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# Probabilistic linear widths of Sobolev space with Jacobi weights on [–1, 1]

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**Abstract**

Optimal asymptotic orders of the probabilistic linear $(n,\delta)$-widths of $\lambda_{n,\delta}(W^r_{2,\alpha,\beta},\nu,L_{q,\alpha,\beta})$ of the weighted Sobolev space $W^r_{2,\alpha,\beta}$ equipped with a Gaussian measure $\nu$ are established, where $L_{q,\alpha,\beta}$, $1\le q\le\infty$, denotes the $L_q$ space on $[-1,1]$ with respect to the measure $(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta>-1/2$.

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## 1 Introduction

This paper mainly focuses on the study of probabilistic linear $(n,\delta)$-widths of a Sobolev space with Jacobi weights on the interval $[-1,1]$. This problem has been investigated only recently. For calculation of probabilistic linear $(n,\delta)$-widths of the Sobolev spaces equipped with Gaussian measure, we refer to [1–5]. Let us recall some definitions.

Let $K$ be a bounded subset of a normed linear space $X$ with the norm $\|\cdot\|_X$. The linear $n$-width of the set $K$ in $X$ is defined by

$$\lambda_n(K,X)=\inf_{L_n}\sup_{x\in K}\|x-L_nx\|_X,$$

where $L_n$ runs over all linear operators from $X$ to $X$ with rank at most $n$.

Let $W$ be equipped with a Borel field $\mathcal{B}$ which is the smallest $\sigma$-algebra containing all open subsets. Assume that $\nu$ is a probability measure defined on $\mathcal{B}$. Let $\delta\in[0,1)$. The probabilistic linear $(n,\delta)$-width is defined by

$$\lambda_{n,\delta}(W,\nu,X)=\inf_{G_\delta}\lambda_n(W\backslash G_\delta,X),$$

where $G_\delta$ runs through all possible $\nu$-measurable subsets of $W$ with measure $\nu(G_\delta)\le\delta$. Compared with the classical case analysis (see [2] or [6]), the probabilistic case analysis, which reflects the intrinsic structure of the class, can be understood as the $\nu$-distribution of the approximation on all subsets of $W$ by $n$-dimensional subspaces and linear operators with rank $n$.

In his recent paper [7], Wang has obtained the asymptotic orders of probabilistic linear $(n,\delta)$-widths of the weighted Sobolev space on the ball with a Gaussian measure in a weighted $L_q$ space. Motivated by Wang's work, this paper considers the probabilistic linear $(n,\delta)$-widths on the interval $[-1,1]$ with Jacobi weights and determines the asymptotic orders of the probabilistic linear $(n,\delta)$-widths. The difference between the work of Wang and ours lies in the different choices of the weighted points for the proofs of discretization theorems.

## 2 Main results

Consider the Jacobi weights

$$w_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1/2.$$

Denote by $L_{p,\alpha,\beta} \equiv L_p(w_{\alpha,\beta})$, $1 \le p < \infty$, the space of measurable functions defined on $[-1,1]$ with the finite norm

$$\|f\|_{p,\alpha,\beta} := \left(\int_{-1}^{1} \left|f(x)\right|^p w_{\alpha,\beta}(x)\,dx\right)^{1/p}, \quad 1 \le p < \infty,$$

and for $p=\infty$ we assume that $L_{\infty,\alpha,\beta}$ is replaced by the space $C[-1,1]$ of continuous functions on $[-1,1]$ with the uniform norm. Let $\Pi_n$ be the space of all polynomials of degree at most $n$. Denote by $\mathbb{P}_n$ the space of all polynomials of degree $n$ which are orthogonal to polynomials of low degree in $L_2(w_{\alpha,\beta})$. It is well known that the classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ form an orthogonal basis for $L_{2,\alpha,\beta} := L_2([-1,1], w_{\alpha,\beta})$ and are normalized by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ (see [8]). In particular,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)w_{\alpha,\beta}(x)\,dx = \delta_{n,m}h_n(\alpha,\beta),$$

where

$$h_n(\alpha,\beta) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \sim n^{-1}$$

with constants of equivalence depending only on $\alpha$ and $\beta$. Then the normalized Jacobi polynomials $P_n(x)$, defined by

$$P_n(x) = \left(h_n^{(\alpha,\beta)}\right)^{-1/2} P_n^{(\alpha,\beta)}(x), \quad n = 0,1,\ldots,$$

form an orthonormal basis for $L_{2,\alpha,\beta}$, where the inner product is defined by

$$\langle f,g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}w_{\alpha,\beta}(x)\,dx.$$

Denote by $S_n$ the orthogonal projector of $L_2(w_{\alpha,\beta})$ onto $\Pi_n$ in $L_2(w_{\alpha,\beta})$, which is called the Fourier partial summation operator. Consequently, for any $f \in L_2(W_{\alpha,\beta})$,

$$f = \sum_{l=0}^{\infty} \langle f,P_l\rangle P_l, \qquad S_nf := \sum_{l=0}^{n} \langle f,P_l\rangle P_l. \tag{2.1}$$

It is well known that (see Proposition 1.4.15 in [9]) $P_n^{(\alpha,\beta)}$ is just the eigenfunction corresponding to the eigenvalues $-n(n+\alpha+\beta+1)$ of the second-order differential operator

$$D_{\alpha,\beta} := \bigl(1-x^2\bigr)D^2 - \bigl(\alpha-\beta+(\alpha+\beta+2)x\bigr)D,$$

which means that

$$D_{\alpha,\beta}P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x).$$

Given $r>0$, we define the fractional power $(-D_{\alpha,\beta})^{r/2}$ of the operator $-D_{\alpha,\beta}$ on $f$ by

$$(-D_{\alpha,\beta})^{r/2}(f) = \sum_{k=0}^{\infty}\bigl(k(k+\alpha+\beta+1)\bigr)^{r/2}\langle f,P_k\rangle P_k,$$

in the sense of distribution. We call $f^{(r)} := (-D_{\alpha,\beta})^{r/2}$ the $r$th order derivative of the distribution $f$. It then follows that for $f\in L_{2,\alpha,\beta}$, $r\in R$, the Fourier series of the distribution $f^{(r)}$ is

$$f^{(r)} = \sum_{k=1}^{\infty}\bigl(k(k+\alpha+\beta+1)\bigr)^{r/2}\langle f,P_k\rangle P_k.$$

Using this operator, we define the weighted Sobolev class as follows: For $r>0$ and $1\le p\le\infty$,

$$W_{p,\alpha,\beta}^r\bigl([-1,1]\bigr) \equiv W_{p,\alpha,\beta}^r := \Bigl\{f\in L_{p,\alpha,\beta} : \|f\|_{W_{p,\alpha,\beta}^r} := \|f\|_{p,\alpha,\beta} + \bigl\|(-D_{\alpha,\beta})^{\frac{r}{2}}(f)\bigr\|_{p,\alpha,\beta} < \infty\Bigr\},$$

while the weighted Sobolev class $BW_{p,\alpha,\beta}^r$ is defined to be the unit ball of $W_{p,\alpha,\beta}^r$. When $p=2$, the norm $\|\cdot\|_{W_{2,\alpha,\beta}^r}$ is equivalent to the norm $\|\cdot\|_{\overline{W}_{2,\alpha,\beta}^r}$, and we can rewrite $W_{2,\alpha,\beta}^r$ as

$$\begin{aligned} W_{2,\alpha,\beta}^r &= \overline{W}_{2,\alpha,\beta}^r \\ &:= \Biggl\{f(x) = \sum_{l=0}^{\infty}\langle f,P_n\rangle P_n(x) : \|f\|_{\overline{W}_{2,\alpha,\beta}^r}^2 := \langle f,P_0\rangle^2 + \bigl\langle f^{(r)},f^{(r)}\bigr\rangle \\ &\qquad = \langle f,P_0\rangle^2 + \sum_{k=1}^{\infty}\bigl(k(k+\alpha+\beta+1)\bigr)^r\langle f,P_k\rangle^2 < \infty\Biggr\} \end{aligned}$$

with the inner product

$$\langle f,g\rangle_r := \langle f,P_0\rangle\langle g,P_0\rangle + \bigl\langle f^{(r)},g^{(r)}\bigr\rangle.$$

