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Janowski type close-to-convex functions associated with conic regions

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Abstract

The analytic functions, mapping the open unit disk onto petal and oval type regions, introduced by Noor and Malik (Comput. Math. Appl. 62:2209-2217, 2011), are considered to define and study their associated close-to-convex functions. This work includes certain geometric properties like sufficiency criteria, coefficient estimates, arc length, the growth rate of coefficients of Taylor series, integral preserving properties of these functions.

MSC: 30C45; 30C50

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1 Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Furthermore, $\mathcal{S}$ represents the class of all functions in $\mathcal{A}$ which are univalent in E.

The convolution (Hadamard product) of functions $f, g \in \mathcal{A}$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in E,$$

where $f(z)$ is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in E.$$

For two functions f and g analytic in E, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$. For more details, see [2].

A function p analytic in E and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

belongs to the class $\mathcal{P}[A,B]$ if and only if

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1.$$

This class was introduced and investigated by Janowski [3]. In particular, if $A = 1$ and $B = -1$, we obtain the class $\mathcal{P}$ of functions with a positive real part (see [4, 5]). The classes $\mathcal{P}$ and $\mathcal{P}[A,B]$ are connected by the relation

$$p(z) \in \mathcal{P} \Leftrightarrow \frac{(A+1)p(z) - (A-1)}{(B+1)p(z) - (B-1)} \in \mathcal{P}[A,B].$$

Now consider, for $k \geq 0$, the classes $k - \mathcal{CV}$ and $k - \mathcal{ST}$ of k-uniformly convex functions and corresponding k-starlike functions, respectively, introduced by Kanas and Wisniowska, respectively. For some details, see [6–8].

Consider the domain

$$\Omega_k = \left\{u + iv; u > k\sqrt{(u-1)^2 + v^2}\right\}. \tag{1.2}$$

For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of a hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$). This domain was studied by Kanas [6–8]. The function p_k, with $p_k(0) = 1$, $p_k'(0) > 0$ plays the role of extremal and is given by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}(\log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2[(\frac{2}{\pi}\arccos k)\arctan h\sqrt{z}], & 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx] + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and t is chosen such that $k = \cos h(\frac{\pi R'(t)}{4R(t)})$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is the complementary integral of $R(t)$ (see [7–9]). Let $\mathcal{P}_{p_k}$ denote the class of all those functions $p(z)$ which are analytic in E with $p(0) = 1$ and $p(z) \prec p_k(z)$ for $z \in E$. Clearly, it can be seen that $\mathcal{P}_{p_k} \subset \mathcal{P}$, where $\mathcal{P}$ is the class of functions with a positive real part (see [4, 5]). For the applications and exclusive study of the class $\mathcal{P}$, we refer to [10–14]. More precisely

$$\mathcal{P}_{p_k} \subset \mathcal{P}\left(\frac{k}{1+k}\right) \subset \mathcal{P},$$

and, for $p \in \mathcal{P}_{p_k}$, we have

$$\left|\arg p(z)\right| \leq \frac{\lambda\pi}{2},$$

where

$$\lambda = \frac{2}{\pi}\arctan\left(\frac{1}{k}\right). \tag{1.4}$$

Therefore, we can write

$$p(z) = h^{\lambda}(z), \quad h \in \mathcal{P}.$$

Definition 1.1 ([1]) A function p analytic in E belongs to the class $k-\mathcal{P}[A,B]$ if and only if

