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$F(\psi, \varphi)$ -Contraction in terms of measure of noncompactness with application for nonlinear integral equations

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Abstract

In this paper, some new generalization of Darbo's fixed point theorem is proved by using a $F(\psi, \varphi)$ -contraction in terms of a measure of noncompactness. Our result extends to obtaining a common fixed point for a pair of compatible mappings. The paper contains an application for nonlinear integral equations as well.

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1 Introduction and preliminaries

A contractive condition in terms of a measure of noncompactness, which was first used by Darbo, is one of the fruitful tools to obtain fixed point and common fixed point theorems. The extensions of these contractions in linear and integral type which are known as generalizations of Darbo's fixed point theorem, are considered by many authors; see, for example, [1–9] and the references therein.

Recently, Khodabakhshi [6] obtained some new common fixed point results with the technique associated with a measure of noncompactness for two commuting operators.

Inspired by the class of α - ψ contractive type mappings which was introduced by Samet *et al.* [10], Ansari [11] presented the weaker class of this contraction named $F(\psi, \varphi)$ -contraction and used it to obtain fixed point and common fixed point results.

This paper mainly aims at employing the $F(\psi, \varphi)$ -contraction and its property in terms of a measure of noncompactness to investigate a fixed point and a common fixed point for a pair of compatible mappings.

Now we present some definitions, notations and results which will be needed later. Throughout this paper we assume that E is an infinite dimensional Banach space. If C is a subset of E then the symbols $\overline{\text{co}}(C)$ and \mathfrak{M}_E and \mathfrak{N}_E denote the closure of convex hull of C and the family of nonempty bounded subsets of E and the subfamily consisting of all relatively compact subsets of E , respectively.

The measure of noncompactness was introduced by Kuratowski [12],

$$\mu(S) := \inf \left\{ \delta > 0 : S = \bigcup_{i=1}^n S_i \text{ for some } S_i \text{ with } \text{diam}(S_i) \leq \delta \text{ for } 1 \leq i \leq n < \infty \right\},$$

for a bounded subset S of a metric space X .

From now on we will use the following definition for the measure of noncompactness.

Definition 1.1 ([13]) A mapping $\mu : \mathfrak{M}_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

- (A1) $\emptyset \neq \text{Ker } \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\} \subseteq \mathfrak{N}_E$;
- (A2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$;
- (A3) $\mu(\overline{X}) = \mu(\text{co } X) = \mu(X)$;
- (A4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
- (A5) If $\{X_n\}$ is a sequence of closed sets from \mathfrak{M}_E such that $X_{n+1} \subseteq X_n$, ($n \geq 1$) and if $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ then the intersection set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

The family $\text{Ker } \mu$ described in (A1) is said to be the kernel of the measure of noncompactness μ . Observe that $X_\infty \in \text{Ker } \mu$, since $\mu(X_\infty) \leq \mu(X_n)$ for any n .

Definition 1.2 ([14]) A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

- (1) ψ is nondecreasing and continuous;
- (2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the class of altering distance functions.

Definition 1.3 ([11]) An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.

We denote by Φ the class of ultra altering distance functions.

Definition 1.4 ([11]) A mapping $F : [0, \infty)^2 \rightarrow \mathbb{R}$ is called a C -class function if it is continuous and satisfies the following axioms:

- (1) $F(s, t) \leq s$;
- (2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note for some F we have $F(0, 0) = 0$.

We denote C -class functions by \mathcal{C} .

Definition 1.5 ([15]) A pair of self-mappings F and G on X is weakly compatible if there exists a point $x \in X$ such that $F(x) = G(x)$ implies $FGx = GFx$ i.e., they commute at their coincidence point.

Proposition 1.6 ([16, Proposition 1.5]) *Let f and g be weakly compatible self-mappings of a set X . If f and g have a unique point of coincidence, $w = f(x) = g(x)$. Then w is the unique common fixed point of f and g .*

Lemma 1.7 ([17]) *Let X be a nonempty set and $f : X \rightarrow X$ be a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \rightarrow X$ is one to one.*

Now, we mention the following two theorems stated in [13, 18].

Theorem 1.8 (Schauder [18]) *Let C be a closed, convex subset of a Banach space E . Then every compact, continuous map $F : C \rightarrow C$ has at least one fixed point.*

As a significant generalization of Schauder’s fixed point theorem, we have the following fixed point theorem.

Theorem 1.9 (Darbo [13]) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(T(X)) \leq k\mu(X)$$

for any subset X of C . Then T has a fixed point.

2 Main results

This section starts by some of the theorems and corollaries related to fixed point are obtained by using $F(\psi, \varphi)$ -contraction in terms of a measure of noncompactness. Next, for a pair of compatible mappings a common fixed point theorem is considered. In the sequel, theorems are proved in integral type to obtain a fixed point and a common fixed point. Our results generalized Darbo’s fixed point theorem and a fixed point theorem which was recently proved.

Theorem 2.1 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping, such that*

$$\psi(\mu(T(M))) \leq F(\psi(\mu(M)), \varphi(\mu(M))),$$

for any subset M of C and where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then T has a fixed point.