Obviously, $\overline{W}_{2,\alpha,\beta}^r$ is a Hilbert space. We equip $\overline{W}_{2,\alpha,\beta}^r = W_{2,\alpha,\beta}^r$ with a Gaussian measure $\nu$ whose mean is zero and whose correlation operator $C_\nu$ has eigenfunctions $P_l(x)$, $l=0,1,2,\ldots$, and eigenvalues

$$\lambda_0 = 1, \qquad \lambda_l = \bigl(l(l+\alpha+\beta+1)\bigr)^{-s/2}, \quad l=1,2,\ldots, s>1,$$

that is,

$$C_\nu P_0 = P_0, \qquad C_\nu P_l = \lambda_l P_l, \quad l = 1, 2, \ldots.$$

Then (see [10], pp.48-49),

$$\langle C_\nu f, g\rangle_r = \int_{\overline{W}^r_{2,\alpha,\beta}} \langle f, h\rangle_r \langle g, h\rangle_r \nu(dh).$$

By Theorem 2.3.1 of [10] the Cameron-Martin space $H(\nu)$ of the Gaussian measure $\nu$ is $\overline{W}^{r+s/2}_{2,\alpha,\beta}$, i.e.,

$$H(\nu) = \overline{W}^{r+s/2}_{2,\alpha,\beta}.$$

See [10] and [11] for more information about the Gaussian measure on Banach spaces.

Throughout the paper, $A(n,\delta) \asymp B(n,\delta)$ means $A(n,\delta) \ll B(n,\delta)$ and $A(n,\delta) \gg B(n,\delta)$, $A(n,\delta) \ll B(n,\delta)$ means that there exists a positive constant $c$ independent of $n$ and $\delta$ such that $A(n,\delta) \le cB(n,\delta)$. If $1 \le q \le \infty$, $r > (2 + 2\min\{0, \max\{\alpha,\beta\}\})(1/p - 1/q)_+$, the space $W^r_{p,\alpha,\beta}$ can be continuously embedded into the space $L_{q,\alpha,\beta}$ (see Lemma 2.3 in [12]).

Set $\rho = r + \frac{s}{2}$. The main result of this paper can be formulated as follows.

**Theorem 2.1** *Let* $1 \le q \le \infty$, $\delta \in (0, 1/2]$, *and let* $\rho > 1/2 + (2\max\{\alpha,\beta\} + 1)(1/2 + 1/q)_+$. *Then*

$$\lambda_{n,\delta}\big(W^r_{2,\alpha,\beta}, \nu, L_{q,\alpha,\beta}\big) \asymp \begin{cases} n^{1/2-\rho}(1 + n^{-\min\{1/2,1/q\}})(\ln(\frac{1}{\delta}))^{\frac{1}{2}}, & 1 \le q < \infty, \\ n^{1/2-\rho}(\ln(\frac{n}{\delta}))^{\frac{1}{2}}, & q = \infty. \end{cases} \tag{2.2}$$

For the proof of Theorem 2.1, the discretization technique is used (see [1, 4, 13, 14]). Since the known results of the probabilistic linear widths of the identity matrix on $\mathbb{R}^m$ are inappropriate here, the probabilistic linear widths of diagonal matrixes on $\mathbb{R}^m$ are adopted for the proof of the upper estimates.

## 3 Main lemmas

Let $\ell^m_q$ $(1 \le q \le \infty)$ denote the space $\mathbb{R}^m$ equipped with the $\ell^m_q$-norm defined by

$$\|x\|_{\ell^m_q} := \begin{cases} (\sum_{i=1}^m |x_i|^q)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \max_{1\le i\le m} |x_i|, & q = \infty. \end{cases}$$

We identify $\mathbb{R}^m$ with the space $\ell^m_2$, denote by $\langle x, y\rangle$ the Euclidean inner product of $x, y \in \mathbb{R}^m$, and write $\|\cdot\|_2$ instead of $\|\cdot\|_{\ell^m_2}$.

Consider in $\mathbb{R}^m$ the standard Gaussian measure $\gamma_m$, which is given by

$$\gamma_m(G) = (2\pi)^{-m/2} \int_G \exp^{\frac{-\|x\|^2}{2}}\, dx,$$

where $G$ is any Borel subset in $\mathbb{R}^m$. Let $1 \le q \le \infty$, $1 \le n < m$, and $\delta \in [0,1)$. The probabilistic linear $(n,\delta)$-width of a linear mapping $T:\mathbb{R}^m \to l_q^m$ is defined by

$$\lambda_{n,\delta}\big(T:\mathbb{R}^m \to l_q^m,\gamma_m\big) = \inf_{G_\delta}\inf_{T_n}\sup_{\mathbb{R}^m\setminus G_\delta}\|Tx - T_n x\|_{l_q^m},$$

where $G_\delta$ runs over all possible Borel subsets of $\mathbb{R}^m$ with measure $\gamma_m(G_\delta) \le \delta$, and $T_n$ runs over all linear operators from $\mathbb{R}^m$ to $l_q^m$ with rank at most $n$.

Throughout the paper, $D$ denotes the $m \times m$ real diagonal matrix $\operatorname{diag}(d_1,\dots,d_m)$ with $d_1 \ge d_2 \ge \cdots \ge d_m > 0$, $D_n$ denotes the $m\times m$ real diagonal matrix $\operatorname{diag}(d_1,\dots,d_n,0,\dots,0)$ with $1 \le n \le m$, and $I_m$ denotes the $m \times m$ identity matrix. Moreover, $\{e_1,\dots,e_m\}$ denotes the standard orthonormal basis in $\mathbb{R}^m$:

$$e_1 = (1,0,\dots,0), \qquad \dots, \qquad e_m = (0,\dots,0,1).$$

Now, we introduce several lemmas which will be used in the proof of Theorem 2.1.

