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Some properties and applications of the Teodorescu operator associated to the Helmholtz equation

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Abstract

In this paper, we first define the Teodorescu operator $T_{\psi,\alpha}$ related to the Helmholtz equation and discuss its properties in quaternion analysis. Then we propose the Riemann boundary value problem related to the Helmholtz equation. Finally we give the integral representation of the boundary value problem by using the previously defined operator.

Keywords: quaternion analysis; Teodorescu operator; Helmholtz equation; Riemann boundary value problem; integral representation

1 Introduction

It is well known that the Helmholtz equation is an elliptic partial differential equation describing the electromagnetic wave, which has important applications in geophysics, medicine, engineering application, and many other fields. Many problems associated with the Helmholtz equation have been studied by many scholars, for example [1–5]. The boundary value problem for partial differential equations is an important and meaningful research topic. The singular integral operator is the core component of the solution of the boundary value problem for a partial differential system. The Teodorescu operator is a generalized solution of the inhomogeneous Dirac equation, which plays an important role in the integral representation of the general solution for the boundary value problem. Many experts and scholars have studied the properties of the Teodorescu operator. For example, Vekua [6] first discussed some properties of the Teodorescu operator on the plane and its application in thin shell theory and gas dynamics. Hile [7] and Gilbert [8] studied some properties of the Teodorescu operator in n-dimensional Euclid space and high complex space, respectively. Yang [9] and Gu [10] studied the boundary value problem associated with the Teodorescu operator in quaternion analysis and Clifford analysis, respectively. Wang [11–15] studied the properties of many Teodorescu operators and related boundary value problems.

In this paper, we will study some properties of the singular integral operator and the Riemann boundary value problem associated to the Helmholtz equation using the quaternion analysis method. The structure of this paper is as follows: in Section 2, we review some basic knowledge of quaternion analysis. In Section 3, we first construct a singular integral operator $T_{\psi,\alpha}$ related to the Helmholtz equation and study some of its properties.

In Section 4, we propose the Riemann boundary value problem related to the Helmholtz equation. Finally we give the integral representation of the boundary value problem by using the previously defined operator.

2 Preliminaries

Let $\{i_1, i_2, i_3\}$ be an orthogonal basis of the Euclidean space R^3 and $\mathbb{H}(\mathbb{C})$ be the set of complex quaternions with basis

$$\{i_0, i_1, i_2, i_3\},$$

where i_0 is the unit and i_1, i_2, i_3 are the quaternionic imaginary units with the following properties:

$$i_0^2 = -i_k^2 = i_0, \qquad i_0 i_k = i_k i_0 = i_k, \quad k = 1, 2, 3,$$
$$i_1 i_2 = -i_2 i_1 = i_3, \qquad i_2 i_3 = -i_3 i_2 = i_1, \qquad i_3 i_1 = -i_1 i_3 = i_2.$$

Then an arbitrary quaternion a can be written as $a = \sum_{k=0}^{3} a_k i_k$, $a_k \in \mathbb{C}$. The quaternionic conjugation is defined by $\bar{a} = a_0 - \sum_{k=1}^{3} a_k \cdot i_k$. The norm for an element $a \in \mathbb{H}(\mathbb{C})$ is taken to be $|a| = \sqrt{\sum_{k=0}^{3} |a_k|^2}$. Moreover, if $a\bar{a} = \bar{a}a = |a|^2$ and $|a| \neq 0$, then we say that a is reversible. Obviously, its inverse element can be written as $a^{-1} = \frac{\bar{a}}{|a|^2}$.

Let $\lambda \in \mathbb{C}\backslash\{0\}$ and let α be its complex square root: $\alpha \in \mathbb{C}$, $\alpha^2 = \lambda$. Suppose $\Omega \subset R^3$ is a domain and $\partial\Omega$ is its boundary. We shall consider functions f defined in $\Omega \subset R^3$ with values in $\mathbb{H}(\mathbb{C})$. Then f can be expressed as $f = \sum_{k=0}^{3} f_k(x) i_k$. Here $f_k(x)$ $(k = 0, 1, 2, 3)$ are complex functions defined on Ω.

Let $C^{(m)}(\Omega, \mathbb{H}(\mathbb{C})) = \{f \mid f : \Omega \to \mathbb{H}(\mathbb{C}), f(x) = \sum_{k=0}^{3} f_k(x) i_k, f_k(x) \in C^m(\Omega, \mathbb{C})\}$. We define the operators as follows:

$$^{\psi}D[f] = \sum_{k=1}^{3} \psi_k \cdot \frac{\partial f}{\partial x_k}, \qquad ^{\psi}\overline{D}[f] = \sum_{k=1}^{3} \overline{\psi}_k \cdot \frac{\partial f}{\partial x_k},$$
$$D^{\psi}[f] = \sum_{k=1}^{3} \frac{\partial f}{\partial x_k} \cdot \psi_k, \qquad \overline{D}^{\psi}[f] = \sum_{k=1}^{3} \frac{\partial f}{\partial x_k} \cdot \overline{\psi}_k,$$

where $\psi = \{\psi_1, \psi_2, \psi_3\} = \{i_1, i_2, i_3\}$.

For the above stated α, let us introduce the following operators:

$$^{\psi}D_{\alpha}[f] = \alpha f + {}^{\psi}D[f], \qquad {}_{\alpha}D^{\psi}[f] = \alpha f + D^{\psi}[f],$$
$$^{\psi}\overline{D}_{\alpha}[f] = \alpha f - {}^{\psi}D[f], \qquad {}_{\alpha}\overline{D}^{\psi}[f] = \alpha f - D^{\psi}[f].$$

f will be called a left (right)-(ψ, α)-hyperholomorphic in the domain Ω, if $^{\psi}D_{\alpha}[f] = 0$ $({}_{\alpha}D^{\psi}[f] = 0)$ in Ω. Let $\alpha \in \mathbb{C}\backslash\{0\}$ and $\operatorname{Im}\alpha \neq 0$. For $x \in R^3\backslash\{0\}$, we introduce the following notation:

$$\theta_{\alpha}(x) = \begin{cases} -\frac{1}{4\pi|x|} e^{i\alpha|x|}, & \operatorname{Im}\alpha > 0, \\ -\frac{1}{4\pi|x|} e^{-i\alpha|x|}, & \operatorname{Im}\alpha < 0. \end{cases}$$

In both cases it is a fundamental solution of the Helmholtz equation with $\lambda = \alpha^2$. Then the fundamental solution to the operator ${}^{\psi}D_\alpha$, $\mathcal{K}_{\psi,\alpha}$ is given by

$$\mathcal{K}_{\psi,\alpha}(x) = {}^{\psi}\overline{D}_\alpha[\theta_\alpha](x) = \begin{cases} \theta_\alpha(x)(\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|}), & \operatorname{Im}\alpha > 0, \\ \theta_\alpha(x)(\alpha + \frac{x}{|x|^2} + i\alpha\frac{x}{|x|}), & \operatorname{Im}\alpha < 0. \end{cases}$$

If $f(x) \in L^{p,\sigma}(R^3, \mathbb{H}(\mathbb{C}))$ means that $f(x) \in L^p(B, \mathbb{H}(\mathbb{C})), f^{(\sigma)}(x) = |x|^{-\sigma} f(\frac{\overline{x}}{|x|^2}) \in L^p(B, \mathbb{H}(\mathbb{C}))$, in which $B = \{x \mid |x| < 1\}$, σ is a real number, $\|f\|_{p,\sigma} = \|f\|_{L^p(B)} + \|f^{(\sigma)}\|_{L^p(B)}$, $p \geq 1$.

Definition 2.1 Suppose that the functions u, v, φ are defined in Ω with values in $\mathbb{H}(\mathbb{C})$ and $u, v \in L^1(\Omega, \mathbb{H}(\mathbb{C}))$. If, for arbitrary $\varphi \in C_0^\infty(\Omega, \mathbb{H}(\mathbb{C}))$, u, v satisfy

$$\int_\Omega \varphi(x)u(x)\,dv_x - \int_\Omega {}_\alpha\overline{D}^{\psi}[\varphi]v(x)\,dv_x = 0,$$

then u is called a generalized derivative of the function v, where we denote $u = {}^{\psi}D_\alpha[v]$.