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad k \geq 0,$$

where $p_k(z)$ is defined by (1.3) and $-1 \leq B < A \leq 1$. Geometrically, the function $p(z) \in k-\mathcal{P}[A,B]$ takes all values in the domain $\Omega_k[A,B]$, $-1 \leq B < A \leq 1$, $k \geq 0$, which is defined as follows:

$$\Omega_k[A,B] = \left\{w : \operatorname{Re}\left(\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}\right) > k\left|\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}-1\right|\right\}$$

or equivalently

$$\begin{aligned}\Omega_k[A,B] = \bigl\{u+iv : &\bigl[\bigl(B^2-1\bigr)\bigl(u^2+v^2\bigr)-2(AB-1)u+\bigl(A^2-1\bigr)\bigr]^2 \\ &> k^2\bigl[\bigl(-2(B+1)\bigl(u^2+v^2\bigr)+2(A+B+2)u-2(A+1)\bigr)^2 \\ &+ 4(A-B)^2v^2\bigr]\bigr\}.\end{aligned}$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega_0[A,B] = \Omega[A,B]$. The impact of $\Omega[A,B]$ on the conic domain Ω_k changes the original shape of the conic regions. The ends of hyperbola and parabola get closer to each other but never meet anywhere and the ellipse gets the shape of oval. When $A \longrightarrow 1$, $B \longrightarrow -1$, the radius of the circular disk defined by $\Omega[A,B]$ tends to infinity; consequently, the arms of hyperbola and parabola expand and the oval turns into ellipse.

Definition 1.2 ([1]) A function $f \in \mathcal{A}$ is said to be in the class $k-\mathcal{CV}[C,D]$, $-1 \leq D < C \leq 1$, if it satisfies the condition

$$\operatorname{Re}\left(\frac{(D-1)\frac{(zf'(z))'}{f'(z)}-(C-1)}{(D+1)\frac{(zf'(z))'}{f'(z)}-(C+1)}\right) > k\left|\frac{(D-1)\frac{(zf'(z))'}{f'(z)}-(C-1)}{(D+1)\frac{(zf'(z))'}{f'(z)}-(C+1)}-1\right| \quad (k \geq 0; z \in E),$$

equivalently, we can write

$$\frac{(zf'(z))'}{f'(z)} \in k-\mathcal{P}[C,D].$$

Definition 1.3 ([1]) The class $k-\mathcal{ST}[C,D]$, $-1 \le D < C \le 1$, is the family of all those functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left(\frac{(D-1)\frac{zf'(z)}{f(z)}-(C-1)}{(D+1)\frac{zf'(z)}{f(z)}-(C+1)}\right) > k\left|\frac{(D-1)\frac{zf'(z)}{f(z)}-(C-1)}{(D+1)\frac{zf'(z)}{f(z)}-(C+1)}-1\right| \quad (k \ge 0; z \in E),$$

or equivalently

$$\frac{zf'(z)}{f(z)} \in k-\mathcal{P}[C,D].$$

These two classes were recently introduced by Noor and Malik [1].

Motivated by the recent work presented by Noor and Malik [1], we define some classes of analytic functions associated with conic domains as follows.

Definition 1.4 Let $f \in \mathcal{A}$. Then $f \in k-\mathcal{UK}[A,B,C,D]$ if and only if there exists $g \in k-\mathcal{ST}[C,D]$ such that

$$\operatorname{Re}\left(\frac{(B-1)\frac{zf'(z)}{g(z)}-(A-1)}{(B+1)\frac{zf'(z)}{g(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{g(z)}-(A-1)}{(B+1)\frac{zf'(z)}{g(z)}-(A+1)}-1\right| \quad (k \ge 0),$$

or equivalently

$$\frac{zf'(z)}{g(z)} \in k-\mathcal{P}[A,B],$$

where $-1 \le D \le C \le 1$ and $-1 \le B < A \le 1$.