**Lemma 3.1**

(1) (*See* [1]) *If* $1 \le q \le 2$, $m \ge 2n$, $\delta \in (0,1/2]$, *then*

$$\lambda_{n,\delta}\big(I_m:\mathbb{R}^m \to l_q^m,\gamma_m\big) \asymp m^{1/q} + m^{1/q-1/2}\sqrt{\ln(1/\delta)}. \tag{3.1}$$

(2) (*See* [4]) *If* $2 \le q < \infty$, $m \ge 2n$, $\delta \in (0,1/2]$, *then*

$$\lambda_{n,\delta}\big(I_m:\mathbb{R}^m \to l_q^m,\gamma_m\big) \asymp m^{1/q} + \sqrt{\ln(1/\delta)}. \tag{3.2}$$

(3) (*See* [5]) *If* $q = \infty$, $m \ge 2n$, $\delta \in (0,1/2]$, *then*

$$\lambda_{n,\delta}\big(I_m:\mathbb{R}^m \to l_q^m,\gamma_m\big) \asymp \sqrt{\ln\big((m-n)/\delta\big)} \asymp \sqrt{\ln m + \ln(1/\delta)}. \tag{3.3}$$

**Lemma 3.2** (See [7]) *Assume that*

$$\sum_{i=1}^m d_i^\beta \le C(m,\beta) \quad \textit{for some } \beta > 0.$$

*Then, for* $2 \le q \le \infty$, $m \ge 2n$, $\delta \in (0,1/2]$, *we have*

$$\lambda_{n,\delta}\big(D:\mathbb{R}^m \to l_q^m,\gamma_m\big) \ll \left(\frac{C(m,\beta)}{n+1}\right)^{\frac{1}{\beta}} \begin{cases} (m^{1/q} + \sqrt{\ln(1/\delta)}, & 2 \le q < \infty, \\ \sqrt{\ln m + \ln(1/\delta)}, & q = \infty. \end{cases} \tag{3.4}$$

Let $\xi_j = \cos\theta_j$, $1 \le j \le 2n$, denote the zeros of the Jacobi polynomial $P_{2n}^{(\alpha,\beta)}(t)$, ordered so that

$$0 =: \theta_0 < \theta_1 < \cdots < \theta_{2n} < \theta_{2n+1} := \pi.$$

Let $\lambda_{2n}(t)$ be the Christoffel function and $b_j = \lambda_{2n}(\xi_j)$. Denote

$$W(n;\xi_j) = \big(1 - x + n^{-2}\big)^{\alpha+\frac{1}{2}}\big(1 - x + n^{-2}\big)^{\beta+\frac{1}{2}}.$$

It is well known uniformly (see [15])

$$\theta_{j+1} - \theta_j \asymp n^{-1}, \qquad \theta_j \asymp jn^{-1} \quad (1 \le j \le 2n),$$

and also

$$b_j \asymp n^{-1} w_{\alpha,\beta}(\xi_j)\bigl(1 - \xi_j^2\bigr)^{1/2} \asymp n^{-1} W(n;\xi_j),$$

where the constants of equivalence depend only on $\alpha$, $\beta$ (see [16] or [17]).

The following lemma is well known as Gaussian quadrature formulae.

**Lemma 3.3** (See [8]) *For each $n \ge 1$, the quadrature*

$$\int_{-1}^{1} f(x) w_{\alpha,\beta}(x)\,dx \asymp \sum_{j=1}^{2n} b_j f(\xi_j) \tag{3.5}$$

*is exact for all polynomials of degree $4n - 1$. Moreover, for any $1 \le p \le \infty$, $f \in \Pi_n$, we have*

$$\|f\|_{p,\alpha,\beta} \asymp \left( \sum_{j=1}^{2n} b_j \bigl|f(\xi_j)\bigr|^p \right)^{1/p}. \tag{3.6}$$

An equivalence like (3.6) is generally called a Marcinkiewicz-Zygmund type inequality.

**Lemma 3.4** (See [12], Lemma 2.7) *Let $\alpha, \beta > -1/2$, $\sigma \in (0, \frac{1}{2\max\{\alpha,\beta\}+1})$ and let $b_j$, $1 \le j \le n$, be defined as in Lemma* 3.3. *Then*

$$\sum_{j=1}^{n} b_j^{-\sigma} \ll n^{1+\sigma}. \tag{3.7}$$

Let

$$L_n(x,y) := \sum_{j=0}^{\infty} \eta\left(\frac{j}{n}\right) P_j(x) P_j(y), \quad x, y \in [-1,1], \tag{3.8}$$

where $\eta \in C^\infty(R)$ is a nonnegative $C^\infty$-function on $[0,\infty)$ supported in $[0,2]$ with the properties that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) > 0$ for $t \in [0,2)$. For any $f \in L_{2,\alpha,\beta}$, we define

$$\delta_1(f) = S_2(f), \qquad \delta_k(f) = S_{2^k}(f) - S_{2^{k-1}}(f) \quad \text{for } k = 2, 3 \ldots, \tag{3.9}$$

where $S_n$ is given in (2.1). Denote by

$$M_k(x,y) = \sum_{l=2^{k-1}+1}^{2^k} P_l(x) P_l(y) \tag{3.10}$$

the reproducing kernel of the Hilbert space $L_{2,\alpha,\beta} \cap \bigoplus_{n=2^{k-1}+1}^{2^k} \mathbb{P}_n$. Then, for $x \in [0,1]$,

$$\delta_k(f)(x) = \sum_{l=2^{k-1}+1}^{2^k} \int_{-1}^{1} f(x) P_l(x) P_l(y) w_{\alpha,\beta}(y)\,dx = \bigl\langle f, M_k(\cdot,x)\bigr\rangle.$$

For $f \in \bigoplus_{n=2^{k-1}+1}^{2^k} \mathbb{P}_n$,

$$f(x) = \delta_k(f)(x) = \langle f, M_k(\cdot, x)\rangle.$$

By Lemma 3.3, there exists a sequence of positive numbers $w_i = b_i \asymp n^{-1} W_{\alpha,\beta}(n;\xi_i)$, $1 \le i \le 2^{k+1}$, for which the following quadrature formula holds for all $f \in \Pi_{2^{k+3}-1}$:

$$\int_{-1}^{1} f(t) W_{\alpha,\beta}(t)\,dt = \sum_{i=1}^{2^{k+1}} w_i f(\xi_i). \tag{3.11}$$

Moreover, for any $1 \le p \le \infty$, $f \in \Pi_{2^k}$, we have

$$\|f\|_{p,\alpha,\beta} \asymp \left(\sum_{i=1}^{2^{k+1}} w_i \left|f(\xi_i)\right|^p\right)^{1/p} = \left\|U_n(f)\right\|_{\ell_{p,w}^{2^{k+1}}},$$

where $w = (w_1, \dots, w_{2^{k+1}})$, $U_k : \Pi_{2^k} \longmapsto \mathbb{R}^{2^{k+1}}$ is defined by

$$U_k(f) = \left(f(\xi_1), \dots, f(\xi_{2^{k+1}})\right), \tag{3.12}$$

and for $x \in \mathbb{R}^{2^{k+1}}$,

$$\|x\|_{\ell_{p,w}^{2^{k+1}}} := \begin{cases} (\sum_{i=1}^{2^{k+1}} |x_i|^p w_i)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{1\le i\le 2^{k+1}} |x_i|, & p = \infty. \end{cases}$$

Let the operator $T_k : \mathbb{R}^{2^{k+1}} \longmapsto \Pi_{2^{k+1}}$ be defined by

$$T_k a(x) := \sum_{i=1}^{2^{k+1}} a_i w_i L_{2^{k+1}}(x, \xi_i), \tag{3.13}$$

where $a := (a_1, \dots, a_{2^{k+1}}) \in \mathbb{R}^{2^{k+1}}$. It is shown in [12] that for $1 \le q \le \infty$,

$$\|T_k a\|_{q,\alpha,\beta} \ll \|v\|_{\ell_{q,w}^{2^{k+1}}}. \tag{3.14}$$

For $f \in \Pi_{2^{k+1}}$, we have

$$f(x) = \int_{-1}^{1} f(y) L_{2^{k+1}}(x,y) w_{\alpha,\beta}(x,y)\,dy = \sum_{i=1}^{2^{k+1}} w_i f(\xi_i) L_{2^{k+1}}(x, \xi_i) = T_k U_k(f)(x).$$

In what follows, we use the letters $S_k$, $R_k$, $V_k$ to denote $u_k \times u_k$ real diagonal matrixes as follows:

$$\begin{aligned} S_k &= \operatorname{diag}\left(w_1^{\frac{1}{2}}, \dots, w_{2^{k+1}}^{\frac{1}{2}}\right), \\ R_k &= \operatorname{diag}\left(w_1^{\frac{1}{q}}, \dots, w_{2^{k+1}}^{\frac{1}{q}}\right), \\ V_k &= \operatorname{diag}\left(w_1^{-\frac{1}{2}+\frac{1}{q}}, \dots, w_{2^{k+1}}^{-\frac{1}{2}+\frac{1}{q}}\right), \end{aligned} \tag{3.15}$$

and use the letter $R_k^{-1}$ to represent the inverse matrix of $R_k$.

