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Quantitative-Voronovskaya and Grüss-Voronovskaya type theorems for Szász-Durrmeyer type operators blended with multiple Appell polynomials

Trapti Neer* and Purshottam Narain Agrawal

*Correspondence:
triptineeriitr@gmail.com
Department of Mathematics, Indian
Institute of Technology Roorkee,
Roorkee, 247667, India

Abstract

In this paper, we establish a link between the Szász-Durrmeyer type operators and multiple Appell polynomials. We study a quantitative-Voronovskaya type theorem in terms of weighted modulus of smoothness using sixth order central moment and Grüss-Voronovskaya type theorem. We also establish a local approximation theorem by means of the Steklov means in terms of the first and the second order modulus of continuity and Voronovskaya type asymptotic theorem. Further, we discuss the degree of approximation by means of the weighted spaces. Lastly, we find the rate of approximation of functions having a derivative of bounded variation.

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1 Introduction

For $f \in C(R_0^+)$ and $x \in R_0^+$ ($R_0^+ = [0, \infty)$), Szász [1] introduced the well-known operators

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad (1)$$

such that $S_n(|f|; x) < \infty$. Several generalizations of Szász operators have been introduced in the literature and authors have studied their approximation properties. In [2], the author considered Baskakov-Szász type operators and studied the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation. In [3], the authors introduced the q -Baskakov-Durrmeyer type operators and studied the rate of convergence and the weighted approximation properties. In [4] the authors proposed the β -operators based on q -integers and established some direct theorems by means of modulus of continuity and also studied the weighted approximation and better approximation using King type approach. For exhaustive literature on approximation by linear positive operators one can refer to [5–7] and the references therein.

Now let us recall some results on multiple Appell polynomials [8]. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 1$, be an analytic function in the disc $|z| \leq r$, $r > 1$ and $p_k(x)$ be the Appell polynomials having the generating function $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$, with $g(1) \neq 0$ and $p_k(x) \geq 0$, $\forall x \in R_0^+$.

Jakimovski and Leviatan [9] proposed a generalization of Szász-Mirakjan operators by means of the Appell polynomials as follows:

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{2}$$

For $g(u) = 1$, these operators reduce to Szász-Mirakjan operators (1).

A set of polynomials $\{p_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$ with degree $k_1 + k_2$ for $k_1, k_2 \geq 0$ is called multiple polynomial system (multiple PS) and a multiple PS is called multiple Appell if it is generated by the relation

$$A(t_1, t_2)e^{x(t_1+t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1, k_2}(x)}{k_1!k_2!} t_1^{k_1} t_2^{k_2}, \tag{3}$$

where A is given by

$$A(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{a_{k_1, k_2}}{k_1!k_2!} t_1^{k_1} t_2^{k_2}, \tag{4}$$

with $A(0, 0) = a_{0,0} \neq 0$.

Theorem 1.1 For multiple PS, $\{p_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$, the following statements are equivalent:

- (a) $\{p_{k_1, k_2}(x)\}_{k_1, k_2=0}^{\infty}$ is a set of multiple Appell polynomials.
- (b) There exists a sequence $\{a_{k_1, k_2}\}_{k_1, k_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$p_{k_1, k_2}(x) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \binom{k_1}{r_1} \binom{k_2}{r_2} a_{k_1-r_1, k_2-r_2} x^{r_1+r_2}.$$

- (c) For every $k_1 + k_2 \geq 1$, we have

$$p'_{k_1, k_2}(x) = k_1 p_{k_1-1, k_2}(x) + k_2 p_{k_1, k_2-1}(x).$$

Varma [10] defined a sequence of linear positive operators for any $f \in C(R_0^+)$, by

$$K_n(f; x) = \frac{e^{-nx}}{A(1, 1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1, k_2}(\frac{nx}{2})}{k_1!k_2!} f\left(\frac{k_1 + k_2}{n}\right), \tag{5}$$

provided $A(1, 1) \neq 0$, $\frac{a_{k_1, k_2}}{A(1, 1)} \geq 0$ for $k_1, k_2 \in \mathbb{N}$, and the series (3) and (4) converge for $|t_1| < R_1$, $|t_2| < R_2$ ($R_1, R_2 > 1$), respectively.