Definition 1.5 Let $f \in \mathcal{A}$. Then $f \in k-\mathcal{UQ}[A,B,C,D]$ if and only if, for $-1 \le D < C \le 1$, $-1 \le B < A \le 1$ and $k \ge 0$, there exists $g \in k-\mathcal{CV}[C,D]$ such that

$$\operatorname{Re}\left(\frac{(B-1)\frac{(zf'(z))'}{g'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{g'(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{g'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{g'(z)}-(A+1)}-1\right|,$$

or equivalently

$$\frac{(zf'(z))'}{g'(z)} \in k-\mathcal{P}[A,B].$$

It can easily be seen that

$$f \in k\text{–}\mathcal{UQ}[A,B,C,D] \Leftrightarrow zf' \in k\text{–}\mathcal{UK}[A,B,C,D]. \tag{1.5}$$

Special cases:

i. $0-\mathcal{UK}[A,B,C,D] = \mathcal{K}[A,B,C,D]$ and $0-\mathcal{UQ}[A,B,1,-1] = \mathcal{Q}[A,B]$, subclasses of close-to-convex and quasi-convex functions studied by Silvia and Noor, respectively, see [15, 16].

ii. $k-\mathcal{UK}[1,-1,1,-1] = k-\mathcal{UK}$ and $k-\mathcal{UQ}[1,-1,1,-1] = k-\mathcal{UQ}$, the class of k-uniformly close-to-convex and the class of k-uniformly quasi-convex functions studied by Acu [17].

iii. $k-\mathcal{UK}[1-2\beta,-1,1-2\gamma,-1] = k-\mathcal{UK}(\beta,\gamma)$ and $k-\mathcal{UQ}[1-2\beta,-1,1-2\gamma,-1] = k-\mathcal{UQ}(\beta,\gamma)$, the well-known class of k-uniformly close-to-convex and the class of k-uniformly quasi-convex functions of order β and type γ, see [18].

iv. 0–$\mathcal{UK}[1-2\beta,-1,1-2\gamma,-1] = \mathcal{K}(\beta,\gamma)$ and $0-\mathcal{UQ}[1-2\beta,-1,1-2\gamma,-1] = \mathcal{Q}(\beta,\gamma)$, well-known classes of close-to-convex and quasi-convex functions of order β type γ, see [19, 20].

v. $0-\mathcal{UK}[1,-1,1,-1] = \mathcal{K}$ and $0-\mathcal{UQ}[1,-1,1,-1] = \mathcal{Q}$, the classes of close-to-convex and quasi-convex functions; for details, see [21, 22].

Throughout this paper, we assume that $-1 \le D < C \le 1$, $-1 \le B < A \le 1$ and $k \ge 0$ unless otherwise specified.

2 A set of lemmas

To prove our main results, we need the following lemmas.

Lemma 2.1 ([23]) *Let* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \prec F(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ *in* E. *If* $F(z)$ *is univalent in* E *and* $F(E)$ *is convex, then*

$$|p_n| \le |d_1|, \quad n \ge 1.$$

Lemma 2.2 ([1]) *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k-\mathcal{P}[A,B]$. *Then*

$$|c_n| \le \bigl|\delta(A,B,k)\bigr|,$$

where

$$\delta(A,B,k) = \frac{(A-B)\delta_k}{2}, \tag{2.1}$$

and

$$\delta_k = \begin{cases} \frac{8(\cos^{-1}k)^2}{\pi^2(1-k^2)}, & 0 \le k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & k > 1. \end{cases} \tag{2.2}$$

Lemma 2.3 ([2]) *Let* f *and* g *be in the class* $\mathcal{C}$ *and* $\mathcal{S}^*$, *respectively. Then, for every function* $F(z)$ *analytic in* E *with* $F(0) = 1$, *we have*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\text{co}}\bigl(F(E)\bigr), \quad z \in E,$$

where "$*$" denotes the well-known convolution of two analytic functions and $\overline{\text{co}}F(E)$ *denotes the closed convex hull* $F(E)$.

Lemma 2.4 ([1]) *Let $g \in k\text{–}\mathcal{ST}[C,D]$ with $k \geq 0$ and be given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$|b_n| \leq \prod_{j=0}^{n-2} \frac{|(C-D)\delta_k - 2jD|}{2(j+1)},$$

where δ_k is defined by (2.2).