**Lemma 3.5** *For any $z = (z_1, \dots, z_{2^{k+1}}) \in \mathbb{R}^{2^{k+1}}$, we have*

$$\left\| \sum_{j=1}^{2^{k+1}} w_j^{\frac{1}{2}} z_j M_k(\cdot, \xi_j) \right\|_{2,\alpha,\beta} \ll \|z\|_{l_2^{2^{k+1}}}, \tag{3.16}$$

*where $M_k(x,y)$ is given in* (3.10), *and $(\xi_1, \dots, \xi_{2^{k+1}})$ is defined as above.*

*Proof* Denote by $K$ the set

$$\left\{ g \in \bigoplus_{j=2^{k-1}-1}^{2^k} \mathbb{P}_j : \|g\|_{2,\alpha,\beta} \le 1 \right\}.$$

Since

$$\sum_{j=1}^{2^{k+1}} w_j^{1/2} z_j M_k(\cdot, \xi_j) \in L_{2,\alpha,\beta} \cap \left( \bigoplus_{j=2^{k-1}-1}^{2^k} \mathbb{P}_j \right).$$

By the Riesz representation theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left\| \sum_{j=1}^{2^{k+1}} w_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_{2,\alpha,\beta} &= \sup_{g \in K} \left| \left\langle \sum_{j=1}^{2^{k+1}} w_j^{1/2} z_j M_k(\cdot, \xi_j), g \right\rangle \right| \\
&= \sup_{g \in K} \left| \sum_{j=1}^{2^{k+1}} w_j^{1/2} z_j g(\xi_j) \right| \\
&\le \sup_{g \in K} \left( \sum_{j=1}^{2^{k+1}} |z_j|^2 \right)^{1/2} \left( \sum_{j=1}^{2^{k+1}} w_j \left| g(\xi_j) \right|^2 \right)^{1/2} \\
&\ll \sup_{g \in K} \left( \sum_{j=1}^{2^{k+1}} |z_j|^2 \right)^{1/2} \|g\|_{2,\alpha,\beta} \\
&\le \|z\|_{l_2^{2^{k+1}}}.
\end{aligned}$$

□

## 4 Proofs of main results

Before Theorem 2.1 is proved, we establish the discretization theorems which give the reduction of the calculation of the probabilistic widths.

**Theorem 4.1** *Let $1 \le q \le \infty$, $\sigma \in (0,1)$, and let the sequences of numbers $\{n_k\}$ and $\{\sigma_k\}$ be such that $0 \le n_k \le 2^{k+1} =: m_k$, $\sum_{k=1}^{\infty} n_k \le n$, $\sigma_k \in (0,1)$, $\sum_{k=1}^{\infty} \sigma_k \le \sigma$. Then*

$$\lambda_{n,\sigma}\left( W_{2,\alpha,\beta}^r, \nu, L_{q,\alpha,\beta} \right) \le \sum_{k=1}^{\infty} 2^{-k\rho} \lambda_{n_k,\sigma_k}\left( V_k : \mathbb{R}^{m_k} \to l_q^{m_k}, \gamma_{m_k} \right). \tag{4.1}$$

*Proof* For convenience, we write

$$\lambda_{n_k,\sigma_k} := \lambda_{n_k,\sigma_k}\left( V_k : \mathbb{R}^{m_k} \to l_q^{m_k}, \gamma_{m_k} \right),$$

where $\gamma_{m_k}$ is the standard Gaussian measure in $\mathbb{R}^{m_k}$. Denote by $L_k$ a linear operator from $\mathbb{R}^{m_k}$ to $\mathbb{R}^{m_k}$ such that the rank of $L_k$ is at most $n_k$ and

$$\gamma_{m_k}\big(\big\{y\in\mathbb{R}^{m_k}\,|\,\|V_ky-L_ky\|>2\lambda_{n_k,\sigma_k}\big\}\big)\le\sigma_k.$$

Then, for any $f\in W^r_{2,\alpha,\beta}$, by (3.8)-(3.10), (3.14) and (3.15) we have

$$\begin{aligned}\big\|\delta_k(f)-T_kR_k^{-1}L_kS_kU_k\delta_k(f)\big\|_{q,\alpha,\beta} &= \big\|T_kU_k\delta_k(f)-T_kR_k^{-1}L_kS_kU_k\delta_k(f)\big\|_{q,\alpha,\beta}\\ &\le \big\|U_k\delta_k(f)-R_k^{-1}L_kS_kU_k\delta_k(f)\big\|_{l^{m_k}_{q,w}}\\ &= \big\|V_kS_kU_k\delta_k(f)-L_kS_kU_k\delta_k(f)\big\|_{l^{m_k}_q}. \end{aligned}\tag{4.2}$$

Let $y=S_kU_k\delta_k(f)=(w_1^{\frac12}\delta_k(f)(\xi_1),\dots,w_{m_k}^{\frac12}\delta_k(f)(\xi_{m_k}))\in\mathbb{R}_{m_k}$, for $x\in[-1,-1]$,

$$\delta_k(f)(x)=\langle f,M_k(\cdot,x)\rangle=\big\langle f^{(-r)},M_k^{(-r,0)}(\cdot,x)\big\rangle_r=\big\langle f,M_k^{(-2r,0)}(\cdot,x)\big\rangle_r,$$

where $M_k^{(r_1,0)}(x,y)$ is the $r_1$-order partial derivative of $M_k(x,y)$ with respect to the variable $x$, $r_1\in\mathbb{R}$. Since the random vector $f$ in $W^r_{2,\alpha,\beta}$ is a centered Gaussian random vector with a covariance operator $C_\nu$, the vector

$$y=S_kU_k\delta_k(f)=\big(\big\langle f,w_1^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_1)\big\rangle_r,\dots,w_{m_k}^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_{m_k})\rangle_r\big)$$

in $\mathbb{R}^{m_k}$ is a random vector with a centered Gaussian distribution $\gamma$ in $\mathbb{R}^{m_k}$, and its covariance matrix $C_\gamma$ is given by

$$C_\gamma=\Big(\big\langle C_\nu\big(w_i^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_i)\big),w_j^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_j)\big\rangle_r\Big)_{i,j=1}^{m_k}.$$

