_1\}}{\alpha}}e^{-s}\,ds=C\rho_\phi(z)^{k+2n}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{C}^n}\rho_\phi(w)^k e^{-pd(z,w)}\,dv(w)\le C\rho_\phi(z)^{k+2n}.$$

The proof is completed. □

Next, we will give the $L^p(\phi)$-norm of the Bergman kernel $K(\cdot,\cdot)$ for $\mathcal{F}^2(\phi)$.

Proposition 3 *For $0<p<\infty$, we have*

$$\|K(\cdot,z)\|_{p,\phi}\le Ce^{\phi(z)}\rho_\phi(z)^{2n(\frac{1}{p}-1)},\quad z\in\mathbb{C}^n.$$

Proof By (3) and Lemma 2, we obtain

$$\begin{aligned}\int_{\mathbb{C}^n}\big|K(w,z)\big|^p e^{-p\phi(w)}\,dv(w)&\le C\frac{e^{p\phi(z)}}{\rho_\phi(z)^{pn}}\int_{\mathbb{C}^n}\rho_\phi(w)^{-pn}e^{-p\epsilon d(z,w)}\,dv(w)\\&\le Ce^{p\phi(z)}\rho_\phi(z)^{2n(1-p)}.\end{aligned}$$

The proof is completed. □

Lemma 4 *For $0<p<\infty$, there is a constant $C>0$ such that for all $r\in(0,1]$, $f\in H(\mathbb{C}^n)$ and $z\in\mathbb{C}^n$, we have*

$$\big|f(z)\big|e^{-\phi(z)}\le\frac{C}{r^{\frac{2n}{p}}\rho_\phi(z)^{\frac{2n}{p}}}\left(\int_{B^r(z)}\big|f(w)e^{-\phi(w)}\big|^p\,dv(w)\right)^{\frac{1}{p}}.\qquad(15)$$

Proof If $p=2$, (15) is just Lemma 13 in [10]. For $p\ne 2$, we borrow the idea in Lemma 19 of [7] and Lemma 13 in [10]. The details are omitted. □

3 Boundedness of Bergman projections

Recall that the Bergman projection P on $L^p(\phi)$ is defined as

$$Pf(z)=\int_{\mathbb{C}^n}K(z,w)f(w)e^{-2\phi(w)}\,dv(w),\quad z\in\mathbb{C}^n.$$

In this section, we focus on the boundedness of Bergman projections P from $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ for $1\le p\le\infty$.

Theorem 5 *Let $1\le p\le\infty$. Then the Bergman projection P is bounded as a map from $L^p(\phi)$ to $\mathcal{F}^p(\phi)$.*

Proof By the definition of P, we can conclude Pf is holomorphic on $\mathbb{C}^n$. Fubini's theorem and Proposition 3 yield

$$\begin{aligned}
\|Pf\|_{1,\phi} &\le \int_{\mathbb{C}^n} e^{-\phi(z)}\,dv(z) \int_{\mathbb{C}^n} \bigl|K(z,w)f(w)\bigr| e^{-2\phi(w)}\,dv(w) \\
&= \int_{\mathbb{C}^n} \bigl|f(w)\bigr| e^{-2\phi(w)}\,dv(w) \int_{\mathbb{C}^n} \bigl|K(z,w)\bigr| e^{-\phi(z)}\,dv(z) \\
&\le C\|f\|_{1,\phi}
\end{aligned}$$

for $f \in L^1(\phi)$. Given $f \in L^\infty(\phi)$, we obtain

$$\begin{aligned}
\|Pf\|_{\infty,\phi} &\le \sup_{z\in\mathbb{C}^n} e^{-\phi(z)} \int_{\mathbb{C}^n} \bigl|K(z,w)f(w)\bigr| e^{-2\phi(w)}\,dv(w) \\
&\le \|f\|_{\infty,\phi} \sup_{z\in\mathbb{C}^n} e^{-\phi(z)} \int_{\mathbb{C}^n} \bigl|K(z,w)\bigr| e^{-\phi(w)}\,dv(w) \\
&\le C\|f\|_{\infty,\phi}.
\end{aligned}$$