For $\alpha > 0$, $\rho > 0$ and $f : R_0^+ \rightarrow \mathbb{R}$, being integrable function, Păltănea [11] defined a modification of the Szász operators by

$$L_{\alpha}^{\rho}(f; x) = \sum_{k=1}^{\infty} s_{\alpha, k}(x) \int_0^{\infty} \frac{\alpha \rho e^{-\alpha \rho t} (\alpha \rho t)^{(k)\rho-1}}{\Gamma(k)\rho} f(t) dt + e^{-\alpha x} f(0), \quad x \in R_0^+. \tag{6}$$

Motivated by [11], for $f \in C_E(\mathbb{R}_0^+)$, the space of all continuous functions satisfying $|f(t)| \leq Ke^{at}$ ($t \geq 0$) for some positive constant K and a , we propose an approximation method by linking the operators (6) and the multiple Appell polynomials by

$$\begin{aligned} \mathcal{L}_n^\rho(f; x) &= \frac{e^{-nx}}{A(1,1)} \sum_{\substack{k_1 \\ k_2 \\ k_1+k_2 \geq 1}} \sum_{k_2} \frac{p_{k_1, k_2}(\frac{nx}{2})}{k_1!k_2!} \int_0^\infty \frac{n\rho e^{-n\rho t} (n\rho t)^{(k_1+k_2)\rho-1}}{\Gamma(k_1+k_2)\rho} f(t) dt \\ &+ \frac{e^{-nx}}{A(1,1)} p_{0,0}\left(\frac{nx}{2}\right) f(0), \end{aligned}$$

and establish a quantitative Voronovskaya type theorem, a Grüss Voronovskaya type theorem, a local approximation theorem by means of the Steklov mean, a Voronovskaya type asymptotic theorem and error estimates for several weighted spaces. Lastly, we study the rate of convergence of functions having a derivative of bounded variation.

2 Basic results

In order to prove the main results of the paper, we shall need the following auxiliary results.

Lemma 2.1 For $K_n(t^i; x)$, $i = 0, 1, 2, 3, 4$, we have

- (i) $K_n(1; x) = 1$,
- (ii) $K_n(t; x) = x + \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{nA(1, 1)}$,
- (iii) $K_n(t^2; x) = x^2 + \frac{x}{n} \left\{ 1 + \frac{2}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\}$
 $+ \frac{1}{n^2 A(1, 1)} \{ A_{t_1}(1, 1) + A_{t_2}(1, 1) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \}$,
- (iv) $K_n(t^3; x) = x^3 + \frac{3x^2}{n} \left\{ 1 + \frac{1}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\} + \frac{x}{n^2} \left\{ 1 \right.$
 $+ \frac{3}{A(1, 1)} (2A_{t_1}(1, 1) + 2A_{t_2}(1, 1) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1)) \left. \right\}$
 $+ \frac{1}{n^3 A(1, 1)} \{ A_{t_1}(1, 1) + A_{t_2}(1, 1) + 3A_{t_1 t_1}(1, 1) + 6A_{t_1 t_2}(1, 1)$
 $+ 3A_{t_2 t_2}(1, 1) + A_{t_1 t_1 t_1}(1, 1) + A_{t_2 t_2 t_2}(1, 1) + 3A_{t_1 t_1 t_2}(1, 1) + 3A_{t_2 t_2 t_1}(1, 1) \}$,
- (v) $K_n(t^4; x) = x^4 + \frac{x^3}{n} \left\{ 6 + \frac{4}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\} + \frac{x^2}{n^2} \left\{ 7 \right.$
 $+ \frac{6}{A(1, 1)} (3A_{t_1}(1, 1) + 3A_{t_2}(1, 1) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1)) \left. \right\}$
 $+ \frac{x}{n^3} \left\{ 1 + \frac{1}{A(1, 1)} (14A_{t_1}(1, 1) + 14A_{t_2}(1, 1) + 18A_{t_1 t_1}(1, 1) + 36A_{t_1 t_2}(1, 1)$
 $+ 18A_{t_2 t_2}(1, 1) + 4A_{t_1 t_1 t_1}(1, 1) + 4A_{t_2 t_2 t_2}(1, 1) + 12A_{t_1 t_1 t_2}(1, 1)$
 $+ 12A_{t_2 t_2 t_1}(1, 1)) \right\} + \frac{1}{n^4 A(1, 1)} \{ A_{t_1}(1, 1) + A_{t_2}(1, 1) + 7A_{t_1 t_1}(1, 1)$
 $+ 14A_{t_1 t_2}(1, 1) + 7A_{t_2 t_2}(1, 1) + 6A_{t_1 t_1 t_1}(1, 1) + 6A_{t_2 t_2 t_2}(1, 1)$