Lemma 2.1 ([16]) *If* $\sigma_1, \sigma_2 > 0$, $0 \leq \gamma \leq 1$, *then we have*

$$\left|\sigma_1^\gamma - \sigma_2^\gamma\right| \leq |\sigma_1 - \sigma_2|^\gamma.$$

Lemma 2.2 ([17]) *Suppose* Ω *is a bounded domain in* R^3 *and let* α', β' *satisfy* $0 < \alpha', \beta' < 3$, $\alpha' + \beta' > 3$. *Then, for all* $x_1, x_2 \in R^3$ *and* $x_1 \neq x_2$, *we have*

$$\int_\Omega |t - x_1|^{-\alpha'}|t - x_2|^{-\beta'}\,dt \leq M_0(\alpha', \beta')|x_1 - x_2|^{3-\alpha'-\beta'}.$$

Lemma 2.3 ([18]) *Let* Ω, $\partial\Omega$ *be as stated above. If* $f \in C^{(m)}(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$ $(m \geq 1)$, *then we have*

$$\int_{\partial\Omega} f(y)\,d\sigma_y \mathcal{K}_{\psi,\alpha}(y - x) + \int_\Omega {}_\alpha\overline{D}^{\psi}[f(y)]\mathcal{K}_{\psi,\alpha}(y - x)\,dv_y = f(x), \quad x \in \Omega.$$

3 Some properties of the singular integral operator $T_{\psi,\alpha}$ for the Helmholtz equation

In this section, we will discuss some properties of the singular integral operators as follows:

$$\begin{aligned}
&\big(T_{\psi,\alpha}[f]\big)(x) \\
&\quad = \int_B \mathcal{K}_{\psi,\alpha}(y - x)f(y)\,dv_y + \int_B \mathcal{K}_{\psi,\alpha}\left(\frac{\overline{y}}{|y|^2} - x\right)f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad = \big(T^{(1)}_{\psi,\alpha}[f]\big)(x) + \big(T^{(2)}_{\psi,\alpha}[f]\big)(x),
\end{aligned} \tag{3.1}$$

where $B = \{x \mid |x| < 1\}$, $\alpha = a + ib$, $b > 0$.

Theorem 3.1 *Assume* B *to be as stated above,* $\alpha = a + ib$, $b > 0$. *If* $f \in L^p(B, \mathbb{H}(\mathbb{C}))$, $3 < p < +\infty$, *then*

(1) $|(T^{(1)}_{\psi,\alpha}[f])(x)| \leq M_1(p)\|f\|_{L^p(B)}$, $x \in R^3$,

(2) $|(T^{(1)}_{\psi,\alpha}[f])(x_1) - (T^{(1)}_{\psi,\alpha}[f])(x_2)| \leq M_2(p)\|f\|_{L^p(B)}|x_1 - x_2| + M_3(p)\|f\|_{L^p(B)}|x_1 - x_2|^\beta$, $x_1, x_2 \in R^3$,

(3) ${}^{\psi}D_{\alpha}(T^{(1)}_{\psi,\alpha}[f])(x) = f(x)$, $x \in B$, ${}^{\psi}D_{\alpha}(T^{(1)}_{\psi,\alpha}[f])(x) = 0$, $x \in R^3\backslash\overline{B}$,
where $0 < \beta = \frac{p-3}{p} < 1$.

Proof (1)

$$\begin{aligned}
\big(T^{(1)}_{\psi,\alpha}[f]\big)(x) &= \int_B \mathcal{K}_{\psi,\alpha}(y-x)f(y)\,dv_y \\
&= -\frac{\alpha}{4\pi}\int_B \frac{e^{i\alpha|y-x|}}{|y-x|}f(y)\,dv_y - \frac{1}{4\pi}\int_B \frac{e^{i\alpha|y-x|}(y-x)}{|y-x|^3}f(y)\,dv_y \\
&\quad + \frac{i\alpha}{4\pi}\int_B \frac{e^{i\alpha|y-x|}(y-x)}{|y-x|^2}f(y)\,dv_y \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

(i) By the Taylor series, we have $|e^{i\alpha|y-x|}| = |e^{i(a+ib)|y-x|}| = e^{-b|y-x|} \leq \frac{1}{b|y-x|}$. By the Hölder inequality, we have

$$\begin{aligned}
|I_1| &\leq \frac{|\alpha|}{4\pi}\int_B \frac{e^{-b|y-x|}}{|y-x|}\big|f(y)\big|\,dv_y \leq J_1\int_B \frac{1}{|y-x|^2}\big|f(y)\big|\,dv_y \\
&\leq J_1\|f\|_{L^p(B)}\left[\int_B \frac{1}{|y-x|^{2q}}\,dv_y\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.2}$$

When $x \in \overline{B}$, because $p > 3$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $1 < q < \frac{3}{2}$. Thus $\int_B \frac{1}{|y-x|^{2q}}\,dv_y$ is bounded. Hence we suppose

$$\int_B \frac{1}{|y-x|^{2q}}\,dv_y \leq J_2. \tag{3.3}$$

When $x \in R^3\backslash\overline{B}$, by Lemma 2.1 and the generalized spherical coordinate, we have

$$\int_B \frac{1}{|y-x|^{2q}}\,dv_y \leq J_3\int_{d_0}^{d_0+2} \rho^{2-2q}\,d\rho \leq J_4, \tag{3.4}$$

where $\rho = |y-x|$, $d_0 = d(x,B)$. Therefore, for arbitrary $x \in R^3$, we obtain

$$|I_1| \leq M^{(1)}_1(p)\|f\|_{L^p(B)}, \tag{3.5}$$

where $M^{(1)}_1(p) = \max\{J_1J_2^{\frac{1}{q}}, J_1J_4^{\frac{1}{q}}\}$.

(ii) Obviously, $e^{-b|y-x|} \leq 1$. By the Hölder inequality, we have

$$\begin{aligned}
|I_2| &\leq \frac{1}{4\pi}\int_B \frac{e^{-b|y-x|}}{|y-x|^2}\big|f(y)\big|\,dv_y \leq J_5\int_B \frac{1}{|y-x|^2}\big|f(y)\big|\,dv_y \\
&\leq J_5\|f\|_{L^p(B)}\left[\int_B \frac{1}{|y-x|^{2q}}\,dv_y\right]^{\frac{1}{q}}.
\end{aligned}$$

Then, by inequality (3.3) and (3.4), we have

$$|I_2| \leq M^{(2)}_1(p)\|f\|_{L^p(B)}, \tag{3.6}$$

where $M^{(2)}_1(p) = \max\{J_5J_2^{\frac{1}{q}}, J_5J_4^{\frac{1}{q}}\}$.

(iii) This case is similar to (ii). We obtain

$$|I_3| \le M_1^{(3)}(p)\|f\|_{L^p(B)}. \tag{3.7}$$

By inequalities (3.5)-(3.7), we obtain

$$\left|\left(T_{\psi,\alpha}^{(1)}[f]\right)(x)\right| \le |I_1| + |I_2| + |I_3| \le M_1(p)\|f\|_{L^p(B)},$$

where $M_1(p) = M_1^{(1)}(p) + M_1^{(2)}(p) + M_1^{(3)}(p)$.

$$\begin{aligned}
(2)\quad & \left(T_{\psi,\alpha}^{(1)}[f]\right)(x_1) - \left(T_{\psi,\alpha}^{(1)}[f]\right)(x_2) \\
&= \int_B \left[\mathcal{K}_{\psi,\alpha}(y-x_1) - \mathcal{K}_{\psi,\alpha}(y-x_2)\right] f(y)\,dv_y \\
&= -\frac{\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|y-x_1|}}{|y-x_1|} - \frac{e^{i\alpha|y-x_2|}}{|y-x_2|}\right] f(y)\,dv_y \\
&\quad - \frac{1}{4\pi}\int_B \left[\frac{e^{i\alpha|y-x_1|}(y-x_1)}{|y-x_1|^3} - \frac{e^{i\alpha|y-x_2|}(y-x_2)}{|y-x_2|^3}\right] f(y)\,dv_y \\
&\quad + \frac{i\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|y-x_1|}(y-x_1)}{|y-x_1|^2} - \frac{e^{i\alpha|y-x_2|}(y-x_2)}{|y-x_2|^2}\right] f(y)\,dv_y \\
&= I_4 + I_5 + I_6.
\end{aligned}$$

Let us consider $e^{i\alpha|y-x|}$. For arbitrary $x \in R^3$, it is easy to prove $|e^{i\alpha|y-x|}| \le 1$ and satisfy $|e^{i\alpha|y-x_1|} - e^{i\alpha|y-x_2|}| \le c|x_1 - x_2|$.