3 The main results and their consequences

This section is about the main results of our defined families $k\text{–}\mathcal{UK}[A,B,C,D]$ and $k - \mathcal{UQ}[A,B,C,D]$. These families will be thoroughly investigated by studying their important properties including coefficient inequalities, sufficient condition, necessary condition, arc length problem, the growth rate of coefficients, convolution preserving properties and the radius of convexity problem. We will also discuss some special cases of our main results.

1. *Coefficient inequalities*

Theorem 3.1 *Let $f \in k\text{–}\mathcal{UK}[A,B,C,D]$, and let it be of the form given by* (1.1). *Then, for $n \geq 2$, we have*

$$|a_n| \leq \frac{1}{n} \prod_{i=0}^{n-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)} + \frac{(A-B)|\delta_k|}{2n} \sum_{j=1}^{n-1} \prod_{i=0}^{j-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)},$$

where δ_k is defined by (2.2). *This result is not sharp.*

Proof Let us take

$$zf'(z) = g(z)p(z), \tag{3.1}$$

where $p \in k - \mathcal{P}[A,B]$ and $g \in k - \mathcal{ST}[C,D]$. Let $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then (3.1) becomes

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

Equating the coefficients of z^n on both sides, we have

$$na_n = b_n + \sum_{j=1}^{n-1} b_j c_{n-j}.$$

This implies that

$$n|a_n| \leq |b_n| + \sum_{j=1}^{n-1} |b_j||c_{n-j}|. \tag{3.2}$$

Since $p \in k - \mathcal{P}[A,B]$ and $g \in k - \mathcal{ST}[C,D]$, therefore by Lemma 2.2 and Lemma 2.4 we have

$$|b_n| \leq \prod_{i=0}^{n-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)}$$

and

$$|c_n| \leq \frac{1}{2}(A-B)|\delta_k|.$$

Hence (3.2) becomes

$$n|a_n| \leq \prod_{i=0}^{n-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)} + \sum_{j=1}^{n-1} \frac{(A-B)|\delta_k|}{2} \prod_{i=0}^{j-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)},$$

which implies that

$$|a_n| \leq \frac{1}{n} \prod_{i=0}^{n-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)} + \frac{(A-B)|\delta_k|}{2n} \sum_{j=1}^{n-1} \prod_{i=0}^{j-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)}.$$

This completes the proof. □

Corollary 3.2 ([9]) *Let* $f \in k - \mathcal{UK}[1,-1,1,-1]$ *which has the form* (1.1). *Then, for* $n \geq 2$,

$$|a_n| \leq \frac{(|\delta_k|)_{n-1}}{n!} + \frac{|\delta_k|}{n} \sum_{j=0}^{n-1} \frac{(|\delta_k|)_{j-1}}{(j-1)!}.$$

Corollary 3.3 *Let* $f \in k - \mathcal{UK}[1-2\beta,-1,1-2\gamma,-1] = f \in k - \mathcal{UK}[\beta,\gamma]$ *which has the form* (1.1). *Then, for* $n \geq 2$,

$$|a_n| \leq \frac{(|\delta_k|)_{n-1}}{n!} + \frac{|\delta_k|}{n} \sum_{j=0}^{n-1} \frac{(|\delta_k|)_{j-1}}{(j-1)!}.$$

Corollary 3.4 ([21]) *Let* $f \in 0 - \mathcal{UK}[1,-1,1,-1] = \mathcal{K}$ *which has the form* (1.1). *Then, for* $n \geq 2$,

$$|a_n| \leq n.$$

Using relation (1.5) and Theorem 3.1, we obtain immediately the following result.

Theorem 3.5 *Let* $f \in k - \mathcal{UQ}[A,B,C,D]$ *which has the form* (1.1). *Then, for* $n \geq 2$,

$$|a_n| \leq \frac{1}{n^2} \prod_{i=0}^{n-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)} + \frac{(A-B)|\delta_k|}{2n^2} \sum_{j=1}^{n-1} \prod_{i=0}^{j-2} \frac{|(C-D)\delta_k - 2iD|}{2(i+1)}.$$

By assigning different permissible values to the parameters, we obtain several known results, see [17, 21, 22].