$$\begin{aligned}
 &+ 18A_{t_1 t_1 t_2}(1, 1) + 18A_{t_2 t_2 t_1}(1, 1) + A_{t_1 t_1 t_1 t_1}(1, 1) + A_{t_2 t_2 t_2 t_2}(1, 1) \\
 &+ 4A_{t_1 t_1 t_1 t_2}(1, 1) + 4A_{t_2 t_2 t_2 t_1}(1, 1) + 6A_{t_1 t_1 t_2 t_2}(1, 1) \}.
 \end{aligned}$$

The values of the moments $K_n(t^i; x)$ for $i = 0, 1, 2$ are given in [10] while the values of $K_n(t^i; x)$ for $i = 3, 4$ have been obtained by us after simple calculations and hence the details are omitted.

Lemma 2.2 For the sequence of linear positive operators $\mathcal{L}_n^\rho(t^i; x)$, $i = 0, 1, 2, 3, 4$, we find

- (i) $\mathcal{L}_n^\rho(1; x) = 1$,
- (ii) $\mathcal{L}_n^\rho(t; x) = x + \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{nA(1, 1)}$,
- (iii) $\mathcal{L}_n^\rho(t^2; x) = x^2 + \frac{x}{n} \left\{ \left(1 + \frac{1}{\rho} \right) + \frac{2}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\} + \frac{1}{n^2 A(1, 1)} \left\{ \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \right\}$,
- (iv) $\mathcal{L}_n^\rho(t^3; x) = x^3 + \frac{3x^2}{n} \left\{ \left(1 + \frac{1}{\rho} \right) + \frac{1}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\} + \frac{x}{n^2} \left\{ \left(1 + \frac{3}{\rho} + \frac{2}{\rho^2} \right) + \frac{3}{A(1, 1)} \left(2 \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \right) \right\} + \frac{1}{n^3 A(1, 1)} \left\{ \left(1 + \frac{3}{\rho} + \frac{2}{\rho^2} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) + 3 \left(1 + \frac{1}{\rho} \right) (A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1)) + A_{t_1 t_1 t_1}(1, 1) + A_{t_2 t_2 t_2}(1, 1) + 3A_{t_1 t_1 t_2}(1, 1) + 3A_{t_2 t_2 t_1}(1, 1) \right\}$,
- (v) $\mathcal{L}_n^\rho(t^4; x) = x^4 + \frac{x^3}{n} \left\{ 6 \left(1 + \frac{1}{\rho} \right) + \frac{4}{A(1, 1)} (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right\} + \frac{x^2}{n^2} \left\{ \left(7 + \frac{18}{\rho} + \frac{11}{\rho^2} \right) + \frac{6}{A(1, 1)} \left(3 \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \right) \right\} + \frac{x}{n^3} \left\{ \left(1 + \frac{6}{\rho} + \frac{11}{\rho^2} + \frac{6}{\rho^3} \right) + \frac{1}{A(1, 1)} \left(\left(14 + \frac{36}{\rho} + \frac{22}{\rho^2} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) + 18 \left(1 + \frac{1}{\rho} \right) (A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1)) + 4A_{t_1 t_1 t_1}(1, 1) + 4A_{t_2 t_2 t_2}(1, 1) + 12A_{t_1 t_1 t_2}(1, 1) + 12A_{t_2 t_2 t_1}(1, 1) \right) \right\}$