Since for any $z=(z_1,\dots,z_{m_k})\in\mathbb{R}^{m_k}$,

$$\sum_{j=1}^{m_k}w_j^{\frac12}z_jM_k(\cdot,\xi_j)\in\bigoplus_{j=2^{k-1}+1}^{2^k}\mathbb{P}_j,$$

and

$$\begin{aligned}\big\langle C_\nu\big(w_i^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_i)\big),w_j^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_j)\big\rangle_r &= \big\langle w_i^{\frac12}M_k^{(-2r-s,0)}(\cdot,\xi_i),w_j^{\frac12}M_k^{(-2r,0)}(\cdot,\xi_j)\big\rangle_r\\ &= \big\langle w_i^{\frac12}M_k^{(-\rho,0)}(\cdot,\xi_i),w_j^{\frac12}M_k^{(-\rho,0)}(\cdot,\xi_j)\big\rangle,\end{aligned}$$

by Lemma 3.5 we get

$$\begin{aligned}\int_{\mathbb{R}^{m_k}}(y,z)^2\gamma(dy) &= zC_\gamma z^T=\sum_{i,j=1}^{m_k}z_iz_j\big\langle w_i^{\frac12}M_k^{(-\rho,0)}(\cdot,\xi_i),w_j^{\frac12}M_k^{(-\rho,0)}(\cdot,\xi_j)\big\rangle\\ &= \left\langle\sum_{j=1}^{m_k}w_j^{\frac12}z_jM_k^{(-\rho,0)}(\cdot,\xi_j),\sum_{j=1}^{m_k}w_j^{\frac12}z_jM_k^{(-\rho,0)}(\cdot,\xi_j)\right\rangle\end{aligned}$$

$$= \left\| \sum_{j=1}^{m_k} w_j^{\frac{1}{2}} z_j M_k^{(-\rho,0)}(\cdot,\xi_j) \right\|_2^2 \asymp 2^{-2k\rho} \left\| \sum_{j=1}^{m_k} w_j^{\frac{1}{2}} z_j M_k(\cdot,\xi_j) \right\|_2^2$$
$$\ll 2^{-2k\rho} \|z\|_{l_2^{m_k}} = 2^{-2k\rho} \int_{\mathbb{R}^{m_k}} (y,z)^2 \gamma_{m_k}(dy). \tag{4.3}$$

Now we consider the subset of $W_{2,\alpha,\beta}^r$

$$G_k := \left\{ f \in W_{2,\alpha,\beta}^r \mid \left\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \right\|_{l_q^{m_k}} > 2c_1 c_2 2^{-k\rho} \lambda_{n_k,\sigma_k} \right\},$$

where $c_1$, $c_2$ are the positive constants given in (4.2), (4.3). Then by (4.2) we get

$$\begin{aligned} \nu(G_k) &\le \nu\left(\left\{ f \in W_{2,\alpha,\beta}^r \mid \left\| V_k S_k U_k \delta_k(f) - L_k S_k U_k \delta_k(f) \right\|_{l_q^{m_k}} > 2c_2 2^{-k\rho} \lambda_{n_k,\sigma_k} \right\}\right) \\ &= \gamma\left(\left\{ y \in \mathbb{R}^{m_k} \mid \| V_k y - L_k y \|_{l_q^{m_k}} > 2c_2 2^{-k\rho} \lambda_{n_k,\sigma_k} \right\}\right). \end{aligned}$$

Note that for any $t > 0$, the set $\{y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{l_q^{m_k}} \le t\}$ is convex symmetric. It then follows by Theorem 1.8.9 in [10] and (4.3), we have

$$\begin{aligned} \nu(G_k) &\le \gamma\left(\left\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{l_q^{m_k}} > 2c_2 2^{-k\rho} \lambda_{n_k,\sigma_k} \right\}\right) \\ &\le \lambda\left(\left\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{l_q^{m_k}} > 2c_2 2^{-k\rho} \lambda_{n_k,\sigma_k} \right\}\right) \\ &\le \gamma_{m_k}\left(\left\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{l_q^{m_k}} > 2\lambda_{n_k,\sigma_k} \right\}\right) \le \sigma_k, \end{aligned}$$

where $\lambda$ is a centered Gaussian measure in $\mathbb{R}^{m_k}$ with covariance matrix $c_2^2 2^{-2k\rho} I_{m_k}$. Consider $G = \bigcup_{k=1}^{\infty} G_k$ and the linear operator $\widetilde{T}_n$ on $W_{2,\alpha,\beta}^r$ which is given by

$$\widetilde{T}_n f = \sum_{k=1}^{\infty} T_k R_k^{-1} L_k S_k U_k \delta_k(f).$$

Then

$$\nu(G) = \nu\left( \bigcup_{k=1}^{\infty} G_k \right) \le \sum_{k=1}^{\infty} \nu(G_k) \le \sum_{k=1}^{\infty} \nu(\sigma_k) \le \sigma,$$

and

$$\begin{aligned} \operatorname{rank} \widetilde{T}_n &\le \sum_{k=1}^{\infty} \operatorname{rank}\left( T_k R_k^{-1} L_k S_k U_k \delta_k \right) \\ &\le \sum_{k=1}^{\infty} n_k \le n. \end{aligned}$$

Thus, according to the definitions of $G$, $\widetilde{T}_n$, and $L_k$, we obtain

$$\begin{aligned} \lambda_{n,\delta}\left( W_{2,\alpha,\beta}^r, \nu, L_{q,\alpha,\beta} \right) &= \sup_{f \in W_{2,\alpha,\beta}^r \setminus G} \| f - \widetilde{T}_n f \|_{q,\alpha,\beta} \\ &\le \sup_{f \in W_{2,\alpha,\beta}^r \setminus G} \sum_{k=1}^{\infty} \left\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \right\|_{q,\alpha,\beta} \end{aligned}$$

$$\le \sum_{k=1}^{\infty} \sup_{f\in W^r_{2,\alpha,\beta}\setminus G} \left\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \right\|_{q,\alpha,\beta}$$
$$\ll \sum_{k=1}^{\infty} 2^{-k\rho} \lambda_{n_k,\sigma_k},$$

which completes the proof of Theorem 4.1. □

Now we turn to the lower estimates. Assume that $m \ge 6$ and $b_1 m \le n \le 2b_1 m$ with $b_1 > 0$ being independent of $n$ and $m$. Set $\{x_j\}_{j=1}^N \subset \{x \in [-1,1] : |x| \le 2/3\}$ and $x_{j+1} - x_j = 3/m$, $j = 1,\dots,N-1$. Then $M \asymp N$ and

$$\{x \in [-1,1] : |x - x_j| \le 1/m\} \cap \{x \in [-1,1] : |x - x_i| \le 1/m\} = \emptyset, \quad \text{if } i \ne j.$$

We may take $b_1 > 0$ sufficiently large so that $N \ge 2n$. Let $\varphi^1$ be a $C^\infty$-function on $\mathbb{R}$ supported in $[-1,1]$, and be equal to 1 on $[-2/3,2/3]$. Let $\varphi^2$ be a nonnegative $C^\infty$-function on $\mathbb{R}$ supported in $[-1/2,1/2]$, and be equal to 1 on $[-1/4,1/4]$. Define

$$\varphi_i(x) = \varphi^1\bigl(m(x-x_i)\bigr) - c_i \varphi^2\bigl(m(x-x_i)\bigr),$$

for some $c_i$ such that $\int_{-1}^1 \varphi_i(x) W_{\alpha,\beta}(x)\,dx = 0$, $i = 1,\dots,N$. Set

$$A_N := \operatorname{span}\{\varphi_1,\dots,\varphi_N\} = \left\{ F_a(x) = \sum_{j=1}^N a_j \varphi_j(x) : a = (a_1,\dots,a_N) \in \mathbb{R}^N \right\}.$$