If $1 < p < \infty$, Hölder's inequality and Fubini's theorem give

$$\begin{aligned}
&\|Pf\|_{p,\phi}^p \\
&\quad\le \int_{\mathbb{C}^n} e^{-p\phi(z)}\,dv(z) \biggl(\int_{\mathbb{C}^n} \bigl|K(z,w)f(w)\bigr| e^{-2\phi(w)}\,dv(w) \biggr)^p \\
&\quad\le \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \bigl|f(w)\bigr|^p e^{-p\phi(w)} \bigl|K(z,w)\bigr| e^{-\phi(w)}\,dv(w) \bigl\|K(z,\cdot)\bigr\|_{1,\phi}^{p-1} e^{-p\phi(z)}\,dv(z) \\
&\quad\le C \int_{\mathbb{C}^n} e^{-\phi(z)}\,dv(z) \int_{\mathbb{C}^n} \bigl|f(w)\bigr|^p e^{-p\phi(w)} \bigl|K(z,w)\bigr| e^{-\phi(w)}\,dv(w) \\
&\quad\le C \int_{\mathbb{C}^n} \bigl|f(w)\bigr|^p e^{-p\phi(w)} e^{-\phi(w)}\,dv(w) \int_{\mathbb{C}^n} \bigl|K(z,w)\bigr| e^{-\phi(z)}\,dv(z) \\
&\quad\le C\|f\|_{p,\phi}^p
\end{aligned}$$

for $f \in L^p(\phi)$. Thus, P is bounded from $L^p(\phi)$ to $\mathcal{F}^p(\phi)$ for $1 \le p \le \infty$. The proof is ended. □

In addition, we observe that the Bergman projection is also well defined and bounded on the weighted Fock space $\mathcal{F}^p(\phi)$ with $p < 1$.

Remark 6 For $p < 1$, the Bergman projection P is bounded on $\mathcal{F}^p(\phi)$.

Proof First, we claim that P is well defined on $\mathcal{F}^p(\phi)$. In fact, given any $f \in \mathcal{F}^p(\phi)$, by (3), (15) and Lemma 2, we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} \bigl|K(z,w)f(w)\bigr| e^{-2\phi(w)}\,dv(w) \\
&\quad\le C\|f\|_{p,\phi} \int_{\mathbb{C}^n} \rho_\phi(w)^{-\frac{2n}{p}} \bigl|K(z,w)\bigr| e^{-\phi(w)}\,dv(w)
\end{aligned}$$

$$\leq Ce^{\phi(z)}\rho_\phi(z)^{-n}\int_{\mathbb{C}^n}\rho_\phi(w)^{-\frac{2n}{p}-n}e^{-\epsilon d(z,w)}\,dv(w)$$
$$\leq Ce^{\phi(z)}\rho_\phi(z)^{-\frac{2n}{p}}<\infty.$$

Now, we deal with the boundedness of P. In fact, let $\{a_k\}_k$ be the lattice. For $f\in\mathcal{F}^p(\phi)$, we get

$$\begin{aligned}\big|Pf(z)\big|^p &\leq \left(\sum_{k=1}^{\infty}\int_{B(a_k)}\big|f(w)K(w,z)\big|e^{-2\phi(w)}\,dv(w)\right)^p\\ &\leq \sum_{k=1}^{\infty}\left(\int_{B(a_k)}\big|f(w)K(w,z)\big|e^{-2\phi(w)}\,dv(w)\right)^p\\ &\leq \sum_{k=1}^{\infty}v\big(B(a_k)\big)^p\Big(\sup_{w\in B(a_k)}\big|f(w)K(w,z)\big|e^{-2\phi(w)}\Big)^p.\end{aligned}$$

Notice that the associated function $\rho_{2\phi}=\frac{\sqrt{2}}{2}\rho_\phi$, which follows from (4). Applying Lemma 4 with weight 2ϕ instead of ϕ, there then is some constant $C>0$ such that $|Pf(z)|^p$ is no more than C times

$$\sum_{k=1}^{\infty}\rho_\phi(a_k)^{2np-2n}\sup_{w\in B(a_k)}\int_{B(w)}\big|f(u)\big|^p\big|K(u,z)\big|^p e^{-2p\phi(u)}\,dv(u).$$