$$\begin{aligned}
 & + \frac{1}{n^4 A(1,1)} \left\{ \left(1 + \frac{6}{\rho} + \frac{11}{\rho^2} + \frac{6}{\rho^3} \right) (A_{t_1}(1,1) + A_{t_2}(1,1)) \right. \\
 & + \left(7 + \frac{18}{\rho} + \frac{11}{\rho^2} \right) (A_{t_1 t_1}(1,1) + 2A_{t_1 t_2}(1,1) + A_{t_2 t_2}(1,1)) \\
 & + 6 \left(1 + \frac{1}{\rho} \right) (A_{t_1 t_1 t_1}(1,1) + A_{t_2 t_2 t_2}(1,1) + 3A_{t_1 t_1 t_2}(1,1) + 3A_{t_2 t_2 t_1}(1,1)) \\
 & + A_{t_1 t_1 t_1 t_1}(1,1) + A_{t_2 t_2 t_2 t_2}(1,1) + 4A_{t_1 t_1 t_1 t_2}(1,1) \\
 & \left. + 4A_{t_2 t_2 t_2 t_1}(1,1) + 6A_{t_1 t_1 t_2 t_2}(1,1) \right\}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \mathcal{L}_n^\rho((t-x)^2; x) & = \frac{x}{n} \left(1 + \frac{1}{\rho} \right) + \frac{1}{n^2 A(1,1)} \left\{ \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1,1) + A_{t_2}(1,1)) \right. \\
 & \quad \left. + A_{t_1 t_1}(1,1) + 2A_{t_1 t_2}(1,1) + A_{t_2 t_2}(1,1) \right\} \\
 & \leq \frac{C}{n} \left(1 + \frac{1}{\rho} \right) (1+x) \\
 & = \delta_{n,\rho}^2(x) \quad (\text{say}), \tag{7}
 \end{aligned}$$

where

$$C = \max \left(1, \frac{|A_{t_1}(1,1)| + |A_{t_2}(1,1)| + |A_{t_1 t_1}(1,1)| + 2|A_{t_1 t_2}(1,1)| + |A_{t_2 t_2}(1,1)|}{|A(1,1)|} \right)$$

and

$$\begin{aligned}
 \mathcal{L}_n^\rho((t-x)^4; x) & = \frac{3x^2}{n^2} \left\{ 1 + \frac{2}{\rho} + \frac{1}{\rho^2} \right\} + \frac{x}{n^3} \left[\left(1 + \frac{6}{\rho} + \frac{11}{\rho^2} + \frac{6}{\rho^3} \right) \right. \\
 & \quad + \frac{1}{A(1,1)} \left\{ \left(13 + \frac{33}{\rho} + \frac{20}{\rho^2} \right) (A_{t_1}(1,1) + A_{t_2}(1,1)) \right. \\
 & \quad + 6 \left(1 + \frac{1}{\rho} \right) (A_{t_1 t_1}(1,1) + 2A_{t_1 t_2}(1,1) + A_{t_2 t_2}(1,1)) \\
 & \quad \left. \left. - 6A_{t_1 t_1 t_1}(1,1) - 6A_{t_2 t_2 t_2}(1,1) \right\} \right].
 \end{aligned}$$

The expression for $\mathcal{L}_n^\rho((t-x)^6; x)$ has not been included in Lemma 2.2 because it is very lengthy and complicated. It will be required to prove the quantitative Voronovskaya type theorem.

Remark 2.3 From Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} n \mathcal{L}_n^\rho((t-x); x) = \frac{A_{t_1}(1,1) + A_{t_2}(1,1)}{A(1,1)}, \tag{8}$$

$$\lim_{n \rightarrow \infty} n \mathcal{L}_n^\rho((t-x)^2; x) = x \left(1 + \frac{1}{\rho} \right), \tag{9}$$

$$\lim_{n \rightarrow \infty} n^2 \mathcal{L}_n^\rho((t-x)^4; x) = 3x^2 \left(1 + \frac{2}{\rho} + \frac{1}{\rho^2} \right), \tag{10}$$

$$\lim_{n \rightarrow \infty} n^3 \mathcal{L}_n^\rho((t-x)^6; x) = 15x^3 \left(1 + \frac{3}{\rho} + \frac{3}{\rho^2} + \frac{1}{\rho^3} \right). \tag{11}$$

3 Main results

Theorem 3.1 *Let $f \in C_E(R_0^+)$. Then $\lim_{n \rightarrow \infty} \mathcal{L}_n^\rho(f; x) = f(x)$ uniformly on each compact subset of R_0^+ .*

Proof Considering Lemma 2.2, it follows that $\lim_{n \rightarrow \infty} \mathcal{L}_n^\rho(t^i; x) = x^i$, $i = 0, 1, 2$, uniformly on every compact subset of R_0^+ . Applying the Bohman Korovkin theorem, we obtain the desired result. □