(i) For arbitrary $x_1, x_2 \in R^3$, by the Hölder inequality, we have

$$\begin{aligned}
|I_4| &\le J_6 \int_B \left|\frac{e^{i\alpha|y-x_1|}}{|y-x_1|} - \frac{e^{i\alpha|y-x_2|}}{|y-x_2|}\right| \left|f(y)\right| dv_y \\
&\le J_6 \int_B \frac{|e^{i\alpha|y-x_1|} - e^{i\alpha|y-x_2|}|}{|y-x_1|}\left|f(y)\right| dv_y + J_6 \int_B \left|e^{i\alpha|y-x_2|}\left(\frac{1}{|y-x_1|} - \frac{1}{|y-x_2|}\right)\right| \left|f(y)\right| dv_y \\
&\le J_7 \int_B \frac{1}{|y-x_1|}\left|f(y)\right| dv_y |x_1 - x_2| + J_6 \int_B \frac{1}{|y-x_1||y-x_2|}\left|f(y)\right| dv_y |x_1 - x_2| \\
&\le \left\{ J_7 \left\{\int_B \frac{1}{|y-x_1|^q}\,dv_y\right\}^{\frac{1}{q}} + J_6 \left\{\int_B \frac{1}{|y-x_1|^q|y-x_2|^q}\,dv_y\right\}^{\frac{1}{q}} \right\} \|f\|_{L^p(B)}|x_1 - x_2|.
\end{aligned}$$

As $1 < q < \frac{3}{2}$, $\int_B \frac{1}{|y-x_1|^q|y-x_2|^q}\,dv_y$ and $\int_B \frac{1}{|y-x_1|^q}\,dv_y$ are bounded. Hence

$$|I_4| \le M_2^{(1)}(p)\|f\|_{L^p(B)}|x_1 - x_2|. \tag{3.8}$$

$$\begin{aligned}
(ii)\quad I_5 &= -\frac{1}{4\pi}\int_B \left[\frac{e^{i\alpha|y-x_1|}(y-x_1)}{|y-x_1|^3} - \frac{e^{i\alpha|y-x_2|}(y-x_2)}{|y-x_2|^3}\right] f(y)\,dv_y \\
&= -\frac{1}{4\pi}\int_B \frac{(e^{i\alpha|y-x_1|} - e^{i\alpha|y-x_2|})(y-x_1)}{|y-x_1|^3} f(y)\,dv_y \\
&\quad - \frac{1}{4\pi}\int_B e^{i\alpha|y-x_2|}\left(\frac{y-x_1}{|y-x_1|^3} - \frac{y-x_2}{|y-x_2|^3}\right) f(y)\,dv_y \\
&= I_5^{(1)} + I_5^{(2)}.
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}\left|I_5^{(1)}\right| &\leq \frac{1}{4\pi}\int_B \frac{|e^{i\alpha|y-x_1|}-e^{i\alpha|y-x_2|}|}{|y-x_1|^2}\left|f(y)\right|dv_y\\ &\leq J_8\int_B \frac{1}{|y-x_1|^2}\left|f(y)\right|dv_y|x_1-x_2|\\ &\leq J_8\|f\|_{L^p(B)}\left[\int_B \frac{1}{|y-x_1|^{2q}}\,dv_y\right]^{\frac{1}{q}}|x_1-x_2|.\end{aligned}$$

As $1 < q < \frac{3}{2}$, $\int_B \frac{1}{|y-x_1|^{2q}}\,dv_y$ is bounded. So we have

$$\left|I_5^{(1)}\right| \leq J_9\|f\|_{L^p(B)}|x_1-x_2|. \tag{3.9}$$

By the Hölder inequality and the Hile lemma, we have

$$\begin{aligned}\left|I_5^{(2)}\right| &\leq \frac{1}{4\pi}\int_B \left|e^{i\alpha|y-x_2|}\right|\left|\frac{y-x_1}{|y-x_1|^3}-\frac{y-x_2}{|y-x_2|^3}\right|\left|f(y)\right|dv_y\\ &\leq J_{10}\int_B \left|\frac{y-x_1}{|y-x_1|^3}-\frac{y-x_2}{|y-x_2|^3}\right|\left|f(y)\right|dv_y\\ &\leq J_{10}\int_B \frac{|y-x_1|+|y-x_2|}{|y-x_1|^2|y-x_2|^2}|x_1-x_2|\left|f(y)\right|dv_y\\ &= J_{10}\left\{\int_B \frac{1}{|y-x_1||y-x_2|^2}\left|f(y)\right|dv_y+\int_B \frac{1}{|y-x_1|^2|y-x_2|}\left|f(y)\right|dv_y\right\}|x_1-x_2|\\ &\leq J_{10}\left\{\left[\int_B \frac{1}{|y-x_1|^q|y-x_2|^{2q}}\,dv_y\right]^{\frac{1}{q}}+\left[\int_B \frac{1}{|y-x_1|^{2q}|y-x_2|^q}\,dv_y\right]^{\frac{1}{q}}\right\}\\ &\quad\times \|f\|_{L^p(B)}|x_1-x_2|.\end{aligned}$$

We suppose $\alpha' = q$, $\beta' = 2q$. As $1 < q < \frac{3}{2}$, we have $\alpha' = q < 3$, $\beta' = 2q < 3$, $\alpha' + \beta' = 3q > 3$. Hence, by Lemma 2.2, we have

$$\begin{aligned}&\int_B \frac{1}{|y-x_1|^q|y-x_2|^{2q}}\,dv_y \leq M_0(\alpha',\beta')|x_1-x_2|^{3-3q},\\ &\int_B \frac{1}{|y-x_1|^{2q}|y-x_2|^q}\,dv_y \leq M_0(\alpha',\beta')|x_1-x_2|^{3-3q}.\end{aligned}$$

So we have

$$\left|I_5^{(2)}\right| \leq J_{11}\|f\|_{L^p(B)}\left(|x_1-x_2|^{3-3q}\right)^{\frac{1}{q}}|x_1-x_2| = J_{11}\|f\|_{L^p(B)}|x_1-x_2|^{\beta}, \tag{3.10}$$

where $0 < \beta = \frac{p-3}{p} < 1$. By inequality (3.9) and (3.10), we have

$$|I_5| \leq J_9\|f\|_{L^p(B)}|x_1-x_2| + J_{11}\|f\|_{L^p(B)}|x_1-x_2|^{\beta}. \tag{3.11}$$

$$\begin{aligned}\text{(iii)}\quad I_6 &= \frac{i\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|y-x_1|}(y-x_1)}{|y-x_1|^2}-\frac{e^{i\alpha|y-x_2|}(y-x_2)}{|y-x_2|^2}\right]f(y)\,dv_y\\ &= \frac{i\alpha}{4\pi}\int_B \frac{(e^{i\alpha|y-x_1|}-e^{i\alpha|y-x_2|})(y-x_1)}{|y-x_1|^2}f(y)\,dv_y\end{aligned}$$

$$\begin{aligned}&+\frac{i\alpha}{4\pi}\int_B e^{i\alpha|y-x_2|}\left(\frac{y-x_1}{|y-x_1|^2}-\frac{y-x_2}{|y-x_2|^2}\right)f(y)\,dv_y\\&=I_6^{(1)}+I_6^{(2)}.\end{aligned}$$

Similar to $I_5^{(1)}$, we have

$$\left|I_6^{(1)}\right|\le J_{12}\|f\|_{L^p(B)}|x_1-x_2|. \tag{3.12}$$

By the Hölder inequality and the Hile lemma, we have

$$\begin{aligned}\left|I_6^{(2)}\right|&\le\frac{|\alpha|}{4\pi}\int_B\left|e^{i\alpha|y-x_2|}\right|\left|\frac{y-x_1}{|y-x_1|^2}-\frac{y-x_2}{|y-x_2|^2}\right|\left|f(y)\right|dv_y\\&\le J_{13}\int_B\left|\frac{y-x_1}{|y-x_1|^2}-\frac{y-x_2}{|y-x_2|^2}\right|\left|f(y)\right|dv_y\\&\le J_{13}\int_B\frac{|x_1-x_2|}{|y-x_1||y-x_2|}\left|f(y)\right|dv_y\\&=J_{13}\int_B\frac{1}{|y-x_1||y-x_2|}\left|f(y)\right|dv_y|x_1-x_2|\\&\le J_{13}\|f\|_{L^p(B)}\left\{\int_B\frac{1}{|y-x_1|^q|y-x_2|^q}\,dv_y\right\}^{\frac{1}{q}}|x_1-x_2|.\end{aligned}$$

As $1<q<\frac{3}{2}$, $\int_B\frac{1}{|y-x_1|^q|y-x_2|^q}\,dv_y$ is bounded. So we have

$$\left|I_6^{(2)}\right|\le J_{14}\|f\|_{L^p(B)}|x_1-x_2|. \tag{3.13}$$

By inequalities (3.12) and (3.13), we have

$$|I_6|\le\left|I_6^{(1)}\right|+\left|I_6^{(2)}\right|\le M_2^{(2)}(p)\|f\|_{L^p(B)}|x_1-x_2|, \tag{3.14}$$

where $M_2^{(2)}(p)=J_{12}+J_{14}$. By inequalities (3.8), (3.11) and (3.14), we have

$$\left|\left(T_{\psi,\alpha}^{(1)}[f]\right)(x_1)-\left(T_{\psi,\alpha}^{(1)}[f]\right)(x_2)\right|\le M_2(p)\|f\|_{L^p(B)}|x_1-x_2|+M_3(p)\|f\|_{L^p(B)}|x_1-x_2|^\beta,$$

where $M_2(p)=M_2^{(1)}(p)+J_9+M_2^{(2)}(p)$, $M_3(p)=J_{11}$.