2. *Sufficient conditions*

Theorem 3.6 *Let $f \in \mathcal{A}$ and be given by* (1.1). *Then $f \in k$–$\mathcal{UK}[A,B,C,D]$ if*

$$\sum_{n=2}^{\infty}\bigl[2(k+1)|b_n - na_n| + \bigl|(B+1)na_n - (A+1)b_n\bigr|\bigr] < |B-A|. \tag{3.3}$$

Proof Let us assume that equation (3.3) holds true. It is sufficient to show that

$$k\left|\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right| - \operatorname{Re}\left(\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right) < 1.$$

Now consider

$$\begin{aligned}
\left|\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right| &= \left|\frac{(B-1)zf'(z) - (A-1)g(z)}{(B+1)zf'(z) - (A+1)g(z)} - 1\right| \\
&= 2\left|\frac{g(z) - zf'(z)}{(B+1)zf'(z) - (A+1)g(z)}\right| \\
&= 2\left|\frac{\sum_{n=2}^{\infty}(b_n - na_n)z^n}{(B-A)z + \sum_{n=2}^{\infty}[(B+1)na_n - (A+1)b_n]z^n}\right| \\
&\leq \frac{2\sum_{n=2}^{\infty}|b_n - na_n|}{(B-A) - \sum_{n=2}^{\infty}|(B+1)na_n - (A+1)b_n|}.
\end{aligned}$$

Since

$$\begin{aligned}
&k\left|\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right| - \operatorname{Re}\left(\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right) \\
&\quad\leq (k+1)\left|\frac{(B-1)\frac{zf'(z)}{g(z)} - (A-1)}{(B+1)\frac{zf'(z)}{g(z)} - (A+1)} - 1\right| \leq \frac{2(k+1)\sum_{n=2}^{\infty}|b_n - na_n|}{(B-A) - \sum_{n=2}^{\infty}|(B+1)na_n - (A+1)b_n|}.
\end{aligned}$$

The last inequality is bounded by 1 if

$$2(k+1)\sum_{n=2}^{\infty}|b_n - na_n| \leq |B-A| - \sum_{n=2}^{\infty}\bigl|(B+1)na_n - (A+1)b_n\bigr|.$$

Hence we have

$$\sum_{n=2}^{\infty}\bigl[2(k+1)|b_n - na_n| + \bigl|(B+1)na_n - (A+1)b_n\bigr|\bigr] \leq |B-A|.$$

This completes the proof. □

Theorem 3.7 *Let $f \in \mathcal{A}$ which has the form* (1.1). *Then $f \in k-\mathcal{UQ}[A,B,C,D]$ if*

$$\sum_{n=2}^{\infty} n\bigl[2(k+1)|b_n - na_n| + \bigl|(B+1)na_n - (A+1)b_n\bigr|\bigr] < |B-A|.$$

The proof follows immediately by using Theorem 3.6 and relation (1.5).

Corollary 3.8 ([24]) *A function is said to be in the class $1-\mathcal{UQ}[1-2\beta,-1,1,-1] = \mathcal{UQ}(\beta)$ for $g(z) = z$ if*

$$\sum_{n=2}^{\infty} n^2|a_n| < \frac{1-\beta}{2}.$$

3. *Necessary condition*

Theorem 3.9 *Let $f \in k-\mathcal{UK}[A,B,C,D]$. Then, for $\theta_1 < \theta_2$, $z \in E$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\left[\frac{C-D}{2k+1-D} + \lambda\right]\pi,$$

where λ is defined by (1.4).