Proof Define a sequence $\{C_n\}_{n=0}^\infty$ setting

$$C_0 := C, \quad C_n := \overline{\text{co}}T(C_{n-1}),$$

where $n = 1, 2, \dots$. Now let us prove that

$$C_{n+1} \subseteq C_n, \quad T(C_n) \subseteq C_n, \tag{1}$$

for every $n = 0, 1, \dots$. The first inclusion will be proved via mathematical induction. Let $n = 0$. Since $C_0 = C$, C is convex and closed, $T(\cdot) : C \rightarrow C$, we have $C_1 = \overline{\text{co}}(T(C_0)) \subseteq C_0$. Now assume that $C_n \subseteq C_{n-1}$. Then $\overline{\text{co}}(T(C_n)) \subseteq \overline{\text{co}}(T(C_{n-1}))$. So we obtain $C_{n+1} \subseteq C_n$. The second inclusion follows immediately from the first one, $T(C_n) \subseteq \overline{\text{co}}(T(C_n)) = C_{n+1} \subseteq C_n$.

If there exists $N \in \mathbb{N}$ such that $\mu(C_N) = 0$ then C_N is compact and Schauder’s fixed point theorem ensures that T has a fixed point in C_N where $C_N \subseteq C$. Suppose $\mu(C_n) > 0$ for each $n \in \mathbb{N}$.

Taking into consideration that $F \in \mathcal{C}$, the property (A3) from Definition (1.1) and using the inequality given in the theorem, we have

$$\begin{aligned} \psi(\mu(C_{n+1})) &= \psi(\mu(\overline{\text{co}}(T(C_n)))) \\ &= \psi(\mu(T(C_n))) \\ &\leq F(\psi(\mu(C_n)), \varphi(\mu(C_n))) \\ &\leq \psi(\mu(C_n)), \end{aligned} \tag{2}$$

for every $n = 1, 2, \dots$. From (1) and condition (A2) of Definition (1.1) we conclude that $\mu(C_{n+1}) \leq \mu(C_n)$ for every $n = 1, 2, \dots$. This means that the sequence $\{\mu(C_n)\}_{n=0}^\infty$ is not increasing, and consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r$. Since ψ is continuous, according to (2) we get

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r),$$

and hence

$$F(\psi(r), \varphi(r)) = \psi(r).$$

The last inequality and the inclusion $F \in \mathcal{C}$ yield $\psi(r) = 0$ or $\varphi(r) = 0$. These equalities and the inclusions $\psi \in \Psi, \varphi \in \Phi$ imply that $r = 0$. So, we obtain

$$\lim_{n \rightarrow \infty} \mu(C_n) = 0. \tag{3}$$

Let $C_\infty = \bigcap_{n=0}^\infty C_n$. Since $C_{n+1} \subset C_n, C_n$ is bounded, closed, and convex for every $n = 0, 1, \dots$, we see that C_∞ is also bounded, closed, and convex, so the equality (3) and property (A5) of Definition (1.1) imply that C_∞ is nonempty and compact. From the inclusion (1) it follows that

$$T(C_\infty) = T\left(\bigcap_{n=1}^\infty C_n\right) \subset \left[\bigcap_{n=1}^\infty T(C_n)\right] \subset \left[\bigcap_{n=1}^\infty C_n\right] = C_\infty.$$

Finally, by virtue of Schauder’s fixed point theorem we see that the map $T : C_\infty \rightarrow C_\infty$ has a fixed point in C_∞ . Since $C_\infty \subset C$, we conclude that the map T has a fixed point in C . The proof is completed. □

If we let $F(s, t) = ks$ in Theorem 2.1 we get the following result.

Corollary 2.2 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping. Assume that*

$$\psi(\mu(TM)) \leq k(\psi(\mu(M))),$$

for $M \subseteq C$. Then T has a fixed point.

If we let $\psi(t) = t$ in Theorem 2.1 we get the following.

Corollary 2.3 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping. Assume that*

$$\mu(TM) \leq F(\mu(M), \varphi(\mu(M))),$$

for $M \subseteq C$. Then T has a fixed point.

Theorem 2.4 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be continuous mapping. Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that the inequality*

$$\psi(\mu(T(C))) \leq F(\psi(M(X, Y)), \varphi(M(X, Y))),$$

is satisfied for every noncompact subset X, Y of C , where

$$M(X, Y) = \max\{\mu(X), \mu(T(X)), \mu(T(Y)), \mu(T(X) \cup T(Y))\}.$$

Then T has a fixed point.

Proof Define a sequence $\{C_n\}_{n=0}^\infty$ setting

$$C_0 := C, \quad C_n := \overline{\text{co}}(T(C_{n-1})),$$

where $n = 1, 2, \dots$. Analogously to Theorem 2.1 it is possible to show that

$$C_{n+1} \subseteq C_n, \quad T(C_n) \subseteq C_n, \tag{4}$$

for every $n = 0, 1, \dots$

If $\mu(C_N) = 0$ for some $N \in \mathbb{N}$ then C_N is a compact set and by virtue of Schauder’s theorem the continuous map $T : C_N \rightarrow C_N$ has a fixed point in $C_N \subset C$.