(3) When $x\in B$, for arbitrary $\varphi\in C_0^\infty(B,\mathbb{H}(\mathbb{C}))$, by Lemma 2.3 and the Fubini theorem, we have

$$\begin{aligned}\int_B {}_\alpha\overline{D}^\psi[\varphi]\left(T_{\psi,\alpha}^{(1)}[f]\right)(x)\,dv_x&=\int_B {}_\alpha\overline{D}^\psi[\varphi]\left[\int_B\mathcal{K}_{\psi,\alpha}(y-x)f(y)\,dv_y\right]dv_x\\&=\int_B\left[\int_B {}_\alpha\overline{D}^\psi[\varphi]\mathcal{K}_{\psi,\alpha}(y-x)\,dv_x\right]f(y)\,dv_y\\&=\int_B\left[\varphi(y)-\int_{\partial B}\varphi(x)\,d\sigma_x\mathcal{K}_{\psi,\alpha}(y-x)\right]f(y)\,dv_y\\&=\int_B\varphi(y)f(y)\,dv_y=\int_B\varphi(x)f(x)\,dv_x.\end{aligned}$$

Hence, in the sense of generalized derivatives, ${}^{\psi}D_{\alpha}(T^{(1)}_{\psi,\alpha}[f])(x) = f(x)$, $x \in B$. When $x \in R^3\backslash\overline{B}$, it is easy to see ${}^{\psi}D_{\alpha}(T^{(1)}_{\psi,\alpha}[f])(x) = 0$. □

Theorem 3.2 *Assume* B *to be as stated above and* $\alpha = a + ib$, $b > 0$.*If* $f \in L^{p,3}(B, \mathbb{H}(\mathbb{C}))$, $3 < p < +\infty$, *then*

(1) $|(T^{(2)}_{\psi,\alpha}[f])(x)| \leq M_4(p)\|f^{(3)}\|_{L^p(B)}$, $x \in R^3$,

(2) $|(T^{(2)}_{\psi,\alpha}[f])(x_1) - (T^{(2)}_{\psi,\alpha}[f])(x_2)| \leq M_5(p)\|f^{(3)}\|_{L^p(B)}|x_1 - x_2| + M_6(p)\|f^{(3)}\|_{L^p(B)}|x_1 - x_2|^{\beta}$, $x_1, x_2 \in R^3$,

(3) ${}^{\psi}D_{\alpha}(T^{(2)}_{\psi,\alpha}[f])(x) = 0$, $x \in B$, ${}^{\psi}D_{\alpha}(T^{(2)}_{\psi,\alpha}[f])(x) = f(x)$, $x \in R^3\backslash\overline{B}$,

where $0 < \beta = \frac{p-3}{p} < 1$.

Proof (1)

$$\begin{aligned}
\bigl(T^{(2)}_{\psi,\alpha}[f]\bigr)(x) &= \int_B \mathcal{K}_{\psi,\alpha}\biggl(\frac{\overline{y}}{|y|^2} - x\biggr) f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\frac{1}{|y|^6}\,dv_y \\
&= -\frac{\alpha}{4\pi}\int_B \frac{e^{i\alpha|\frac{\overline{y}}{|y|^2}-x|}}{|\frac{\overline{y}}{|y|^2} - x|} f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\frac{1}{|y|^6}\,dv_y \\
&\quad - \frac{1}{4\pi}\int_B \frac{e^{i\alpha|\frac{\overline{y}}{|y|^2}-x|}(\frac{\overline{y}}{|y|^2} - x)}{|\frac{\overline{y}}{|y|^2} - x|^3} f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\frac{1}{|y|^6}\,dv_y \\
&\quad + \frac{i\alpha}{4\pi}\int_B \frac{e^{i\alpha|\frac{\overline{y}}{|y|^2}-x|}(\frac{\overline{y}}{|y|^2} - x)}{|\frac{\overline{y}}{|y|^2} - x|^2} f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\frac{1}{|y|^6}\,dv_y \\
&= I_7 + I_8 + I_9.
\end{aligned}$$

As the first step, by the Hölder inequality, we have

$$\begin{aligned}
|I_7| &\leq \frac{|\alpha|}{4\pi}\int_B \frac{e^{-b|\frac{\overline{y}}{|y|^2}-x|}}{|\frac{\overline{y}}{|y|^2} - x|}\biggl|f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\biggr|\frac{1}{|y|^6}\,dv_y \\
&\leq C_1\int_B \frac{1}{|\frac{\overline{y}}{|y|^2} - x|}\biggl|f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\biggr|\frac{1}{|y|^6}\,dv_y \\
&\leq C_1\biggl\{\int_B\biggl[|y|^{-3}\biggl|f\biggl(\frac{\overline{y}}{|y|^2}\biggr)\biggr|\biggr]^p dv_y\biggr\}^{\frac{1}{p}}\biggl\{\int_B \frac{1}{|\frac{\overline{y}}{|y|^2} - x|^q|y|^{3q}}\,dv_y\biggr\}^{\frac{1}{q}} \\
&= C_1\bigl\|f^{(3)}\bigr\|_{L^p(B)}\bigl[O_1(x)\bigr]^{\frac{1}{q}},
\end{aligned} \tag{3.15}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Next we discuss $O_1(x)$ in two cases.

(i) When $|x| \geq \frac{1}{2}$, since

$$\begin{aligned}
\biggl|\frac{\overline{y}}{|y|^2} - x\biggr|^{-q}|y|^{-3q} &= |y|^{-2q}\biggl\{|y|^{-q}\biggl|\frac{\overline{y}}{|y|^2} - x\biggr|^{-q}\biggl|\frac{\overline{x}}{|x|^2}\biggr|^{-q}\biggr\}|x|^{-q} \\
&\leq C_2|y|^{-2q}\biggl|y\biggl(\frac{\overline{y}}{|y|^2} - x\biggr)\frac{\overline{x}}{|x|^2}\biggr|^{-q}|x|^{-q} = C_2|y|^{-2q}\biggl|\frac{\overline{x}}{|x|^2} - y\biggr|^{-q}|x|^{-q},
\end{aligned}$$

we have

$$O_1(x) \leq \int_B C_2 |y|^{-2q} \left| \frac{\overline{x}}{|x|^2} - y \right|^{-q} |x|^{-q}\, dv_y = C_2 |x|^{-q} \int_B |y|^{-2q} \left| \frac{\overline{x}}{|x|^2} - y \right|^{-q} dv_y.$$

We suppose $\alpha' = 2q$, $\beta' = q$. As $1 < q < \frac{3}{2}$, we have $0 < \alpha' < 3$, $0 < \beta' < 3$, $\alpha' + \beta' = 3q > 3$. Thus, by Lemma 2.2, we have

$$O_1(x) \leq C_2 M_0(\alpha', \beta') |x|^{-q} \left| \frac{\overline{x}}{|x|^2} \right|^{3-3q} \leq C_2 M_0(\alpha', \beta') 2^{3-2q} = C_3. \tag{3.16}$$

(ii) When $|x| < \frac{1}{2}$, by $|y| < 1$, we have $|1 - yx| \geq \frac{1}{2}$, thus

$$\begin{aligned} O_1(x) &= \int_B \frac{1}{|\frac{\overline{y}}{|y|^2} - x|^q |y|^{3q}}\, dv_y = \int_B |y|^{-2q} |y|^{-q} \left| \frac{\overline{y}}{|y|^2} - x \right|^{-q} dv_y \\ &\leq C_4 \int_B |y|^{-2q} \left| y \left(\frac{\overline{y}}{|y|^2} - x \right) \right|^{-q} dv_y = C_4 \int_B |y|^{-2q} |1 - yx|^{-q}\, dv_y \\ &\leq C_4 \int_B |y|^{-2q} 2^q\, dv_y \leq C_5 \int_B |y|^{-2q}\, dv_y \leq C_6. \end{aligned} \tag{3.17}$$

Therefore, by (3.15)-(3.17), we have

$$|I_7| \leq M_4^{(1)}(p) \| f^{(3)} \|_{L^p(B)}, \tag{3.18}$$

where $M_4^{(1)}(p) = \max\{C_1 C_3^{\frac{1}{q}}, C_1 C_6^{\frac{1}{q}}\}$.