Proof Since $f \in k-\mathcal{UK}[A,B,C,D]$, there exists $g \in k-\mathcal{CV}[C,D] \subset \mathcal{C}(\beta_1)$,

$$\beta_1 = \frac{2k+1-C}{2k+1-D} \tag{3.4}$$

such that

$$f'(z) = g'(z)p(z),$$

where $p \in k-\mathcal{P}[A,B]$. We can write

$$f'(z) = \bigl(g_1'(z)\bigr)^{1-\beta_1} h^{\lambda}(z), \quad h \in \mathcal{P}[A,B] \subset \mathcal{P}, g_1 \in \mathcal{C}.$$

For $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta_1 \le \theta_2 \le 2\pi$, we have

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta &= (1-\beta_1)\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\theta + \beta_1(\theta_2-\theta_1) \\ &\quad + \lambda\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} d\theta. \end{aligned}\tag{3.5}$$

Also, we observe that, for $h \in \mathcal{P}[A,B]$,

$$\begin{aligned}\frac{\partial}{\partial\theta} \arg h\bigl(re^{i\theta}\bigr) &= \frac{\partial}{\partial\theta} \operatorname{Re}\bigl\{-i\ln h\bigl(re^{i\theta}\bigr)\bigr\} \\ &= \operatorname{Re}\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}.\end{aligned}$$

Therefore

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\} d\theta = \arg h\big(re^{i\theta_2}\big) - \arg h\big(re^{i\theta_1}\big),$$

this implies that

$$\max_{h\in P[A,B]}\left|\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\} d\theta\right| = \max_{h\in P[A,B]}\left|\arg h\big(re^{i\theta_2}\big) - \arg h\big(re^{i\theta_1}\big)\right|. \tag{3.6}$$

Since $h \in \mathcal{P}[A,B]$, so

$$\left|h(z) - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}.$$

From (3.6), we observe that

$$\begin{aligned}\max_{h\in P[A,B]}\left|\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\} d\theta\right| &\leq 2\sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right)\\ &\leq \pi - 2\cos^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right).\end{aligned} \tag{3.7}$$

Also, for $g_1 \in \mathcal{C}$, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\theta > -\pi.$$

Using (3.6) and (3.7) in (3.5), we obtain

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta &> -(1-\beta_1)\pi + \beta_1(\theta_2-\theta_1) - \lambda\pi + 2\lambda\cos^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right)\\ &> -\left[\frac{(C-D)}{2k+1-D} + \lambda\right]\pi\,(r\to 1).\end{aligned}$$

This completes the required result. □

Remark 3.10 Let $f \in k-\mathcal{UK}[A,B,C,D]$. Then, for $\frac{C-D}{2k+1-D} < 1-\lambda$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\pi,$$

and hence f is univalent in E, see [21].

Corollary 3.11 ([21]) *Let $f \in 0-\mathcal{UK}[1,-1,1,-1]$. Then, for $\theta_1 < \theta_2$, $z \in E$,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\pi.$$

4. *Arc length problem*

Theorem 3.12 *Let* $f \in k-\mathcal{UK}[A,B,C,D]$ *which has the form* (1.1). *Then*

$$\mathcal{L}_r(f) \le C(\lambda, C, D)\left(\frac{1}{1-r}\right)^{2(1-\beta_1)-1} \quad (r \to 1),$$

where β_1 *is defined by* (3.4) *and* $C(\lambda, A, B)$ *is a constant depending upon* λ, A *and* B.

Proof Let

$$zf'(z) = g(z)h^{\lambda}(z), \tag{3.8}$$

where $g \in k-\mathcal{ST}[C,D]$ and $h \in \mathcal{P}[A,B] \subset \mathcal{P}$. Since $k-\mathcal{ST}[C,D] \subseteq \mathcal{S}^*(\beta_1)$, see [1]. We can write

$$g(z) = z\left(\frac{g_1(z)}{z}\right)^{1-\beta_1}, \quad g_1 \in \mathcal{S}^*.$$

Equation (3.8) gives

$$zf'(z) = z^{\beta_1}\big(g_1(z)\big)^{1-\beta_1}h^{\lambda}(z).$$

Now, for $z = re^{i\theta}$,

$$\mathcal{L}_r(f) = \int_0^{2\pi} \left|zf'(z)\right| d\theta = \int_0^{2\pi} \left|z^{\beta_1}\big(g_1(z)\big)^{1-\beta_1}h^{\lambda}(z)\right| d\theta.$$