Now suppose that $\mu(C_n) > 0$ for every $n \in \mathbb{N}$. From (4) and condition (A2) of Definition 1.1 we conclude that $\mu(C_{n+1}) \leq \mu(C_n)$ for every $n = 1, 2, \dots$. This means that the sequence $\{\mu(C_n)\}_{n=0}^\infty$ is not increasing, and consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r$.

Taking into consideration that $F \in \mathcal{C}$, condition (A3), we have

$$\begin{aligned} \psi(\mu(C_{n+1})) &= \psi(\mu(\overline{\text{co}}(T(C_n)))) = \psi(T(C_n)) \\ &\leq F(\psi(M(C_n, C_{n+1})), \varphi(M(C_n, C_{n+1}))) \\ &\leq \psi(M(C_n, C_{n+1})), \end{aligned} \tag{5}$$

where

$$M(C_n, C_{n+1}) = \max\{\mu(C_n), \mu(T(C_n)), \mu(T(C_{n+1})), \mu(T(C_n) \cup T(C_{n+1}))\}.$$

Since $T(C_n) \subset C_n$ and $C_{n+1} \subset C_n$ we have

$$\begin{aligned} \mu(T(C_n)) &\leq \mu(C_n), \quad \mu(T(C_{n+1})) \leq \mu(T(C_n)), \\ \mu(T(C_n) \cup T(C_{n+1})) &\leq \mu(C_n \cup C_{n+1}) = \mu(C_n), \end{aligned}$$

for every $n \geq 1$. Thus we obtain

$$M(C_n, C_{n+1}) = \mu(C_n),$$

and consequently

$$\psi(M(C_n, C_{n+1})) = \psi(\mu(C_n)), \quad \varphi(M(C_n, C_{n+1})) = \varphi(\mu(C_n)),$$

for every $n \geq 1$. The last equalities and (5) yield

$$\psi(\mu(C_{n+1})) \leq F(\psi(M(C_n, C_{n+1})), \varphi(M(C_n, C_{n+1}))) \leq \psi(\mu(C_n)),$$

and hence

$$\psi(\mu(C_{n+1})) \leq F(\psi(\mu(C_n)), \varphi(\mu(C_n))) \leq \psi(\mu(C_n)),$$

for every $n \geq 1$. Since $\lim_{n \rightarrow \infty} \mu(C_n) = r$; ψ , φ and F are continuous functions, we get

$$F(\psi(r), \varphi(r)) = \psi(r).$$

The inclusion $F \in \mathcal{C}$ yields $\psi(r) = 0$ or $\varphi(r) = 0$. These equalities and inclusions $\psi \in \Psi$, $\varphi \in \Phi$ imply that $r = 0$. So, we obtain

$$\lim_{n \rightarrow \infty} \mu(C_n) = 0.$$

From now on the proof repeats the proof of Theorem 2.1. The theorem is proved. □

By taking $\psi(t) = t$ we have the following.

Corollary 2.5 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping. Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that the inequality*

$$\mu(T(C)) \leq F(\psi(M(X, Y)), \varphi(M(X, Y))),$$

is satisfied for every noncompact subset X, Y of C , where

$$M(X, Y) = \max\{\mu(X), \mu(T(X)), \mu(T(Y)), \mu(T(X) \cup T(Y))\}.$$

Then the map T has a fixed point.

Theorem 2.6 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T, S : C \rightarrow C$ be continuous mappings. Assume that:*

- (a). *The range of T contains the range of S .*
- (b). *For any $M \subset C$:*

$$\psi(\mu(T(M))) \leq F(\psi(\mu(S(M))), \varphi(\mu(S(M)))) \tag{6}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$.

Then:

- (i). The sets $A = \{x \in C : S(x) = x\}$ and $B = \{x \in C : T(x) = x\}$ are nonempty and closed.
- (ii). T and S have a coincidence point.
- (iii). If T and S are weakly compatible. Then S and T have a unique common fixed point.

Proof Let $C_0 = C$, choose $C_1 \subset E$ such that $S(C_0) \subseteq T(C_1)$ and $C_1 := \overline{\text{co}}S(C_0)$. This can be done since the range of T contains the range of S . We have

$$S(C_0) \subseteq \overline{\text{co}}S(C_0) \subseteq R_S \subseteq R_T,$$

so there exists C_1 such that $S(C_0) \subseteq T(C_1)$.

Continuing this process having chosen C_n in E we obtain C_{n+1} in E such that $S(C_n) \subseteq T(C_{n+1})$ and $C_{n+1} := \overline{\text{co}}SC_n$.