As the second step, by the Hölder inequality, we have

$$\begin{aligned} |I_8| &\leq \frac{1}{4\pi} \int_B \frac{e^{-b|\frac{\overline{y}}{|y|^2} - x|}}{|\frac{\overline{y}}{|y|^2} - x|^2} \left| f\left(\frac{\overline{y}}{|y|^2} \right) \right| \frac{1}{|y|^6}\, dv_y \\ &\leq C_7 \int_B \frac{1}{|\frac{\overline{y}}{|y|^2} - x|^2} \left| f\left(\frac{\overline{y}}{|y|^2} \right) \right| \frac{1}{|y|^6}\, dv_y \\ &\leq C_7 \left\{ \int_B \left[|y|^{-3} \left| f\left(\frac{\overline{y}}{|y|^2} \right) \right| \right]^p dv_y \right\}^{\frac{1}{p}} \left\{ \int_B \frac{1}{|\frac{\overline{y}}{|y|^2} - x|^{2q} |y|^{3q}}\, dv_y \right\}^{\frac{1}{q}} \\ &= C_7 \| f^{(3)} \|_{L^p(B)} \left[O_2(x) \right]^{\frac{1}{q}}. \end{aligned} \tag{3.19}$$

Similar to $O_1(x)$, we find that $O_2(x)$ is bounded. Suppose $O_2(x) \leq C_8$. Then

$$|I_8| \leq M_4^{(2)}(p) \| f^{(3)} \|_{L^p(B)}. \tag{3.20}$$

As the third step, similar to I_7, we have

$$|I_9| \leq M_4^{(3)}(p) \| f^{(3)} \|_{L^p(B)}. \tag{3.21}$$

By inequalities (3.18), (3.20), and (3.21),

$$\left| \left(T^{(2)}_{\psi,\alpha}[f] \right)(x) \right| \leq |I_7| + |I_8| + |I_9| \leq M_4(p) \| f^{(3)} \|_{L^p(B)},$$

where $M_4(p) = M_4^{(1)}(p) + M_4^{(2)}(p) + M_4^{(3)}(p)$.

$$\begin{aligned}
(2)\quad & \big(T^{(2)}_{\psi,\alpha}[f]\big)(x_1) - \big(T^{(2)}_{\psi,\alpha}[f]\big)(x_2) \\
&= -\frac{\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|}}{|\frac{\bar{y}}{|y|^2}-x_1|} - \frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}}{|\frac{\bar{y}}{|y|^2}-x_2|}\right] f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad - \frac{1}{4\pi}\int_B \left[\frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|}(\frac{\bar{y}}{|y|^2}-x_1)}{|\frac{\bar{y}}{|y|^2}-x_1|^3} - \frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}(\frac{\bar{y}}{|y|^2}-x_2)}{|\frac{\bar{y}}{|y|^2}-x_2|^3}\right] f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad + \frac{i\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|}(\frac{\bar{y}}{|y|^2}-x_1)}{|\frac{\bar{y}}{|y|^2}-x_1|^2} - \frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}(\frac{\bar{y}}{|y|^2}-x_2)}{|\frac{\bar{y}}{|y|^2}-x_2|^2}\right] f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= I_{10} + I_{11} + I_{12}.
\end{aligned}$$

Firstly, we discuss I_{10}. We have

$$\begin{aligned}
I_{10} &= -\frac{\alpha}{4\pi}\int_B \left[\frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|}}{|\frac{\bar{y}}{|y|^2}-x_1|} - \frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}}{|\frac{\bar{y}}{|y|^2}-x_2|}\right] f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= -\frac{\alpha}{4\pi}\int_B \frac{e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|} - e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}}{|\frac{\bar{y}}{|y|^2}-x_1|} f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad - \frac{\alpha}{4\pi}\int_B e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}\left(\frac{1}{|\frac{\bar{y}}{|y|^2}-x_1|} - \frac{1}{|\frac{\bar{y}}{|y|^2}-x_2|}\right) f\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= I_{10}^{(1)} + I_{10}^{(2)}.
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}
\big|I_{10}^{(1)}\big| &\le \frac{|\alpha|}{4\pi}\int_B \frac{|e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_1|} - e^{i\alpha|\frac{\bar{y}}{|y|^2}-x_2|}|}{|\frac{\bar{y}}{|y|^2}-x_1|}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&\le \frac{|\alpha|}{4\pi}\int_B \frac{c|x_1-x_2|}{|\frac{\bar{y}}{|y|^2}-x_1|}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&= C_9\int_B \frac{1}{|\frac{\bar{y}}{|y|^2}-x_1|}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y|x_1-x_2| \\
&\le C_9\left\{\int_B \left[|y|^{-3}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\right]^p dv_y\right\}^{\frac{1}{p}}\left\{\int_B \frac{1}{|\frac{\bar{y}}{|y|^2}-x_1|^q|y|^{3q}}\,dv_y\right\}^{\frac{1}{q}}|x_1-x_2| \\
&= C_9\big\|f^{(3)}\big\|_{L^p(B)}\big[O_1(x)\big]^{\frac{1}{q}}|x_1-x_2|.
\end{aligned}$$

By (3.16) and (3.17), we have $O_1(x) \le \max\{C_3, C_6\}$. Therefore

$$\big|I_{10}^{(1)}\big| \le C_{10}\big\|f^{(3)}\big\|_{L^p(B)}|x_1-x_2|, \tag{3.22}$$

where $C_{10} = \max\{C_9 C_3^{\frac{1}{q}}, C_9 C_6^{\frac{1}{q}}\}$.

By the Taylor series, we have $|e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_2|}| = |e^{-b|\frac{\overline{y}}{|y|^2}-x_2|}| \le \frac{1}{b|\frac{\overline{y}}{|y|^2}-x_2|}$. Therefore

$$\begin{aligned}
\big|I_{10}^{(2)}\big| &\le \frac{|\alpha|}{4\pi}\int_B e^{-b|\frac{\overline{y}}{|y|^2}-x_2|}\left|\frac{1}{|\frac{\overline{y}}{|y|^2}-x_1|}-\frac{1}{|\frac{\overline{y}}{|y|^2}-x_2|}\right|\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&\le \frac{|\alpha|}{4\pi}\int_B \frac{1}{b|\frac{\overline{y}}{|y|^2}-x_2|}\frac{||\frac{\overline{y}}{|y|^2}-x_2|-|\frac{\overline{y}}{|y|^2}-x_1||}{|\frac{\overline{y}}{|y|^2}-x_1||\frac{\overline{y}}{|y|^2}-x_2|}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&\le C_{11}\int_B \frac{|x_1-x_2|}{|\frac{\overline{y}}{|y|^2}-x_1||\frac{\overline{y}}{|y|^2}-x_2|^2}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&= C_{11}\int_B \frac{1}{|\frac{\overline{y}}{|y|^2}-x_1||\frac{\overline{y}}{|y|^2}-x_2|^2|y|^3}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right||y|^{-3}\,dv_y|x_1-x_2| \\
&\le C_{11}\big\|f^{(3)}\big\|_{L^p(B)}\left\{\int_B \frac{1}{|\frac{\overline{y}}{|y|^2}-x_1|^q|\frac{\overline{y}}{|y|^2}-x_2|^{2q}|y|^{3q}}\,dv_y\right\}^{\frac{1}{q}}|x_1-x_2| \\
&= C_{11}\big\|f^{(3)}\big\|_{L^p(B)}\big[O_3(x)\big]^{\frac{1}{q}}|x_1-x_2|.
\end{aligned} \tag{3.23}$$

Since

$$\begin{aligned}
\left|\frac{\overline{y}}{|y|^2}-x_1\right|^{-q}\left|\frac{\overline{y}}{|y|^2}-x_2\right|^{-2q}|y|^{-3q} &= |y|^{-q}\left|\frac{\overline{y}}{|y|^2}-x_1\right|^{-q}|y|^{-2q}\left|\frac{\overline{y}}{|y|^2}-x_2\right|^{-2q} \\
&\le C_{12}\left|y\left(\frac{\overline{y}}{|y|^2}-x_1\right)\right|^{-q}\left|y\left(\frac{\overline{y}}{|y|^2}-x_2\right)\right|^{-2q} \\
&= C_{12}|1-yx_1|^{-q}|1-yx_2|^{-2q},
\end{aligned}$$

we have

$$O_3(x) \le C_{12}\int_B \frac{1}{|1-yx_1|^q|1-yx_2|^{2q}}\,dv_y = C_{12}O_4(x).$$

By (3.23), we have

$$\big|I_{10}^{(2)}\big| \le C_{13}\big\|f^{(3)}\big\|_{L^p(B)}\big[O_4(x)\big]^{\frac{1}{q}}|x_1-x_2|. \tag{3.24}$$

In the following, we discuss $O_4(x)$ in four cases.