For $f \in C_B(R_0^+)$, the space of bounded and continuous functions on R_0^+ endowed with the norm $\|f\| = \sup_{x \in R_0^+} |f(x)|$, the first and second order modulus of continuity are, respectively, defined as

$$\begin{aligned} \omega(f; \delta) &= \sup_{0 < |h| \leq \delta} \sup_{x, x+h \in R_0^+} |f(x+h) - f(x)|, \\ \omega_2(f; \delta) &= \sup_{0 < |h| \leq \delta} \sup_{x, x+h, x+2h \in R_0^+} |f(x+2h) - 2f(x+h) + f(x)|, \quad \delta > 0. \end{aligned}$$

Further, for $f \in C_B(R_0^+)$, the Steklov mean of second order [12] is defined as

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))] du dv, \quad h > 0. \tag{12}$$

Hence

$$\begin{aligned} f(x) - f_h(x) &= \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \Delta_{u+v}^2 f(x) du dv, \quad \text{and} \\ f_h''(x) &= \frac{1}{h^2} (8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)). \end{aligned}$$

Thus, it follows that

$$\|f_h - f\| \leq \omega_2(f, h). \tag{13}$$

Further, $f_h', f_h'' \in C_B(R_0^+)$ and

$$\|f_h'\| \leq \frac{5}{h} \omega(f, h), \quad \|f_h''\| \leq \frac{9}{h^2} \omega_2(f, h). \tag{14}$$

Theorem 3.2 *For $f \in C_B(R_0^+)$ and $x \in R_0^+$, we have*

$$|\mathcal{L}_n^\rho(f; x) - f(x)| \leq 5\omega(f; \delta_{n,\rho}(x)) + \frac{13}{2}\omega_2(f; \delta_{n,\rho}(x)),$$

where $\delta_{n,\rho}(x)$ is defined by equation (7).

Proof Using the Steklov mean f_h defined by (12), we may write

$$\begin{aligned} |\mathcal{L}_n^\rho(f; x) - f(x)| &\leq |\mathcal{L}_n^\rho(f - f_h; x)| + |\mathcal{L}_n^\rho(f_h(t) - f_h(x); x)| \\ &\quad + |f_h(x) - f(x)|. \end{aligned} \tag{15}$$

Applying Lemma 2.2, we have

$$\|\mathcal{L}_n^\rho(f)\| \leq \|f\|, \tag{16}$$

Using inequality (16) and equation (13), we have

$$\begin{aligned} |\mathcal{L}_n^\rho(f - f_h; x)| &\leq \|f - f_h\| \\ &\leq \omega_2(f, h). \end{aligned}$$

Now by Taylor’s expansion and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\mathcal{L}_n^\rho(f_h(t) - f_h(x); x)| &\leq |\mathcal{L}_n^\rho((t - x)f'_h(x); x)| + \left| \mathcal{L}_n^\rho\left(\int_x^t (t - u)f''_h(u) du; x\right) \right| \\ &\leq \|f'_h\| |\mathcal{L}_n^\rho(|t - x|; x)| + \|f''_h\| \mathcal{L}_n^\rho\left(\left|\int_x^t |t - u| du\right|; x\right) \\ &= \|f'_h\| \sqrt{\mathcal{L}_n^\rho((t - x)^2; x)} + \frac{1}{2} \|f''_h\| \mathcal{L}_n^\rho((t - x)^2; x). \end{aligned}$$

Applying Lemma 2.2, equations (13), (14) and choosing h as $\delta_{n,\rho}(x)$, we get the required result. □

Theorem 3.3 For $f \in C_E^2(\mathbb{R}_0^+)$, we obtain

$$\lim_{n \rightarrow \infty} n[\mathcal{L}_n^\rho(f; x) - f(x)] = \frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{A(1, 1)} f'(x) + \frac{x}{2} \left(1 + \frac{1}{\rho}\right) f''(x),$$

uniformly in $x \in [0, a]$, $a > 0$.

Proof By Taylor’s expansion of f for some fixed $x \in [0, a]$, we obtain

$$f(t) - f(x) = (t - x)f'(x) + \frac{1}{2}(t - x)^2 f''(x) + \xi(t, x)(t - x)^2, \tag{17}$$

where $\xi(t, x) \in C_E(\mathbb{R}_0^+)$ and $\lim_{t \rightarrow x} \xi(t, x) = 0$.