Using Holder's inequality, we have

$$\mathcal{L}_r(f) \le 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\big(g_1(z)\big)^{(1-\beta_1)(\frac{2}{2-\lambda})}\right| d\theta\right)^{\frac{2-\lambda}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h(z)\right|^2 d\theta\right)^{\frac{\lambda}{2}}. \tag{3.9}$$

Since $h \in \mathcal{P}[A,B] \subset \mathcal{P}$, so

$$\frac{1}{2\pi}\int_0^{2\pi}\left|h(z)\right|^2 d\theta \le \frac{1+\{(A-B)^2-1\}r^2}{1-r^2}. \tag{3.10}$$

Using (3.10) and the distortion result for a starlike function in (3.9), we obtain

$$\begin{aligned}\mathcal{L}_r(f) &\le 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{r^{(1-\beta_1)(\frac{2}{2-\lambda})}}{|1-re^{i\theta}|^{\frac{4(1-\beta_1)}{2-\lambda}}}\, d\theta\right)^{\frac{2-\lambda}{2}}\left(\frac{1+\{(A-B)^2-1\}r^2}{1-r^2}\right)^{\frac{\lambda}{2}} \\ &\le C(\lambda, A, B)\left(\frac{1}{1-r}\right)^{2(1-\beta_1)+\lambda-1} \quad (r \to 1),\end{aligned}$$

where $C(\lambda, A, B) = \pi^{\frac{\lambda}{2}}(A-B)^{\lambda}$ and $2(1-\beta_1)+\lambda > 1$. This completes the proof. □

Corollary 3.13 ([25]) *Let* $f \in 0-\mathcal{UK}[1,-1,1,-1]$. *Then, for* $0 < r < 1$,

$$\mathcal{L}_r(f) \leq \mathcal{O}\left(\mathcal{M}(r)\log\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

where $\mathcal{O}$*-notation denotes that the constant is absolute and* $\mathcal{M}(r) = \max_{|z|=r}|f(z)|$.

5. *Growth rate of coefficients*

Theorem 3.14 *Let* $f \in k-\mathcal{UK}[A,B,C,D]$ *which has the form* (1.1). *Then, for* $k \geq 0$, *we have*

$$|a_n| \leq C(\lambda, A, B)n^{2(1-\beta_1)+\lambda-2},$$

where β_1 *is defined by* (3.4).

Proof From Cauchy's theorem with $z = re^{i\theta}$, one can easily have

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}\,d\theta.$$

This implies that

$$\begin{aligned}|na_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi}\left|zf'(z)\right|d\theta \\ &= \frac{1}{2\pi r^n}\mathcal{L}_r(f).\end{aligned}$$

Using Theorem 3.12 and putting $r = 1 - \frac{1}{n}$, we obtain the required result. □

6. *Convolution properties*

Theorem 3.15 *If* $f \in k-\mathcal{ST}[C,D]$ *and* $\varphi \in \mathcal{C}$, *then* $\varphi * f \in k-\mathcal{ST}[C,D]$.

Proof To prove the result, we need to prove

$$\frac{z(\varphi(z) * f(z))'}{\varphi(z) * f(z)} \in k-\mathcal{P}[A,B].$$

Consider

$$\begin{aligned}\frac{z(\varphi(z) * f(z))'}{\varphi(z) * f(z)} &= \frac{\varphi(z) * \frac{zf'(z)}{f(z)}f(z)}{\varphi(z) * f(z)} \\ &= \frac{\varphi(z) * \Psi(z)f(z)}{\varphi(z) * f(z)},\end{aligned}$$

where $\frac{zf'(z)}{f(z)} = \Psi(z) \in k-\mathcal{P}[C,D]$. Applying Lemma 2.3, we obtain the required result. □

Theorem 3.16 *Let* $f \in k-\mathcal{UK}[A,B,C,D]$ *and* $\varphi \in \mathcal{C}$. *Then* $\varphi * f \in k-\mathcal{UK}[A,B,C,D]$.