If we put $C_{n+1} := \overline{\text{co}}S(C_n)$, then

$$S(C_n) \subseteq C_{n+1} = \overline{\text{co}}S(C_n) \subseteq C_0 \quad \text{and} \quad T(C_{n+1}) \subseteq C_0, \quad \forall n \in \mathbb{N} \cup \{0\},$$

so

$$S(C_n) \cap T(C_{n+1}) \subseteq C_0,$$

therefore $S(C_n) \subseteq T(C_{n+1})$ for every $n \in \mathbb{N} \cup \{0\}$ because the cases

$$S(C_n) \cap T(C_{n+1}) = \emptyset \quad \text{and} \quad T(C_{n+1}) \subseteq S(C_n)$$

are impossible, since $R_S \subseteq R_T$.

We observe that $C_{n+1} \subseteq C_n$ and $SC_n \subseteq C_n$ for $n \in \mathbb{N} \cup \{0\}$, because

$$C_1 = \overline{\text{co}}(S(C_0)) \subseteq C_0.$$

Let $C_n \subseteq C_{n-1}$ so

$$C_{n+1} = \overline{\text{co}}(S(C_n)) \subseteq \overline{\text{co}}(S(C_{n-1})) = C_n.$$

And also

$$S(C_n) \subset S(C_{n-1}) \subset \overline{\text{co}}(S(C_{n-1})) = C_n.$$

If $\mu(C_N) = 0$, for some $N \in \mathbb{N}$, then T has a fixed point in C , because Schauder's fixed point theorem guarantees this. Suppose $\mu(C_n) > 0$ for each $n \in \mathbb{N}$. Therefore we get

$$\begin{aligned} \psi(\mu(C_{n+1})) &= \psi(\mu(\overline{\text{co}}(SC_n))) \\ &= \psi(\mu(S(C_n))) \\ &\leq \psi(\mu(T(C_{n+1}))) \\ &\leq F(\psi(\mu(S(C_{n+1}))), \varphi(\mu(S(C_{n+1})))) \end{aligned}$$

$$\begin{aligned} &\leq F(\psi(\mu(C_{n+1})), \varphi(\mu(C_{n+1}))) \\ &\leq \psi(\mu(C_{n+1})). \end{aligned} \tag{7}$$

Since $C_{n+1} \subset C_n$ for every $n = 1, 2, \dots$, the condition (A2) of Definition 1.1 implies that $\mu(C_{n+1}) \leq \mu(C_n)$ for every $n = 1, 2, \dots$. This means that the sequence $\{\mu(C_n)\}_{n=0}^\infty$ is not increasing, and consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r$. By (7) we find that

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r),$$

so

$$F(\psi(r), \varphi(r)) = \psi(r),$$

according to the property of F we have $\psi(r) = 0$ or $\varphi(r) = 0$. Hence

$$r = \lim_{n \rightarrow \infty} \mu(C_n) = 0.$$

Also since $C_{n+1} \subseteq C_n$ by property (A5) of Definition 1.1 $C_\infty = \bigcap_{n=1}^\infty C_n$ is nonempty and compact.

Moreover, since C_n and C are convex, and $S(C_n) \subset C_n, S : C_n \rightarrow C_n$ for $n = 0, 1, 2, \dots$ and so $S : C_\infty \rightarrow C_\infty$, now Schauder’s fixed point theorem ensures S has a fixed point and the set $A = \{x \in C : S(x) = x\}$ is nonempty and closed.

Similarly to S ; T has a fixed point and by continuity of $T, B = \{x \in C : T(x) = x\}$ is nonempty and closed. By Lemma 1.7, take

$$D := \{x \in C : S(x) = x \text{ or } T(x) = x\}$$

and define a map $g : S(D) \rightarrow S(D)$ by $g(Sx) = Tx$. Clearly g is well defined. Now if we put $X : S(D)$ and $E := B$ in Lemma 1.7, then $g(E) = g(X)$, so g is one to one. Now by using (6) we have

$$\psi(\mu(g(S(M)))) = \psi(\mu(T(M))) \leq F(\psi(\mu(S(M))), \varphi(\mu(S(M)))),$$

so according to Theorem 2.1 there exists $z \in E$ and it is unique, since g is one to one, such that $g(Sz) = Sz$, which implies $Tz = Sz$.

Hence T and S have a unique coincidence point thus from Proposition 1.6 and it follows that T and S have a unique common fixed point. □

Example 2.7 Let $C = [0, 2]$ be a subset of \mathbb{R} . Take $S, T : [0, 2] \rightarrow [0, 2]$ defined by

$$T(x) = \frac{1}{2}x \quad \text{and} \quad S(x) = \frac{1}{3}x,$$

also let

$$M = [1, 2], \quad \varphi(t) = \frac{t}{6}, \quad \psi(t) = t, \quad F(s, t) = \frac{s}{s+t},$$

and

$$\mu(X) = \text{diam}(X).$$

It is clear that $\psi \in \Psi$ and $\varphi \in \Phi$. To verify the hypotheses of Theorem 2.6:

(a). $\mathbf{R}_S = [0, \frac{2}{3}] \subset \mathbf{R}_T = [0, 1]$.