(i) When $|x_1| \le \frac{1}{2}$, $|x_2| \le \frac{1}{2}$, as $|y| \le 1$, we have $|1-yx_1| \ge \frac{1}{2}$, $|1-yx_2| \ge \frac{1}{2}$, $|x_1-x_2| \le 1$. Hence

$$O_4(x) \le \int_B 2^q 2^{2q}\,dv_y = 2^{3q}\int_B dv_y = C_{14}.$$

As $|x_1-x_2| \le 1$, $0 < \beta = \frac{p-3}{p} < 1$, we have $|x_1-x_2| \le |x_1-x_2|^\beta$. Therefore, by (3.24), we have

$$\big|I_{10}^{(2)}\big| \le C_{15}\big\|f^{(3)}\big\|_{L^p(B)}|x_1-x_2|^\beta. \tag{3.25}$$

(ii) When $|x_1| \geq \frac{1}{2}$, $|x_2| \leq \frac{1}{2}$, we have $|1 - yx_2| \geq \frac{1}{2}$, $\frac{1}{|x_1|} \leq 2$, $\frac{|x_2|}{|x_1|} \leq 1$. Thus

$$\begin{aligned} O_4(x) &\leq 2^{2q} \int_B \frac{1}{|1-yx_1|^q}\, dv_y = 2^{2q}|x_1|^{-q} \int_B \frac{1}{|1-yx_1|^q |\frac{\overline{x}_1}{|x_1|^2}|^q}\, dv_y \\ &\leq C_{16} 2^{2q}|x_1|^{-q} \int_B \frac{1}{|(1-yx_1)\frac{\overline{x}_1}{|x_1|^2}|^q}\, dv_y = C_{16}2^{2q}|x_1|^{-q}\int_B \frac{1}{|y - \frac{\overline{x}_1}{|x_1|^2}|^q}\, dv_y. \end{aligned}$$

Again, since

$$\begin{aligned} \frac{1}{|x_1|} &= \frac{1}{|x_1|^\beta}\left|\frac{\overline{x}_1}{|x_1|^2}\right|^{1-\beta} = \frac{1}{|x_1|^\beta}\left|\frac{\overline{x}_1(x_1-x_2)(\overline{x}_1-\overline{x}_2)}{|x_1|^2|x_1-x_2|^2}\right|^{1-\beta} \\ &\leq C_{17}\frac{1}{|x_1|^\beta}\left|\frac{\overline{x}_1(x_1-x_2)}{|x_1|^2}\right|^{1-\beta}\frac{1}{|x_1-x_2|^{1-\beta}} = C_{17}\frac{1}{|x_1|^\beta}\left|1 - \frac{\overline{x}_1 x_2}{|x_1|^2}\right|^{1-\beta}|x_1-x_2|^{\beta-1} \\ &\leq C_{17}|x_1|^{-\beta}\left(1 + \frac{|x_2|}{|x_1|}\right)^{1-\beta}|x_1-x_2|^{\beta-1} \leq C_{18}|x_1-x_2|^{\beta-1}, \end{aligned}$$

we have $|x_1|^{-q} \leq C_{19}|x_1 - x_2|^{(\beta-1)q}$. Again from the notion that $1 < q < \frac{3}{2}$, we know $\int_B \frac{1}{|y - \frac{\overline{x}_1}{|x_1|^2}|^q}\, dv_y$ is bounded. Hence,we obtain

$$O_4(x) \leq C_{20}|x_1 - x_2|^{(\beta-1)q}.$$

Therefore, by (3.24), we have

$$\begin{aligned} \left|I_{10}^{(2)}\right| &\leq C_{13}\left\|f^{(3)}\right\|_{L^p(B)}\left[C_{25}|x_1-x_2|^{(\beta-1)q}\right]^{\frac{1}{q}}|x_1-x_2| \\ &= C_{21}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta. \end{aligned} \tag{3.26}$$

(iii) When $|x_1| \leq \frac{1}{2}$, $|x_2| \geq \frac{1}{2}$, similar to (ii), we have

$$\left|I_{10}^{(2)}\right| \leq C_{22}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta. \tag{3.27}$$

(iv) When $|x_1| \geq \frac{1}{2}$, $|x_2| \geq \frac{1}{2}$, we have $\frac{1}{|x_1|} \leq 2$, $\frac{1}{|x_2|} \leq 2$. Since

$$\begin{aligned} |1-yx_1|^{-q} &= |1-yx_1|^{-q}|x_1|^q|x_1|^{-q} = |1-yx_1|^{-q}\left|\frac{\overline{x}_1}{|x_1|^2}\right|^{-q}|x_1|^{-q} \\ &\leq C_{23}\left|(1-yx_1)\frac{\overline{x}_1}{|x_1|^2}\right|^{-q}|x_1|^{-q} = C_{23}\left|y - \frac{\overline{x}_1}{|x_1|^2}\right|^{-q}|x_1|^{-q}, \\ |1-yx_2|^{-2q} &= |1-yx_2|^{-2q}|x_2|^{2q}|x_2|^{-2q} = |1-yx_2|^{-2q}\left|\frac{\overline{x}_2}{|x_2|^2}\right|^{-2q}|x_2|^{-2q} \\ &\leq C_{24}\left|(1-yx_2)\frac{\overline{x}_2}{|x_2|^2}\right|^{-2q}|x_2|^{-2q} = C_{24}\left|y - \frac{\overline{x}_2}{|x_2|^2}\right|^{-2q}|x_2|^{-2q}. \end{aligned}$$

We have

$$O_4(x) \leq C_{25}\int_B \frac{1}{|y - \frac{\overline{x}_1}{|x_1|^2}|^q|y - \frac{\overline{x}_2}{|x_2|^2}|^{2q}}\, dv_y.$$

Suppose $\alpha' = q$, $\beta' = 2q$. Then $0 < \alpha' < 3$, $0 < \beta' < 3$, $\alpha' + \beta' = 3q > 3$. Thus, by Lemma 2.2, we have

$$\begin{aligned}
O_4(x) &\le C_{26}\left|\frac{\overline{x}_1}{|x_1|^2} - \frac{\overline{x}_2}{|x_2|^2}\right|^{3-3q} = C_{26}\left|\frac{\overline{x}_1|x_2|^2 - \overline{x}_2|x_1|^2}{|x_1|^2|x_2|^2}\right|^{3-3q} \\
&= C_{26}\left|\frac{\overline{x}_1|x_2|^2 - \overline{x}_2|x_2|^2 + \overline{x}_2|x_2|^2 - \overline{x}_2|x_1|^2}{|x_1|^2|x_2|^2}\right|^{3-3q} \\
&= C_{26}\left|\frac{\overline{x}_1 - \overline{x}_2}{|x_1|^2} + \frac{\overline{x}_2(|x_2|^2 - |x_1|^2)}{|x_1|^2|x_2|^2}\right|^{3-3q} \\
&\le C_{26}\left(\frac{1}{|x_1|^2} + \frac{|x_1| + |x_2|}{|x_1|^2|x_2|}\right)^{3-3q}|x_1 - x_2|^{3-3q} \\
&= C_{26}\left(\frac{1}{|x_1|^2} + \frac{1}{|x_1|^2} + \frac{1}{|x_1||x_2|}\right)^{3-3q}|x_1 - x_2|^{3-3q} \\
&\le C_{27}|x_1 - x_2|^{3-3q}.
\end{aligned}$$

Therefore, by (3.24), we have

$$\begin{aligned}
\left|I_{10}^{(2)}\right| &\le C_{13}\left\|f^{(3)}\right\|_{L^p(B)}\left[C_{27}|x_1 - x_2|^{3-3q}\right]^{\frac{1}{q}}|x_1 - x_2| \\
&= C_{28}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta},
\end{aligned} \tag{3.28}$$

where $0 < \beta = \frac{p-3}{p} < 1$. From (3.25)-(3.28), we obtain

$$\left|I_{10}^{(2)}\right| \le M_6^{(1)}(p)\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta}, \tag{3.29}$$

where $M_6^{(1)}(p) = \max\{C_{15}, C_{21}, C_{22}, C_{28}\}$.

By (3.22), (3.29), we obtain

$$|I_{10}| \le C_{10}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2| + M_6^{(1)}(p)\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta}. \tag{3.30}$$

Secondly, we discuss I_{11}. We have

$$\begin{aligned}
I_{11} &= -\frac{1}{4\pi}\int_B\left[\frac{e^{i\alpha|\frac{\overline{y}}{|y|^2} - x_1|}(\frac{\overline{y}}{|y|^2} - x_1)}{|\frac{\overline{y}}{|y|^2} - x_1|^3} - \frac{e^{i\alpha|\frac{\overline{y}}{|y|^2} - x_2|}(\frac{\overline{y}}{|y|^2} - x_2)}{|\frac{\overline{y}}{|y|^2} - x_2|^3}\right]f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= -\frac{1}{4\pi}\int_B\frac{(e^{i\alpha|\frac{\overline{y}}{|y|^2} - x_1|} - e^{i\alpha|\frac{\overline{y}}{|y|^2} - x_2|})(\frac{\overline{y}}{|y|^2} - x_1)}{|\frac{\overline{y}}{|y|^2} - x_1|^3}f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad - \frac{1}{4\pi}\int_B e^{i\alpha|\frac{\overline{y}}{|y|^2} - x_2|}\left(\frac{\frac{\overline{y}}{|y|^2} - x_1}{|\frac{\overline{y}}{|y|^2} - x_1|^3} - \frac{\frac{\overline{y}}{|y|^2} - x_2}{|\frac{\overline{y}}{|y|^2} - x_2|^3}\right)f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= I_{11}^{(1)} + I_{11}^{(2)}.
\end{aligned}$$