Clearly,

$$\varphi_j \in W^2_{2,\alpha,\beta}, \quad \operatorname{supp}\varphi_j \subset \{x \in [-1,1] : |x - x_j| \le 1/m\} \subset \{x \in [-1,1] : |x| \le 5/6\},$$
$$\|\varphi_j\|_{q,\alpha,\beta} \asymp \left( \int_{-2/3}^{2/3} \bigl|\varphi_j(x)\bigr|^q\,dx \right)^{1/q} = \left( \int_{-2/3}^{2/3} \bigl|\varphi^1\bigl(m(x-x_j)\bigr) - c_j\varphi^2\bigl(m(x-x_j)\bigr)\bigr|^q\,dx \right)^{1/q}$$
$$\asymp m^{-1/q}, \quad 1 \le q \le \infty, j = 1,\dots,N,$$

and

$$\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_i = \emptyset \quad (i \ne j).$$

It follows that for $F_a \in A_n$, $a = (a_1,\dots,a_N) \in \mathbb{R}^N$, we have

$$\|F_a\|_{q,\alpha,\beta} \asymp \left( m^{-1} \sum_{j=1}^N |a_j|^q \right)^{1/q} = m^{-1/q} \|a\|_{l_q^N}. \tag{4.4}$$

For a nonnegative integer $\nu = 0,1,\dots$, and $F_a \in A_N$, $a = (a_1,\dots,a_N) \in \mathbb{R}^N$, it follows from the definition of $-D_{\alpha,\beta}$ that

$$\operatorname{supp}(-D_{\alpha,\beta})^\nu(\varphi_j) \subset \{x \in [-1,1] : |x - x_j| \le 1/m\}$$

and

$$\big\|(D_{\alpha,\beta})^{\nu}(\varphi_j)\big\|_{q,\alpha,\beta} \le m^{2\nu-1/q}.$$

Hence, for $1 \le q \le \infty$ and $F_a = \sum_{j=1}^{N} a_j\varphi_j \in A_N$,

$$\big\|(-(D_{\alpha,\beta})^{\nu}(F_a)\big\|_{q,\alpha,\beta} \le m^{2\nu-1/q}\|a\|_{l_q^N}.$$

It then follows by the Kolmogorov type inequality (see Theorem 8.1 in [18]) that

$$\begin{aligned}\big\|F_a^{(\rho)}\big\|_{q,\alpha,\beta} &= \big\|(-D_{\alpha,\beta})^{\rho/2}(F_a)\big\|_{q,\alpha,\beta} \\ &\ll \big\|(-D_{\alpha,\beta})^{1+[\rho]}(F_a)\big\|_{q,\alpha,\beta}^{\frac{\rho}{2+2[\rho]}}\|F_a\|_{q,\alpha,\beta}^{1-\frac{\rho}{2+2[\rho]}} \\ &\ll m^{\rho-1/q}\|a\|_{l_q^N} \ll m^{\rho}\|F_a\|_{q,\alpha,\beta}. \end{aligned} \tag{4.5}$$

For $f \in L_{1,\alpha,\beta}$ and $x \in [-1,1]$, we define

$$P_N(f)(x) = \sum_{j=1}^{N} \frac{\varphi_j(x)}{\|\varphi_j\|_{2,\alpha,\beta}^2}\int_{-1}^{1} f(y)\varphi_j(y)W_{\alpha,\beta}(y)\,dy$$

and

$$Q_N(f)(x) = \sum_{j=1}^{N} \frac{\varphi_j(x)}{\|\varphi_j\|_{2,\alpha,\beta}^2}\int_{-1}^{1} f(y)\varphi_j^{(\rho)}(y)W_{\alpha,\beta}(y)\,dy.$$

Clearly, the operator $P_N$ is the orthogonal projector from $L_{2,\alpha,\beta}$ to $A_N$, and if $f \in W_{2,\alpha,\beta}^{\rho}$, then $Q_N(f)(x) = P_N(f^{\rho})(x)$. Also, using the method in [19], we can prove that $P_N$ is the bounded operator from $L_{q,\alpha,\beta}$ to $A_N \cap L_{q,\mu}$ for $1 \le q \le \infty$,

$$\big\|P_N(f)\big\|_{q,\alpha,\beta} \ll \|f\|_{q,\alpha,\beta}. \tag{4.6}$$

Since $Q_N(f) \in A_N$ for $f \in W_{2,\alpha,\beta}^{\rho}$, we have

$$\big\|Q_N(f)^{(\rho)}\big\|_{2,\alpha,\beta} \ll m^{\rho}\big\|Q_N(f))\big\|_{2,\alpha,\beta} = m^{\rho}\big\|P_N(f)^{(\rho)}\big\|_{2,\alpha,\beta} \ll m^{\rho}\big\|f^{(\rho)}\big\|_{2,\alpha,\beta}. \tag{4.7}$$

**Theorem 4.2** *Let $1 \le q \le \infty$, $\delta \in (0,1)$, and let $N$ be given above. Then*

$$\lambda_{n,\delta}\big(W_{2,\alpha,\beta}^{r}, \nu, L_{q,\alpha,\beta}\big) \gg n^{1/2-\rho-1/q}\lambda_{n,\delta}\big(I_N : \mathbb{R}^N \to l_q^N, \gamma_N\big),$$

*where $N \asymp n$, $N \ge 2n$ and $\gamma_N$ is the standard Gaussian measure in $\mathbb{R}^N$.*

*Proof* Let $T_n$ be a bounded linear operator on $W_{2,\alpha,\beta}^{r}$ with rank $T_n \le n$ such that

$$\nu\big(\big\{f \in W_{2,\alpha,\beta}^{r} : \|f - T_n f\|_{q,\alpha,\beta} > 2\lambda_{n,\delta}\big\}\big) \le \delta,$$

where $\lambda_{n,\delta} := \lambda_{n,\delta}(W^r_{2,\alpha,\beta}, \nu, L_{q,\alpha,\beta})$. Note that if $A$ is a bounded linear operator from $W^r_{2,\alpha,\beta}$ to $W^r_{2,\alpha,\beta}$ and from $H(\nu)$ to $H(\nu)$, then the image measure $\lambda$ of $\nu$ under $A$ is also a centered Gaussian measure on $W^r_{2,\alpha,\beta}$ with covariance

$$R_\lambda(f)(f) = \langle A^*C_\nu f, A^*C_\nu f\rangle_{H(\nu)}, \quad f \in W^r_{2,\alpha,\beta},$$

where $C_\nu$ is the covariance of the measure $\nu$, $H(\nu) = W^\rho_{2,\alpha,\beta}$ is the Camera-Martin space of $\nu$, and $A^*$ is the adjoint of $A$ in $H(\nu)$ (see Theorem 3.5.1 of [10]). Furthermore, if the operator $A$ also satisfies

$$\|Af\|_{H(\nu)} \le \|f\|_{H(\nu)},$$

then

$$R_\lambda(f)(f) = \|A^*C_\nu f\|^2_{H(\nu)} \le \|A^*\|^2\|C_\nu f\| \le \langle C_\nu f, C_\nu f\rangle_{H(\nu)} = R_\nu(f)(f).$$

By Theorem 3.3.6 in [10], we get that for any absolutely convex Borel set $E$ of $W^r_{2,\alpha,\beta}$ there holds the inequality

$$\nu(E) \le \lambda(E).$$

Applying (4.7) we assert that

$$\|Q_N(f)\|_{H(\nu)} = \|(Q_N(f))^{(\rho)}\|_{2,\alpha,\beta} \ll m^\rho\|f^{(\rho)}\|_{2,\alpha,\beta} = m^\rho\|f\|_{H(\nu)}.$$