Hence by linearity of the operators \mathcal{L}_n^ρ , from equation (17), we get

$$\begin{aligned} n[\mathcal{L}_n^\rho(f; x) - f(x)] &= n\mathcal{L}_n^\rho(t - x; x)f'(x) + \frac{1}{2}n\mathcal{L}_n^\rho((t - x)^2; x)f''(x) \\ &\quad + n\mathcal{L}_n^\rho(\xi(t, x)(t - x)^2; x). \end{aligned} \tag{18}$$

Applying the Cauchy-Schwarz inequality in the last term of equation (18), we have

$$n\mathcal{L}_n^\rho(\xi(t, x)(t - x)^2; x) \leq \sqrt{n^2 \mathcal{L}_n^\rho((t - x)^4; x) \mathcal{L}_n^\rho(\xi^2(t, x); x)}. \tag{19}$$

From Remark 2.3, it follows that

$$\lim_{n \rightarrow \infty} n^2 \mathcal{L}_n^\rho((t-x)^4; x) = 3x^2 \left(1 + \frac{2}{\rho} + \frac{1}{\rho^2} \right), \tag{20}$$

uniformly in $x \in [0, a]$.

Further, let $\xi^2(t, x) = v(t, x)$, $x \geq 0$, then $v(t, x) \in C_E(R_0^+)$ and hence from Theorem 1.1, we get

$$\lim_{n \rightarrow \infty} \mathcal{L}_n^\rho(\xi^2(t, x); x) = \lim_{n \rightarrow \infty} \mathcal{L}_n^\rho(v(t, x); x) = v(x, x) = 0, \tag{21}$$

Hence from equation (19), we obtain

$$\lim_{n \rightarrow \infty} (n \mathcal{L}_n^\rho(\xi(t, x)(t-x)^2; x)) = 0, \tag{22}$$

uniformly in $x \in [0, a]$. Now taking the limit $n \rightarrow \infty$ in (18) and using Remark 2.3, we get the desired result. This completes the proof. \square

4 Weighted approximation

Let $\theta(x) \geq 1$ be a weight function on R_0^+ . We consider the weighted space defined on R_0^+ :

$$B_\theta(R_0^+) := \{f : |f(x)| \leq M_f \theta(x), \forall x \in R_0^+ \text{ and } M_f > 0\}.$$

Further, let

$$C_\theta(R_0^+) := \{f \in B_\theta(R_0^+) : f \text{ is a continuous function on } R_0^+\}$$

and

$$C_\theta^*(R_0^+) := \left\{ f \in C_\theta(R_0^+) : \lim_{x \rightarrow \infty} \frac{f(x)}{\theta(x)} = K_f < \infty \right\}.$$

We define the norm in the space $B_\theta(R_0^+)$ as

$$\|f\|_\theta = \sup_{x \in R_0^+} \frac{|f(x)|}{\theta(x)}.$$

The usual modulus of continuity of the function f on $[0, p]$ is defined as

$$\omega_p(f; \delta) = \sup_{|t-x| \leq \delta} \sup_{t, x \in [0, p]} |f(t) - f(x)|. \tag{23}$$

Let us denote $\|\cdot\|_{C[a, b]}$ as the supremum norm on $[a, b]$. Throughout the paper we have taken $\theta(x) = 1 + x^2$.

Theorem 4.1 For $x \in [0, c]$ and $f \in C_\theta(R_0^+)$, we have

$$\|\mathcal{L}_n^\rho(f; \cdot) - f\|_{C[0, c]} \leq 4M_f(1 + c^2)\eta_{n, \rho}^2 + 2\omega_{c+1}(f; \eta_{n, \rho}),$$

where $\eta_{n, \rho}^2 = \max_{x \in [0, c]} (\mathcal{L}_n^\rho((t-x)^2; x))$.