Similar to $I_{10}^{(1)}$, we get

$$\left|I_{11}^{(1)}\right| \le C_{29}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|. \tag{3.31}$$

By the Hölder inequality and the Hile lemma, we have

$$
\begin{aligned}
\left|I_{11}^{(2)}\right| &\leq \frac{1}{4\pi}\int_B e^{-b|\frac{\overline{y}}{|y|^2}-x_2|}\left|\frac{\frac{\overline{y}}{|y|^2}-x_1}{|\frac{\overline{y}}{|y|^2}-x_1|^3}-\frac{\frac{\overline{y}}{|y|^2}-x_2}{|\frac{\overline{y}}{|y|^2}-x_2|^3}\right|\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&\leq C_{30}\int_B \frac{|\frac{\overline{y}}{|y|^2}-x_1|+|\frac{\overline{y}}{|y|^2}-x_2|}{|\frac{\overline{y}}{|y|^2}-x_1|^2|\frac{\overline{y}}{|y|^2}-x_2|^2}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y|x_1-x_2| \\
&= C_{30}\int_B \frac{1}{|\frac{\overline{y}}{|y|^2}-x_1||\frac{\overline{y}}{|y|^2}-x_2|^2}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y|x_1-x_2| \\
&\quad + C_{30}\int_B \frac{1}{|\frac{\overline{y}}{|y|^2}-x_1|^2|\frac{\overline{y}}{|y|^2}-x_2|}\left|f\left(\frac{\overline{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y|x_1-x_2| \\
&\leq C_{31}\left\|f^{(3)}\right\|_{L^p(B)}\left[\int_B \frac{1}{|1-yx_1|^q|1-yx_2|^{2q}}\,dv_y\right]^{\frac{1}{q}}|x_1-x_2| \\
&\quad + C_{32}\left\|f^{(3)}\right\|_{L^p(B)}\left[\int_B \frac{1}{|1-yx_1|^{2q}|1-yx_2|^q}\,dv_y\right]^{\frac{1}{q}}|x_1-x_2| \\
&= C_{31}\left\|f^{(3)}\right\|_{L^p(B)}\left[O_4(x)\right]^{\frac{1}{q}}|x_1-x_2| + C_{32}\left\|f^{(3)}\right\|_{L^p(B)}\left[O_5(x)\right]^{\frac{1}{q}}|x_1-x_2|.
\end{aligned} \tag{3.32}
$$

This is similar to $I_{10}^{(2)}$ and it is easy to prove the following:

$$
\begin{aligned}
&C_{31}\left\|f^{(3)}\right\|_{L^p(B)}\left[O_4(x)\right]^{\frac{1}{q}}|x_1-x_2| \leq C_{33}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta, \\
&C_{32}\left\|f^{(3)}\right\|_{L^p(B)}\left[O_5(x)\right]^{\frac{1}{q}}|x_1-x_2| \leq C_{34}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta.
\end{aligned}
$$

Therefore, we obtain

$$
\left|I_{11}^{(2)}\right| \leq C_{35}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta. \tag{3.33}
$$

By (3.31) and (3.33), we have

$$
|I_{11}| \leq C_{29}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2| + C_{35}\left\|f^{(3)}\right\|_{L^p(B)}|x_1-x_2|^\beta. \tag{3.34}
$$

Finally, we discuss I_{12}. We have

$$
\begin{aligned}
I_{12} &= \frac{i\alpha}{4\pi}\int_B\left[\frac{e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_1|}(\frac{\overline{y}}{|y|^2}-x_1)}{|\frac{\overline{y}}{|y|^2}-x_1|^2}-\frac{e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_2|}(\frac{\overline{y}}{|y|^2}-x_2)}{|\frac{\overline{y}}{|y|^2}-x_2|^2}\right]f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= \frac{i\alpha}{4\pi}\int_B \frac{(e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_1|}-e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_2|})(\frac{\overline{y}}{|y|^2}-x_1)}{|\frac{\overline{y}}{|y|^2}-x_1|^2}f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&\quad + \frac{i\alpha}{4\pi}\int_B e^{i\alpha|\frac{\overline{y}}{|y|^2}-x_2|}\left(\frac{\frac{\overline{y}}{|y|^2}-x_1}{|\frac{\overline{y}}{|y|^2}-x_1|^2}-\frac{\frac{\overline{y}}{|y|^2}-x_2}{|\frac{\overline{y}}{|y|^2}-x_2|^2}\right)f\left(\frac{\overline{y}}{|y|^2}\right)\frac{1}{|y|^6}\,dv_y \\
&= I_{12}^{(1)} + I_{12}^{(2)}.
\end{aligned}
$$

Similar to $I_{10}^{(1)}$, we get

$$\left|I_{12}^{(1)}\right| \le C_{36}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|. \tag{3.35}$$

By the Hile lemma and the Hölder inequality, we have

$$\begin{aligned}
\left|I_{12}^{(2)}\right| &\le \frac{|\alpha|}{4\pi}\int_B e^{-b|\frac{\bar{y}}{|y|^2}-x_2|}\left|\frac{\frac{\bar{y}}{|y|^2}-x_1}{|\frac{\bar{y}}{|y|^2}-x_1|^2} - \frac{\frac{\bar{y}}{|y|^2}-x_2}{|\frac{\bar{y}}{|y|^2}-x_2|^2}\right|\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&\le C_{37}\int_B \frac{|x_1-x_2|}{|\frac{\bar{y}}{|y|^2}-x_1||\frac{\bar{y}}{|y|^2}-x_2|^2}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|\frac{1}{|y|^6}\,dv_y \\
&= C_{37}\int_B \frac{1}{|\frac{\bar{y}}{|y|^2}-x_1||\frac{\bar{y}}{|y|^2}-x_2|^2|y|^3}|y|^{-3}\left|f\left(\frac{\bar{y}}{|y|^2}\right)\right|dv_y|x_1-x_2| \\
&\le C_{37}\left\|f^{(3)}\right\|_{L^p(B)}\left[\int_B \frac{1}{|\frac{\bar{y}}{|y|^2}-x_1|^q|\frac{\bar{y}}{|y|^2}-x_2|^{2q}|y|^{3q}}\,dv_y\right]^{\frac{1}{q}}|x_1-x_2| \\
&= C_{37}\left\|f^{(3)}\right\|_{L^p(B)}\left[O_3(x)\right]^{\frac{1}{q}}|x_1-x_2|.
\end{aligned}$$

Therefore

$$\left|I_{12}^{(2)}\right| \le C_{38}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta}, \tag{3.36}$$

by (3.35) and (3.36), so we have

$$|I_{12}| \le C_{36}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2| + C_{38}\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta}. \tag{3.37}$$

By (3.30), (3.34), and (3.37), we have

$$\begin{aligned}
&\left|\left(T_{\psi,\alpha}^{(2)}[f]\right)(x_2) - \left(T_{\psi,\alpha}^{(2)}[f]\right)(x_1)\right| \\
&\quad\le M_5(p)\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2| + M_6(p)\left\|f^{(3)}\right\|_{L^p(B)}|x_1 - x_2|^{\beta},
\end{aligned}$$

where $M_5(p) = C_{10} + C_{29} + C_{36}$, $M_6(p) = M_6^{(1)}(p) + C_{35} + C_{38}$.

(3) This case is similar to Theorem 3.1, and it is easy to prove. □

Remark 3.1 Assume B to be as stated above and $\alpha = a + ib, b > 0$. If $f \in L^{p,3}(B, \mathbb{H}(\mathbb{C}))$, $3 < p < +\infty$, then

(1) $|(T_{\psi,\alpha}[f])(x)| \le M_7(p)\|f\|_{p,3}$, $x \in R^3$,

(2) $|(T_{\psi,\alpha}[f])(x_1) - (T_{\psi,\alpha}[f])(x_2)| \le M_8(p)\|f\|_{p,3}|x_1 - x_2| + M_9(p)\|f\|_{p,3}|x_1 - x_2|^{\beta}$, $x_1, x_2 \in R^3$,

(3) ${}^{\psi}D_{\alpha}(T_{\psi,\alpha}[f])(x) = f(x)$, $x \in R^3 \setminus \partial B$,

where $0 < \beta = \frac{p-3}{p} < 1$.