Combining (7) with (8), we obtain

$$\begin{aligned}\big|Pf(z)\big|^p &\leq C\sum_{k=1}^{\infty}\int_{B^{m_2}(a_k)}\rho_\phi(u)^{2np-2n}\big|f(u)\big|^p\big|K(u,z)\big|^p e^{-2p\phi(u)}\,dv(u)\\ &\leq CN\int_{\mathbb{C}^n}\rho_\phi(u)^{2np-2n}\big|f(u)\big|^p\big|K(u,z)\big|^p e^{-2p\phi(u)}\,dv(u).\end{aligned}$$

Therefore, applying Fubini's theorem and Proposition 3, we get

$$\begin{aligned}&\int_{\mathbb{C}^n}\big|Pf(z)\big|^p e^{-p\phi(z)}\,dv(z)\\ &\quad\leq C\int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\big|K(u,z)\big|^p e^{-p\phi(z)}\,dv(z)\rho_\phi(u)^{2np-2n}\big|f(u)\big|^p e^{-2p\phi(u)}\,dv(u)\\ &\quad\leq C\int_{\mathbb{C}^n}\big|f(u)\big|^p e^{-p\phi(u)}\,dv(u).\end{aligned}$$

This means that P is bounded on $\mathcal{F}^p(\phi)$. The proof is ended. □

4 Conclusion

In this paper, we show the boundedness of Bergman projection from the pth Lebesgue space $L^p(\phi)$ to the weighted Fock space $\mathcal{F}^p(\phi)$ for $1\leq p\leq\infty$. We also remak that the Bergman projection is bounded on $\mathcal{F}^p(\phi)$ with $p<1$. Meanwhile, we get the estimates for the distance induced by ϕ and the $L^p(\phi)$-norm of Bergman kernel for $\mathcal{F}^2(\phi)$.

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Competing interests
The author declares that she has no competing interests.

Authors' contributions
The author wrote this paper by herself. She has read and approved the final manuscript.

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References

1. Hu, ZJ, Lv, XF: Toeplitz operators from one Fock space to another. Integral Equ. Oper. Theory **70**, 541-559 (2011)
2. Isralowitz, J, Zhu, KH: Toeplitz operators on the Fock space. Integral Equ. Oper. Theory **66**(4), 593-611 (2010)
3. Zhu, KH: Analysis on Fock Spaces. Springer, New York (2012)
4. Hu, ZJ, Lv, XF: Toeplitz operators on Fock spaces $F^p(\varphi)$. Integral Equ. Oper. Theory **80**(1), 33-59 (2014)
5. Schuster, AP, Varolin, D: Toeplitz operators and Carleson measures on generalized Bargmann-Fock spaces. Integral Equ. Oper. Theory **72**, 363-392 (2012)
6. Christ, M: On the $\overline{\partial}$ equation in weighted L^2 norms in $\mathbb{C}$. J. Geom. Anal. **1**(3), 193-230 (1991)
7. Marco, N, Massanedv, X, Ortega-Cerdà, J: Interpolating and sampling sequences for entire functions. Geom. Funct. Anal. **13**(4), 862-914 (2003)
8. Marzo, J, Ortega-Cerdà, J: Pointwise estimates for the Bergman kernel of the weighted Fock space. J. Geom. Anal. **19**, 890-910 (2009)
9. Delin, H: Pointwise estimates for the weighted Bergman projection kernel in $\mathbb{C}^n$, using a weighted L^2 estimate for the $\overline{\partial}$ equation. Ann. Inst. Fourier (Grenoble) **48**(4), 967-997 (1998)
10. Dall'Ara, GM: Pointwise estimates of weighted Bergman kernels in several complex variables. Adv. Math. **285**, 1706-1740 (2015)
11. Hedenmalm, H, Korenblum, B, Zhu, KH: Theory of Bergman Spaces. GTM, vol. 199. Springer, New York (2000)
12. Mattila, P: Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability. Cambridge Studies in Advanced Mathematics, vol. 44. Cambridge University Press, Cambridge (1995)