By taking $C_0 = [0, 2]$ and $C_1 = [0, \frac{4}{3}]$ we have

$$S([0, 2]) = \left[0, \frac{2}{3}\right] = T\left(\left[0, \frac{4}{3}\right]\right),$$

the algorithm of C_n follows by

$$C_3 = \left[0, \frac{16}{27}\right], \quad C_4 = \left[0, \frac{32}{81}\right], \quad C_5 = \left[0, \frac{64}{243}\right], \quad \dots,$$

such that $S(C_n) \subseteq T(C_{n+1})$.

(b). $\psi(\mu(T[1, 2])) = \frac{1}{2}$, $\psi(\mu(S[1, 2])) = \frac{1}{3}$, $\varphi(\mu(S[1, 2])) = \frac{1}{18}$, and also

$$F\left(\frac{1}{3}, \frac{1}{18}\right) = \frac{6}{7},$$

therefore

$$\psi\left(\mu(T(M)) = \frac{1}{2} \leq F(\psi(\mu(S(M))), \varphi(\mu(S(M))))\right) = \frac{6}{7}, \tag{8}$$

thus the sets $F = \{x \in C : S(x) = x\}$ and $K = \{x \in C : T(x) = x\}$ are nonempty and closed. Also $T(0) = S(0)$, so 0 is a coincidence point. Finally since T and S commute at 0, that is, $ST(0) = TS(0)$, so T and S are weakly compatible and 0 is a common fixed point of S and T .

Theorem 2.8 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping such that*

$$\int_0^{\varphi(\mu(T(M)))} f(t) dt \leq F\left(\int_0^{\varphi(\mu(M))} f(t) dt, \int_0^{\psi(\mu(M))} f(t) dt\right), \tag{9}$$

for any subset M of C and where $f : [0, \infty) \rightarrow [0, \infty)$ be a Lebesgue integrable function, which is summable on each compact of $[0, \infty)$ and $\int_0^\varepsilon f(t) dt > 0$ for each $\varepsilon > 0$ and $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then T has a fixed point.

Proof Define a sequence $\{C_n\}$ as follows:

$$C_0 := C, \quad C_n := \overline{\text{co}}TC_{n-1}, \quad \text{for } n = 1, 2, \dots$$

If $\mu(C_N) = 0$ for some $N \in \mathbb{N}$. Then T has a fixed point C by the proof of previous theorems. Suppose $\mu(C_n) > 0$ for all $n \in \mathbb{N}$. Since $C_{n+1} \subset C_n$ for every $n = 1, 2, \dots$, the condition (A2) of Definition 1.1 implies that $\mu(C_{n+1}) \leq \mu(C_n)$ $n = 1, 2, \dots$. This means that

the sequence $\{\mu(C_n)\}_{n=0}^\infty$ is not increasing, and consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r$. From the inclusions $\psi \in \Psi, \varphi \in \Phi$ we obtain $\lim_{n \rightarrow \infty} \psi(\mu(C_n)) = \psi(r)$ and $\lim_{n \rightarrow \infty} \varphi(\mu(C_n)) = \varphi(r)$. Since $f(\cdot)$ is Lebesgue integrable on each compact subset of $[0, \infty)$, we see that

$$\lim_{n \rightarrow \infty} \int_0^{\psi(\mu(C_n))} f(t) dt = \int_0^{\psi(r)} f(t) dt, \quad \lim_{n \rightarrow \infty} \int_0^{\varphi(\mu(C_n))} f(t) dt = \int_0^{\varphi(r)} f(t) dt.$$

The last equalities, inclusion $F \in \mathcal{C}$ and (9) imply that

$$\int_0^{\varphi(r)} f(t) dt \leq F\left(\int_0^{\varphi(r)} f(t) dt, \int_0^{\psi(r)} f(t) dt\right) \leq \int_0^{\varphi(r)} f(t) dt,$$

and hence

$$F\left(\int_0^{\varphi(r)} f(t) dt, \int_0^{\psi(r)} f(t) dt\right) = \int_0^{\varphi(r)} f(t) dt.$$

Thus we see that

$$\int_0^{\varphi(r)} f(t) dt = 0, \quad \text{or} \quad \int_0^{\psi(r)} f(t) dt = 0,$$

and consequently

$$\varphi(r) = 0 \quad \text{or} \quad \psi(r) = 0.$$

From the last equalities it follows that $r = 0$, i.e. $\lim_{n \rightarrow \infty} \mu(C_n) = r$. Also since $C_{n+1} \subseteq C_n$ by property (A5) of Definition 1.1 $C_\infty = \bigcap_{n=1}^\infty C_n$ is a nonempty, closed, and convex subset of C . Moreover, we know that C_∞ belongs to $\ker(\mu)$. So C_∞ is compact and invariant by the mapping T . Consequently, Schauder’s fixed point theorem implies that T has a fixed point in C_∞ . Since $C_\infty \subset C$ the proof is complete. \square

Remark 2.9 Put $f(t) = 1, \varphi(t) = t$ and $F(s, t) = ks$ for $t \in [0, \infty)$ in Theorem 2.8. Then

$$\begin{aligned} \mu(T(X)) &= \int_0^{\varphi(\mu(T(X)))} f(t) dt \\ &\leq F\left(\int_0^{\varphi(\mu(X))} f(t) dt, \int_0^{\psi(\mu(X))} f(t) dt\right) \\ &= k\left(\int_0^{\varphi(\mu(X))} f(t) dt\right) \\ &= k(\mu(X)), \end{aligned}$$

thus we get Darbo’s fixed point theorem.