Then there exists a positive constant $c_3$ such that

$$\left\|\frac{1}{c_3 m^\rho}Q_N(f)\right\|_{H(\nu)} \le \|f\|_{H(\nu)}.$$

Note that, for any $t > 0$, the set $\{f \in W^r_{2,\alpha,\beta} : \|f - T_nf\|_{q,\alpha,\beta} \le t\}$ is absolutely convex. It then follows that

$$\nu(\{f \in W^r_{2,\alpha,\beta} : \|f - T_nf\|_{q,\alpha,\beta} < 2\lambda_{n,\delta}\}) \le \lambda(\{f \in W^r_{2,\alpha,\beta} : \|f - T_nf\|_{q,\alpha,\beta} < 2\lambda_{n,\delta}\}),$$

which leads to

$$\begin{aligned}&\nu(\{f \in W^r_{2,\alpha,\beta} : \|f - T_nf\|_{q,\alpha,\beta} > 2\lambda_{n,\delta}\})\\&\quad\ge \nu(\{f \in W^r_{2,\alpha,\beta} : \|Q_Nf - T_nQ_Nf\|_{q,\alpha,\beta} > 2c_3m^\rho\lambda_{n,\delta}\}).\end{aligned}$$

Let $L_N : \mathbb{R}^N \to A_N$ and $J_N : A_N \to \mathbb{R}^N$ be defined by

$$L_N(a)(x) = \sum_{i=1}^N \frac{a_i\varphi_i(x)}{\|\varphi_i\|_{2,\alpha,\beta}}, \quad a = (a_1, \dots, a_N) \in \mathbb{R}^N$$

and

$$J_N(F_a) = (a_1\|\varphi_1\|_{2,\alpha,\beta}, \dots, a_N\|\varphi_N\|_{2,\alpha,\beta}), \quad F_a \in A_N.$$

We see at once that $L_N J_N(F_a) = F_a$ for any $F_a \in A_N$. Set $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, where $y_j = \frac{1}{\|\varphi_j\|_{2,\alpha,\beta}} \langle f, \varphi_j^{(\rho)} \rangle$. Then $y = J_N Q_N(f)$. Thus by (4.4) and $\|\varphi_j\|_{2,\alpha,\beta} \asymp m^{-\frac{1}{2}}$, we obtain

$$\left\| L_N(a) \right\|_{q,\alpha,\beta} \asymp m^{-\frac{1}{q}+\frac{1}{2}} \|a\|_{l_q^N}. \tag{4.8}$$

Combining (4.6) with (4.8), we conclude that for any $f \in W_{2,\alpha,\beta}^r$,

$$\begin{aligned}
\left\| Q_N(f) - T_N Q_N(f) \right\|_{q,\alpha,\beta} &\gg \left\| P_N\big(Q_N(f)\big) - P_N T_n Q_N(f) Q \right\|_{q,\alpha,\beta} \\
&= \left\| L_N J_N Q_N(f) - L_N J_N P_N T_N L_N J_N Q_N(f) \right\|_{q,\alpha,\beta} \\
&\gg m^{-\frac{1}{q}+\frac{1}{2}} \left\| J_N Q_N(f) - J_N P_N T_n L_N J_N Q_N(f) \right\|_{l_q^N} \\
&\gg m^{-\frac{1}{q}+\frac{1}{2}} \|y - J_N P_N T_n L_N y\|_{l_q^N}.
\end{aligned}$$

Remark that $g_k = \frac{\varphi_k}{\|\varphi_k\|_{2,\alpha,\beta}}$, $k = 1, 2, \dots, N$, is an orthonormal system in $L_{2,\alpha,\beta}$ and $g_k \in H(\nu) = W_{2,\alpha,\beta}^\rho$. Then the random vector $(\langle f, g_1^{(\rho)} \rangle, \dots, \langle f, g_N^{(\rho)} \rangle) = y$ in $\mathbb{R}^N$ on the measurable space $(W_{2,\alpha,\beta}^r, \nu)$ has the standard Gaussian distribution $r_N$ in $\mathbb{R}^N$. It then follows that

$$\begin{aligned}
&\nu\big(\big\{f \in W_{2,\alpha,\beta}^r : \left\| Q_N(f) - T_n Q_N(f) \right\|_{q,\alpha,\beta} > 2c_3 m^\rho \lambda_{n,\delta}\big\}\big) \\
&\quad \ge \nu\big(\big\{f \in W_{2,\alpha,\beta}^r : \|y - T_J N P_N T_n L_N y\|_{l_q^N} > c_4 m^{\rho+\frac{1}{q}-\frac{1}{2}} \lambda_{n,\delta}\big\}\big) \\
&\quad = r_N\big(\big\{y \in \mathbb{R}^N : \|y - T_J N P_N T_n L_N y\|_{l_q^N} > c_4 m^{\rho+\frac{1}{q}-\frac{1}{2}} \lambda_{n,\delta}\big\}\big) \\
&\quad =: r_N(G),
\end{aligned}$$

where $c_4$ is a positive constant. Clearly, $\operatorname{rank}(J_N P_N T_n L_N) \le n$ and

$$r_N(G) \le \nu\big(\big\{f \in W_{2,\alpha,\beta}^r : \|f - T_n f\|_{q,\alpha,\beta} > 2\lambda_{n,\delta}\big\}\big) \le \delta.$$

Consequently,

$$\begin{aligned}
\lambda_{n,\delta}\big(I_N : \mathbb{R}^N \to l_q^N, r_N\big) &= \inf_G \inf_{I_N} \sup_{x \in \mathbb{R}^N \setminus G} \|I_N x - T_n x\|_{l_q^N} \\
&\le \sup_{y \in \mathbb{R}^N \setminus G} \|I_N y - J_N P_N T_n L_N y\|_{l_q^N} \\
&\ll m^{\rho+\frac{1}{q}-\frac{1}{2}} \lambda_{n,\delta},
\end{aligned}$$

which implies

$$\begin{aligned}
\lambda_{n,\delta}\big(W_{2,\alpha,\beta}^r, \nu, L_{q,\alpha,\beta}\big) &\ll m^{-\rho-\frac{1}{q}+\frac{1}{2}} \lambda_{n,\delta}\big(I_N : \mathbb{R}^N \to l_q^N, r_N\big) \\
&\asymp n^{-\rho-\frac{1}{q}+\frac{1}{2}} \lambda_{n,\delta}\big(I_N : \mathbb{R}^N \to l_q^N, r_N\big).
\end{aligned}$$

This completes the proof of Theorem 4.2. □

Now, we are in a position to prove Theorem 2.1.