4 Integral representation of solution of Riemann boundary problem to inhomogeneous partial differential system

In this section, we will discuss the inhomogeneous partial differential system of first order equations as follows:

$$\begin{cases} \alpha w_0 - w_{1_{x_1}} - w_{2_{x_2}} - w_{3_{x_3}} = c_0(x), \\ \alpha w_1 + w_{0_{x_1}} - w_{2_{x_3}} + w_{3_{x_2}} = c_1(x), \\ \alpha w_2 + w_{0_{x_2}} + w_{1_{x_3}} - w_{3_{x_1}} = c_2(x), \\ \alpha w_3 + w_{0_{x_3}} - w_{1_{x_2}} + w_{2_{x_1}} = c_3(x), \end{cases} \tag{4.1}$$

where $w_j(x)$, $c_j(x)$ $(j = 0, 1, 2, 3)$ are real-value functions.

Problem P Let $B \subset R^3$ be as stated above. The Riemann boundary value problem for system (4.1) is to find a solution $w(x)$ of (4.1) that satisfies the boundary condition

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B,$$

where $w^{\pm}(\tau) = \lim_{x \in B^{\pm}, x \to \tau} w(x)$, $B^+ = B$, $B^- = R^3 \backslash \overline{B}$, G is a quaternion constant, G^{-1} exists, and $f \in H^{\nu}_{\partial B}$ $(0 < \nu < 1)$.

In fact,

$$\begin{aligned} {}^{\psi}D_{\alpha}[w] &= \sum_{j=1}^{3} i_j \frac{\partial w}{\partial x_j} + \alpha w \\ &= \sum_{j=1}^{3} \left(i_j i_0 \frac{\partial w_0}{\partial x_j} + i_j i_1 \frac{\partial w_1}{\partial x_j} + i_j i_2 \frac{\partial w_2}{\partial x_j} + i_j i_3 \frac{\partial w_3}{\partial x_j} \right) + \alpha \sum_{k=0}^{3} w_k i_k \\ &= (\alpha w_0 - w_{1_{x_1}} - w_{2_{x_2}} - w_{3_{x_3}}) i_0 + (\alpha w_1 + w_{0_{x_1}} - w_{2_{x_3}} + w_{3_{x_2}}) i_1 \\ &\quad + (\alpha w_2 + w_{0_{x_2}} + w_{1_{x_3}} - w_{3_{x_1}}) i_2 + (\alpha w_3 + w_{0_{x_3}} - w_{1_{x_2}} + w_{2_{x_1}}) i_3. \end{aligned} \tag{4.2}$$

Let

$$g(x) = c_0(x) i_0 + c_1(x) i_1 + c_2(x) i_2 + c_3(x) i_3 = \sum_{j=0}^{3} c_j(x) i_j. \tag{4.3}$$

By (4.2) and (4.3), the inhomogeneous partial differential system (4.1) can be transformed to the following equation:

$${}^{\psi}D_{\alpha}[w] = \sum_{j=0}^{3} c_j(x) i_j = g(x). \tag{4.4}$$

Therefore Problem P as stated above can be transformed to Problem Q.

Problem Q Let $B \subset R^3$ be as stated above. The Riemann boundary value problem for system (4.1) is to find a solution $w(x)$ of (4.4) that satisfies the boundary condition

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B,$$

where $w^{\pm}(\tau) = \lim_{x\in B^{\pm},x\to\tau} w(x)$, $B^+ = B$, $B^- = R^3\backslash\overline{B}$, G is a quaternion constant, G^{-1} exists, and $f \in H^{\nu}_{\partial B}$ $(0 < \nu < 1)$.

Theorem 4.1 *Let B be as stated above. Find a quaternion-valued function $u(x)$ satisfying the system ${}^{\psi}D_{\alpha}[u] = 0(x \in R^3\backslash\partial B)$ and vanishing at infinity with the boundary condition*

$$u^+(\tau) = u^-(\tau)G + f(\tau), \quad \tau \in \partial B, \tag{4.5}$$

where $u^{\pm}(\tau) = \lim_{x\in B^{\pm},x\to\tau} u(x)$, G is a quaternion constant, G^{-1} exists, and $f \in H^{\lambda}_{\partial B}$ $(0 < \lambda < 1)$. Then the solution can be expressed as

$$u(x) = \begin{cases} \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\,d\sigma_y f(y), & x \in B^+, \\ \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\,d\sigma_y f(y)G^{-1}, & x \in B^-. \end{cases}$$

Proof Define

$$\varphi(x) = \begin{cases} u(x), & x \in B^+, \\ u(x)G, & x \in B^-. \end{cases}$$

Then it is obvious that ${}^{\psi}D_{\alpha}[\varphi] = 0$ $(x \in R^3\backslash\partial B)$ and the Riemann boundary condition (4.5) is equivalent to

$$\varphi^+(\tau) = \varphi^-(\tau) + f(\tau), \quad \tau \in \partial B.$$

Suppose $\Psi(x) = \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\,d\sigma_y f(y)$. Then ${}^{\psi}D_{\alpha}[\Psi] = 0$ $(x \in R^3\backslash\partial B)$. By the Plemelj formula, we have

$$\Psi^+(\tau) - \Psi^-(\tau) = f(\tau), \quad \tau \in \partial B.$$

Hence $\varphi^+(\tau) - \Psi^+(\tau) = \varphi^-(\tau) - \Psi^-(\tau)$ $(\tau \in \partial B)$. Thus ${}^{\psi}D_{\alpha}[\varphi - \Psi] = 0$ and by Theorem 3.12 in [10] we obtain $\varphi(x) = \Psi(x)$. So the solution can be expressed as

$$u(x) = \begin{cases} \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\,d\sigma_y f(y), & x \in B^+, \\ \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\,d\sigma_y f(y)G^{-1}, & x \in B^-. \end{cases} \qquad \square$$

Theorem 4.2 *Let B be as stated above and $g(x) \in L^{p,3}(R^3, \mathbb{H}(\mathbb{C}))$, $3 < p < +\infty$. Find a quaternion-valued function $w(x)$ satisfying the system ${}^{\psi}D_{\alpha}[w](x) = g(x)$ $(x \in R^3\backslash\partial B)$ and vanishing at infinity with the boundary condition*

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B, \tag{4.6}$$

where $w^{\pm}(\tau) = \lim_{x\in B^{\pm},x\to\tau} w(x)$, G is a quaternion constant, G^{-1} exists, and $f \in H^{\lambda}_{\partial B}$ $(0 < \lambda < 1)$. Then the solution has the form

$$w(x) = \Psi(x) + \left(T_{\psi,\alpha}[g]\right)(x),$$

in which ${}^{\psi}D_{\alpha}[\Psi] = 0$ *and*

$$\Psi(x) = \begin{cases} \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y) G^{-1}, & x \in B^-, \end{cases}$$

where $\tilde{f} = f + (T_{\psi,\alpha}[g])(G-1)$.

Proof By Remark 3.1, we know ${}^{\psi}D_{\alpha}[w] = {}^{\psi}D_{\alpha}[\Psi(x) + (T_{\psi,\alpha}[g])(x)] = g(x)$. The boundary condition (4.6) is equivalent to

$$\bigl(\Psi + T_{\psi,\alpha}[g]\bigr)^+(\tau) = \bigl(\Psi + T_{\psi,\alpha}[g]\bigr)^-(\tau)G + f(\tau), \quad \tau \in \partial B. \tag{4.7}$$

Again from Remark 3.1, we know that $(T_{\psi,\alpha}[g])(x)$ has continuity in R^3. Thus $(T_{\psi,\alpha}[g])^+ = (T_{\psi,\alpha}[g])^- = T_{\psi,\alpha}[g]$, so (4.7) is equivalent to

$$\Psi^+(\tau) = \Psi^-(\tau)G + \bigl(T_{\psi,\alpha}[g]\bigr)(\tau)(G-1) + f(\tau), \quad \tau \in \partial B. \tag{4.8}$$

Suppose $\tilde{f} = f + (T_{\psi,\alpha}[g])(G-1)$. Then (4.8) has the following form:

$$\Psi^+(\tau) = \Psi^-(\tau)G + \tilde{f}(\tau), \quad \tau \in \partial B. \tag{4.9}$$

Again from Theorem 4.1, the solutions which satisfy the system ${}^{\psi}D_{\alpha}[\Psi] = 0$ and boundary condition (4.9) can be represented in the form

$$\Psi(x) = \begin{cases} \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y) G^{-1}, & x \in B^-, \end{cases}$$

where $\tilde{f} = f + (T_{\psi,\alpha}[g])(G-1)$. □

Remark 4.1 By Theorem 4.2, the solution of problem P can be expressed as

$$w(x) = \Psi(x) + \bigl(T_{\psi,\alpha}[g]\bigr)(x),$$

in which ${}^{\psi}D_{\alpha}[\Psi] = 0$ and

$$\Psi(x) = \begin{cases} \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \int_{\partial B} \mathcal{K}_{\psi,\alpha}(y-x)\, d\sigma_y \tilde{f}(y) G^{-1}, & x \in B^-, \end{cases}$$

where $\tilde{f} = f + (T_{\psi,\alpha}[g])(G-1)$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

LPW has presented the main purpose of the article. All authors read and approved the final manuscript.

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