Proof Since $f \in k-\mathcal{UK}[A,B,C,D]$, there exists $g \in k-\mathcal{ST}[C,D]$ such that $\frac{zf'(z)}{g(z)} \in k-\mathcal{P}[A,B]$. It follows from Lemma 2.3 that $\varphi * g \in k-\mathcal{ST}[C,D]$. Now

$$\begin{aligned}\frac{z(\varphi(z)*f(z))'}{\varphi(z)*g(z)} &= \frac{\varphi(z)*zf'(z)}{\varphi(z)*g(z)} = \frac{\varphi(z)*\frac{zf'(z)}{g(z)}g(z)}{\varphi(z)*g(z)}\\ &= \frac{\varphi(z)*F(z)g(z)}{\varphi(z)*g(z)},\end{aligned}$$

where $F(z) \in k-\mathcal{P}[A,B]$. Applying Lemma 2.3, we have $\frac{z(\varphi(z)*f(z))'}{\varphi(z)*g(z)} \in k-\mathcal{UK}[A,B,C,D]$ for $z \in E$. □

7. *Radius of convexity problem*

Theorem 3.17 *Let $f \in k-\mathcal{UK}[A,B,C,D]$ in E. Then $f \in \mathcal{C}$ for $|z| < r_1$, where*

$$r_1 = \frac{2}{2(1-\beta_1)+\lambda(A-B)+\sqrt{([2(1-\beta_1)+\lambda(A-B)]^2+4[2\beta_1-1])}}, \tag{3.11}$$

where λ and β_1 are defined by (1.4) *and* (3.4), *respectively.*

Proof Let

$$zf'(z) = g(z)p(z),$$

where $g \in k-\mathcal{ST}[C,D]$ and $p \in k-\mathcal{P}[A,B]$. Since $k-\mathcal{ST}[C,D] \subset \mathcal{S}^*(\beta_1)$, it is known [26] that there exists $g_1 \in \mathcal{S}^*$ such that

$$g(z) = z\left(\frac{g_1(z)}{z}\right)^{1-\beta_1}.$$

We can write

$$zf'(z) = z^{\beta_1}\big(g_1(z)\big)^{1-\beta_1}h^{\lambda}(z), \quad h \in \mathcal{P}[A,B] \subset \mathcal{P}. \tag{3.12}$$

The logarithmic differentiation of (3.12) yields

$$\frac{(zf'(z))'}{f'(z)} = \beta_1 + (1-\beta_1)\frac{zg_1'(z)}{g_1(z)} + \lambda\frac{zh'(z)}{h(z)}.$$

Using the distortion result for the classes $\mathcal{S}^*$ and $\mathcal{P}[A,B]$, we obtain

$$\begin{aligned}\operatorname{Re}\frac{(zf'(z))'}{f'(z)} &\geq \beta_1 + (1-\beta_1)\frac{1-r}{1+r} - \frac{\lambda(A-B)r}{1-r^2}\\ &\geq \frac{1+(1-2\beta_1)r^2-[2(1-\beta_1)+\lambda(A-B)]r}{1-r^2}.\end{aligned} \tag{3.13}$$

The right-hand side of (3.13) is positive for $|z| < r_1$, where r_1 is given by (3.11). □

We note the following cases:

(i). For $A = 1$, $B = -1$, $C = 1$ and $D = -1$, we obtain the radius of convexity problem for the class $k-\mathcal{UK}$.

(ii). For $k = 0$, we have the radius of convexity for the class $\mathcal{K}[A,B,C,D]$.

(iii). For $A = 1$, $B = -1$, $C = 1$, $D = -1$ and $k = 0$, we have the radius of convexity problem for the well-known class of close-to-convex functions studied by Kaplan [21].

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The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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