*Proof* For the lower estimates, using Theorem 4.2 and Lemma 3.1, we have for $1 \leq q \leq 2$

$$\begin{aligned}\lambda_{n,\delta}\left(W^r_{2,\alpha,\beta}, \nu, L_{q,\alpha,\beta}\right) &\gg n^{-\rho+1/2-1/q}\lambda_{n,\delta}\left(I_N : \mathbb{R}^N \to l^N_q, \gamma_N\right) \\ &\asymp n^{-\rho+1/2-1/q}\left(N^{1/q} + N^{1/q-1/2}\left(\ln\left(\frac{1}{\delta}\right)\right)^{1/2}\right) \\ &\asymp n^{1/2-\rho}\left(1 + n^{-1/2}\left(\ln\left(\frac{1}{\delta}\right)\right)^{1/2}\right).\end{aligned}$$

For $2 \leq q < \infty$, we have

$$\begin{aligned}\lambda_{n,\delta}\left(W^r_{2,\alpha,\beta}, \nu, L_{q,\alpha,\beta}\right) &\gg n^{-\rho+1/2-1/q}\left(n^{1/q} + \left(\ln\left(\frac{1}{\delta}\right)\right)^{1/2}\right) \\ &\asymp n^{1/2-\rho}\left(1 + n^{-1/q}\left(\ln\left(\frac{1}{\delta}\right)\right)^{1/2}\right).\end{aligned}$$

And for $q = \infty$,

$$\begin{aligned}\lambda_{n,\delta}\left(W^r_{2,\alpha,\beta}, \nu, L_{q,\alpha,\beta}\right) &\gg n^{-\rho+1/2-1/q}\left(\ln m + \ln\left(\frac{1}{\delta}\right)\right)^{1/2} \\ &= n^{1/2-\rho}\left(\ln\left(\frac{m}{\delta}\right)\right)^{1/2}.\end{aligned}$$

It remains to prove the upper estimates. For $2 \leq q \leq \infty$ and any fixed natural number $n$, assume $C_1 2^m \leq n \leq C_1^2 2^m$ with $C_1 > 0$ to be specified later. We may take sufficiently small positive numbers $\varepsilon > 0$ such that $\rho > \frac{1}{2} + (1+\varepsilon)(2\max\{\alpha,\beta\} + 1 + \varepsilon)(\frac{1}{2} - \frac{1}{q})$. Set

$$n_j = \begin{cases} 2^{j+1}, & \text{if } j \leq m, \\ 2^{j+1}2^{(1+\varepsilon)(m-j)-1}, & \text{if } j > m, \end{cases}$$

and

$$\delta_j = \begin{cases} 0, & \text{if } j \leq m, \\ \delta 2^{m-j}, & \text{if } j > m. \end{cases}$$

Then

$$\sum_{j\geq 0} n_j \ll \sum_{j\leq m} 2^j + \sum_{j>m} 2^{m(1+\varepsilon)-\varepsilon j} \ll 2^m$$

and

$$\sum_{j\geq 0} \delta_j = \delta \sum_{j\leq m} 2^{m-j} \leq \delta.$$

Thus, we can take $C_1$ sufficiently large so that

$$\sum_{j=0}^{\infty} n_j \leq C_1 2^m \leq n.$$

It follows from Lemma 3.4 for $\tau \in (0, \frac{1}{(2\max\{\alpha,\beta\}+1)(1/2-1/q)})$, $2 \le q \le \infty$,

$$\sum_{j=1}^{n} b_j^{-\tau(1/2-1/q)} \ll 2^{k[1+\tau(1/2-1/q)]} = 2^{k+k\tau(1/2-1/q)}.$$

If $j \le m$, then $n_j = 2^{j+1}$, and thence $\lambda_{n_j,\delta_j}(V_j : \mathbb{R}^{2^{j+1}} \to l_q^{2^{j+1}}, \gamma_{2^{j+1}}) = 0$. If $j > m$, then taking $\frac{1}{\tau} = (2\max\{\alpha,\beta\} + 1 + \varepsilon)(1/2 - 1/q)$ and applying Lemma 3.2, Theorem 4.1, we obtain for $2 \le q < \infty$,

$$\begin{aligned}
&\lambda_{n_j,\delta_j}\big(V_j : \mathbb{R}^{2^{j+1}} \to l_q^{2^{j+1}}, \gamma_{2^{j+1}}\big) \\
&\quad \ll \left(\frac{C(m,\tau)}{n_j+1}\right)^{1/\tau} \left(2^{(j+1)/q} + \sqrt{\ln\frac{1}{\delta}}\right) \\
&\quad \ll 2^{j(1/2-1/q)} 2^{-(1+\varepsilon)(m-j)(2\max\{\alpha,\beta\}+1+\varepsilon)(\frac{1}{2}-\frac{1}{q})} \left(2^{\frac{j}{q}} + \sqrt{\ln\frac{1}{\delta}}\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\lambda_{n,\delta}\big(W_{2,\alpha,\beta}^r, \nu, L_{q,\alpha,\beta}\big) \\
&\quad \ll \sum_{j=m+1}^{\infty} 2^{-j\rho} 2^{j(1/2-1/q)} 2^{-(1+\varepsilon)(m-j)(2\max\{\alpha,\beta\}+1+\varepsilon)(\frac{1}{2}-\frac{1}{q})} 2^{1/2-1/q} \left(2^{\frac{j}{q}} + \sqrt{\ln\frac{1}{\delta}}\right) \\
&\quad \ll 2^{-m(\rho-\frac{1}{2}+\frac{1}{q})} \left(2^{\frac{m}{q}} + \sqrt{\ln\frac{1}{\delta}}\right) \asymp n^{1/2-\rho}\left(1 + n^{-1/q}\sqrt{\ln\frac{1}{\delta}}\right).
\end{aligned} \tag{4.9}$$

For $q = \infty$, by Lemma 3.2 we get

$$\begin{aligned}
\lambda_{n_j,\delta_j}\big(V_j : \mathbb{R}^{2^{j+1}} \to l_q^{2^{j+1}}, \gamma_{2^{j+1}}\big) &\ll \left(\frac{C(2^{j+1},\tau)}{n_j+1}\right)^{1/\tau} \sqrt{\ln 2^{j+1} + \ln\frac{1}{\delta}} \\
&= 2^{j/2-(1+\varepsilon)(m-j)(2\max\{\alpha,\beta\}+1+\varepsilon)/2} \sqrt{j + \ln\frac{1}{\delta}},
\end{aligned}$$

then applying Theorem 4.1, we obtain

$$\begin{aligned}
\lambda_{n,\delta}\big(W_{2,\alpha,\beta}^r, \nu, L_{\infty,\alpha,\beta}\big) &\ll \sum_{j=m+1}^{\infty} 2^{-j\rho} 2^{j/2-(1+\varepsilon)(m-j)(2\max\{\alpha,\beta\}+1+\varepsilon)/2} \sqrt{j + \ln\frac{1}{\delta}} \\
&\ll 2^{-m(\rho-\frac{1}{2})} \sqrt{m + \ln\frac{1}{\delta}} \asymp n^{1/2-\rho}\sqrt{\ln\frac{n}{\delta}}.
\end{aligned} \tag{4.10}$$

To finish the proof of the upper estimates, we only need to show that, for $1 \le q < 2$,

$$\lambda_{n,\delta}\big(W_{2,\alpha,\beta}^r, \nu, L_{q,\alpha,\beta}\big) \ll \lambda_{n,\delta}\big(W_{2,\alpha,\beta}^r, \nu, L_{2,\alpha,\beta}\big) \ll n^{1/2-\rho}\left(1 + n^{-1}\sqrt{\ln\frac{1}{\delta}}\right)^{1/2}.$$

Theorem 2.1 is proved. □

## 5 Conclusions

In this paper, optimal estimates of the probabilistic linear $(n,\delta)$-widths of the weighted Sobolev space $W^r_{2,\alpha,\beta}$ on $[-1,1]$ are established. This kind of estimates play an important role in the widths theory and have a wide range of applications in the approximation theory of functions, numerical solutions of differential and integral equations, and statistical estimates.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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