Corollary 2.10 Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be a continuous mapping such that

$$\int_0^{\varphi(\|Tx - Ty\|)} f(t) dt \leq F\left(\int_0^{\varphi(\|x - y\|)} f(t) dt, \int_0^{\psi(\|x - y\|)} f(t) dt\right),$$

for any $x, y \in C$ where $f : [0, \infty) \rightarrow [0, \infty)$ be a Lebesgue integrable function which is summable on each compact set of $[0, \infty)$ and $\int_0^\varepsilon f(t) dt > 0$ for each $\varepsilon > 0$ and $\psi \in \Psi, \varphi \in \Phi$ and $F \in C$. Then T has a fixed point.

Proof Define $\mu : m_E \rightarrow \mathbb{R}_+$ with $\mu(X) = \text{diam}(X)$ for any $X \in m_E$, where $\text{diam}(X)$ is a diameter of the set X . It is easy to verify that μ is a measure of noncompactness on space E . By assumption, we have

$$\int_0^{\varphi(\sup_{x,y \in X} \|Tx - Ty\|)} f(t) dt \leq F \left(\int_0^{\varphi(\sup_{x,y \in X} \|x - y\|)} f(t) dt, \int_0^{\psi(\sup_{x,y \in X} \|x - y\|)} f(t) dt \right),$$

thus we get

$$\int_0^{\varphi(\mu(TX))} f(t) dt \leq F \left(\int_0^{\varphi(\mu(X))} f(t) dt, \int_0^{\psi(\mu(X))} f(t) dt \right),$$

so according to Theorem 2.8 we get the result. □

Corollary 2.11 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \rightarrow C$ be continuous mapping such that*

$$\int_0^{\varphi(\mu(T(X)))} f(t) dt \leq F \left(\int_0^{\varphi(\mu(X))} f(t) dt, \int_0^{\psi(\mu(X))} f(t) dt \right) - \psi^* \left(\int_0^{\psi(\mu(X))} f(t) dt \right),$$

for any subset X of C and where $f : [0, \infty) \rightarrow [0, \infty)$ be a Lebesgue integrable function, which is summable on each compact subset of $[0, \infty)$ and $\int_0^\varepsilon f(t) dt > 0$ for each $\varepsilon > 0$ and $\psi, \psi^* \in \Psi, \varphi \in \Phi$ and $F \in C$. Then T has a fixed point.

Proof Define a sequence $\{C_n\}_{n=0}^\infty$ setting

$$C_0 := C, \quad C_n := \overline{c\bar{o}}T(C_{n-1}),$$

where $n = 1, 2, \dots$. Analogously to Theorem 2.1 it is possible to show that

$$C_{n+1} \subseteq C_n, \quad T(C_n) \subseteq C_n, \tag{10}$$

for every $n = 0, 1, \dots$. From condition (A2) of Definition 1.1 we conclude that $\mu(C_{n+1}) \leq \mu(C_n)$ for every $n = 1, 2, \dots$. This means that the sequence $\{\mu(C_n)\}_{n=0}^\infty$ is not increasing, and consequently there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \mu(C_n) = r$.

If $\mu(C_N) = 0$ for some $N \in \mathbb{N}$ then C_N is a compact set and using Schauder's theorem we conclude that the continuous map $T : C_N \rightarrow C_N$ has a fixed point in $C_N \subset C$.

Now suppose that $\mu(C_N) > 0$ for every $n \in \mathbb{N}$. Taking into consideration that $F \in C, \psi^* \in \psi$ and condition (A3), we have

$$\begin{aligned} \int_0^{\varphi(\mu(C_{n+1}))} f(t) dt &= \int_0^{\varphi(\mu(\overline{c\bar{o}}(TC_n)))} f(t) dt \\ &= \int_0^{\varphi(\mu(TC_n))} f(t) dt \end{aligned}$$

$$\begin{aligned} &\leq F\left(\int_0^{\varphi(\mu(C_n))} f(t) dt, \int_0^{\psi(\mu(C_n))} f(t) dt\right) - \psi^*\left(\int_0^{\psi(\mu(C_n))} f(t) dt\right) \\ &\leq \int_0^{\varphi(\mu(C_n))} f(t) dt - \psi^*\left(\int_0^{\psi(\mu(C_n))} f(t) dt\right). \end{aligned} \tag{11}$$

Since $\lim_{n \rightarrow \infty} \mu(C_n) = r$, $F \in \mathcal{C}$, $\varphi \in \Phi$, $\psi^* \in \Psi$, the function $f(\cdot)$ is Lebesgue summable on the compact subset of $[0, \infty)$, and from (11) we obtain

$$\int_0^{\varphi(r)} f(t) dt \leq \int_0^{\varphi(r)} f(t) dt - \psi^*\left(\int_0^{\psi(r)} f(t) dt\right),$$

and hence

$$\psi^*\left(\int_0^{\psi(r)} f(t) dt\right) \leq 0.$$

From the last inequality it follows that $\psi(r) = 0$ and consequently $r = 0$, i.e. $\lim_{n \rightarrow \infty} \mu(C_n) = 0$.

Continuing the proof analogously to the proof of the theorem 2.1 we obtain the result. \square

Theorem 2.12 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T, S : C \rightarrow C$ be continuous mappings. Assume that:*

- (a). *The range of T contains the range of S .*
- (b). *For any $M \subseteq C$:*

$$\int_0^{\varphi(\mu(T(M)))} f(t) dt \leq F\left(\int_0^{\varphi(\mu(S(M)))} f(t) dt, \int_0^{\psi(\mu(S(M)))} f(t) dt\right), \tag{12}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$.

Then:

- (i). *The sets $A = \{x \in C : S(x) = x\}$ and $B = \{x \in C : T(x) = x\}$ are nonempty and closed.*
- (ii). *T and S have a coincidence point.*
- (iii). *If T and S are weakly compatible, then S and T have a common fixed point.*

Proof Let $C_0 = C$, $C_n := \overline{\text{co}}SC_{n-1}$ for $n = 1, 2, \dots$. As is pointed out in Theorem 2.6, C_{n+1} in E such that $T(C_{n+1}) = S(C_n)$. We have

$$T(C_n) \subseteq C_n \quad \text{and also} \quad S(C_n) \subset S(C_{n-1}) \subset \overline{\text{co}}(S(C_{n-1})) = C_n,$$

for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} \int_0^{\varphi(\mu(C_{n+1}))} f(t) dt &= \int_0^{\varphi(\mu(\overline{\text{co}}(S(C_n))))} f(t) dt \\ &= \int_0^{\varphi(\mu(S(C_n)))} f(t) dt \\ &= \int_0^{\varphi(\mu(T(C_{n+1})))} f(t) dt \\ &\leq F\left(\int_0^{\varphi(\mu(S(C_{n+1})))} f(t) dt, \int_0^{\psi(\mu(S(C_{n+1})))} f(t) dt\right) \end{aligned}$$

$$\begin{aligned}
 &\leq F\left(\int_0^{\varphi(\mu(C_{n+1}))} f(t) dt, \int_0^{\psi(\mu(C_{n+1}))} f(t) dt\right) \\
 &\leq \int_0^{\varphi(\mu(C_{n+1}))} f(t) dt \\
 &\leq \int_0^{\varphi(\mu(C_n))} f(t) dt.
 \end{aligned} \tag{13}$$

The remaining part is similar to the previous proofs. □

3 Application

In this section, since integral equations arise in different problems of theory and applications (see, e.g., [19–21] and the references therein). We consider the existence of solutions for the following nonlinear integral equation in $BC(\mathbb{R}^+)$, the space of bounded and continuous functions $x(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$:

$$x(t) = \lambda f\left(t, \int_0^t g(t, s, x(s)) ds\right) + (1 - \lambda) \int_0^t g(t, s, x(s)) ds, \quad t \geq 0, \lambda \in (0, 1) \tag{14}$$

for any nonempty bounded subset X of $BC(\mathbb{R}^+)$, $x \in X$ and $T > 0$ and $\epsilon > 0$, let

$$\begin{aligned}
 \omega^T(x, \epsilon) &= \sup\{|x(t) - x(s)| : s, t \in [0, T], |t - s| \leq \epsilon\}, \\
 \omega^T(X, \epsilon) &= \sup\{\omega^T(x, \epsilon) : x \in X\}, \quad \omega_0^T(X) = \lim_{\epsilon \rightarrow 0} \omega^T(X, \epsilon), \\
 \omega_0(X) &= \lim_{T \rightarrow \infty} \omega_0^T(X), \quad X(t) = \{x(t) : x \in X\}, \\
 \text{diam } X(t) &= \sup\{|x(t) - y(t)| : x, y \in X\},
 \end{aligned}$$

and

$$\mu(X) = \omega_0(X) + \limsup_{t \rightarrow \infty} \text{diam } X(t). \tag{15}$$

Banaś has shown in [13] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+)$.

Theorem 3.1 *The nonlinear integral equation (14) has at least one solution in the space $BC(\mathbb{R}^+)$, if the following conditions are satisfied:*

- (A₀). *The function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, $\alpha(t) \rightarrow \infty$ as $t \rightarrow \infty$.*
- (A₁). *The function $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and*

$$|f(t, x) - f(t, y)| \leq \psi(|x - y|),$$

moreover, ψ and φ are an altering distance function and an ultra altering distance function, respectively which ψ satisfies for all $t, s \in \mathbb{R}^+$, $\psi(t) + \psi(s) \leq \psi(s + t)$ and $\psi(t) < t$.

- (A₂). $L = \sup\{f(t, 0) : t \in \mathbb{R}^+\} < \infty$.

(A₃). The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a continuous function $b : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is increasing with respect to the first component, satisfying

$$|g(t, s, x)| \leq b(t, s),$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ where

$$\lim_{t \rightarrow \infty} \int_0^t b(t, s) ds = 0.$$

(A₄). $f(t, Kx(t)) = K(f(t, x(t)))$,

where

$$Kx(t) = \int_0^t g(t, s, x(s)) ds.$$

For the following remark, by definition, commuting mappings means for a pair of self-mappings $K, L : X \rightarrow X$ that there exists a point $x \in X$ such that $KL(x) = LK(x)$.

Remark 3.2 Note that by hypothesis (A₃) there exists constant $V > 0$ such that

$$V = \sup_{t \geq 0} v(t) = \sup_{t \geq 0} \left[\int_0^t b(t, s) ds \right].$$

Proof Put

$$Q = \{x \in BC(\mathbb{R}^+) : \|x\| \leq r = L + V\},$$

$$(Fx)(t) = f(t, x(t)), \quad (Kx)(t) = \int_0^t g(t, s, x(s)) ds.$$

Thus equation (14) becomes

$$x(t) = (Hx)(t) = \lambda FKx(\alpha(t)) + (1 - \lambda)(Kx)(t)$$

we define the operation $G : C(\mathbb{R}^+) \rightarrow C(\mathbb{R}^+)$ by

$$G(x) = \frac{Hx - (1 - \lambda)(Kx)(t)}{\lambda} = FKx(\alpha(t)),$$

where $C(\mathbb{R}^+)$ is the space of continuous functions on \mathbb{R}^+ .

Khodabakhshi and Vaezpour in [6] have shown that G, K are continuous on Q , bounded, commuting mappings, $R_G \subseteq R_K$ and also G, K have a common fixed point. The conditions (a) and (b) of Theorem 2.6 hold.

By referring to [6] we get

$$\mu(G(X)) \leq \psi(\mu(K(X))) \leq \mu(K(X)).$$

On the other hand

$$|g(t, s, x)| \leq b(t, s),$$

for all $t, s \in \mathbb{R}^+$. And for $T > 0$ such that $t_1, t_2 \in [0, T]$ and $x \in X$ we have

$$\begin{aligned} |Kx(t_2) - Kx(t_1)| &= \left| \int_0^{t_2} g(t_2, s, x(s)) ds - \int_0^{t_1} g(t_1, s, x(s)) ds \right| \\ &\leq \left| \int_0^{t_2} g(t_2, s, x(s)) ds \right| \\ &\leq \int_0^{t_2} b(t_2, s) ds \\ &\leq \int_0^T b(T, s) ds, \end{aligned} \tag{16}$$

so

$$\omega_0^T(KX) \leq \int_0^T b(T, s) ds; \tag{17}$$

by taking $T \rightarrow \infty$

$$\omega_0(KX) \leq \lim_{T \rightarrow \infty} \int_0^T b(T, s) ds = 0. \tag{18}$$

Also

$$\begin{aligned} |Kx(t) - Ky(t)| &= \left| \int_0^t g(t, s, x(s)) ds - \int_0^t g(t, s, y(s)) ds \right| \\ &\leq \left| \int_0^t g(t, s, x(s)) ds \right| \\ &\leq \int_0^t b(t, s) ds \\ &\leq \int_0^T b(T, s) ds, \end{aligned} \tag{19}$$

so we have

$$\limsup_{T \rightarrow \infty} \text{diam}(KX(T)) \leq \limsup_{T \rightarrow \infty} \int_0^T b(T, s) ds = 0; \tag{20}$$

by (18) and (20) we have

$$\mu(KX) = \mu\left(\omega_0(KX) + \limsup_{T \rightarrow \infty} \text{diam}(KX)\right) = \mu(0) + 0 = 0,$$

therefore

$$\psi(\mu(KX)) = 0,$$

thus for the \mathcal{C} -class function F and the ultra altering distance function φ , defined in Definitions 1.4 and 1.3, respectively, we can get

$$\psi(\mu(G(X))) \leq \psi(\mu(K(X))) = F(\psi(\mu(KX)), \varphi(\mu(KX))),$$

since K and G commutes, so they are weakly compatible; therefore according to Theorem 2.6, G and K have a common fixed point so H has a fixed point and thus the functional integral equation (14) has at least one solution. \square

4 Conclusion

In Theorem 3.2 of [6] the condition $TS = ST$ for two self-mappings T, S is used. But in this article for achieving a common fixed point from two self-maps the hypothesis of the common range which is weaker than the hypotheses of a commuting map is utilized.

In this article the measure of noncompactness is used instead of the metric d , which is used in [4] and [11]. Also, contractions associated with a measure of noncompactness in two linear and integral types and the application in solving integral equations are considered, while in the two mentioned article just the linear contractions in terms of the metric d is used for achieving a fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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