Proof Let $x \in [0, c]$ and $t > c + 1$ then $t - x > 1$. Then, for $f \in C_\theta(R_0^+)$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + t^2 + x^2) \\ &= M_f(2 + 2x^2 + (t - x)^2 + 2x(t - x)) \\ &\leq M_f(t - x)^2(3 + 2x^2 + 2x) \leq 4M_f(1 + x^2)(t - x)^2. \end{aligned} \tag{24}$$

For $x \in [0, c]$ and $t \in [0, c + 1]$, we have

$$|f(t) - f(x)| \leq \omega_{c+1}(f; |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{c+1}(f; \delta). \tag{25}$$

From equations (24) and (25), for $x \in [0, c]$ and $t \geq 0$, we have

$$|f(t) - f(x)| \leq 4M_f(1 + x^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{c+1}(f; \delta).$$

Applying the Cauchy-Schwarz inequality and choosing $\delta = \sqrt{\eta_{n,\rho}}$, we get

$$\begin{aligned} |\mathcal{L}_n^\rho(f; x) - f(x)| &\leq 4M_f(1 + x^2)\mathcal{L}_n^\rho((t - x)^2; x) + \left(1 + \frac{1}{\delta}\mathcal{L}_n^\rho(|t - x|; x)\right) \omega_{c+1}(f; \delta) \\ &\leq 4M_f(1 + c^2)\eta_{n,\rho}^2(c) + 2\omega_{c+1}(f; \eta_{n,\rho}(c)). \end{aligned}$$

This completes the proof. □

Theorem 4.2 For $f \in C_\theta(R_0^+)$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in R_0^+} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1 + x^2)^{1+\eta}} = 0,$$

where η is some positive constant.

Proof Since $|f(x)| \leq \|f\|_\theta(1 + x^2)$, for fixed $y > 0$, we may write

$$\begin{aligned} \sup_{x \in R_0^+} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1 + x^2)^{1+\eta}} &\leq \sup_{x \in [0, y]} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1 + x^2)^{1+\eta}} + \sup_{x \in (y, \infty)} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1 + x^2)^{1+\eta}} \\ &\leq \|\mathcal{L}_n^\rho(f; \cdot) - f\|_{C[0, y]} + \frac{\|f\|_\theta}{(1 + y^2)^\eta} \\ &\quad + \|f\|_\theta \sup_{x \in (y, \infty)} \frac{|\mathcal{L}_n^\rho(1 + t^2; x)|}{(1 + x^2)^{1+\eta}}. \end{aligned} \tag{26}$$

Using Theorem 3.1, for a given $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$|\mathcal{L}_n^\rho(1 + t^2; x) - 1 + x^2| < \frac{\epsilon}{3\|f\|_\theta}, \quad \forall n \geq k.$$

or

$$\mathcal{L}_n^\rho(1 + t^2; x) < 1 + x^2 + \frac{\epsilon}{3\|f\|_\theta}, \quad \forall n \geq k.$$

Hence,

$$\begin{aligned} \|f\|_\theta \frac{\mathcal{L}_n^\rho(1+t^2;x)}{(1+x^2)^{1+\eta}} &< \frac{\|f\|_\theta}{(1+x^2)^{1+\eta}} \left(1+x^2 + \frac{\epsilon}{3\|f\|_\theta}\right) \\ &< \frac{\|f\|_\theta}{(1+y^2)^\eta} + \frac{\epsilon}{3}, \quad \forall n \geq k. \end{aligned}$$

Therefore,

$$\|f\|_\theta \sup_{x \in [y, \infty)} \frac{\mathcal{L}_n^\rho(1+t^2;x)}{(1+x^2)^{1+\eta}} \leq \frac{\|f\|_\theta}{(1+y^2)^\eta} + \frac{\epsilon}{3}, \quad \text{for all } n \geq k. \tag{27}$$

Let us choose y so large that

$$\frac{\|f\|_\theta}{(1+y^2)^\eta} \leq \frac{\epsilon}{6}. \tag{28}$$

Also, in view of Theorem 4.1, for $\epsilon > 0$ there exists a $n \geq l$ such that

$$\|\mathcal{L}_n^\rho(f; \cdot) - f\|_{C[0,y]} < \frac{\epsilon}{3}, \quad n \geq l. \tag{29}$$

Taking $m = \max(k, l)$ and combining equations (26)-(29), we get

$$\sup_{x \in R_0^+} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1+x^2)^{1+\eta}} < \epsilon, \quad n \geq m.$$

This completes the proof. □

Following [13], the weighted modulus of continuity $\bar{\omega}(g; \delta)$ for $g \in C_\theta(R_0^+)$ is defined as

$$\bar{\omega}(g; \delta) = \sup_{0 < |h| \leq \delta, x \in R_0^+} \frac{|g(x+h) - g(x)|}{(1+h^2)(1+x^2)}. \tag{30}$$

Also, for $g \in C_\theta^*(R_0^+)$, the weighted modulus of continuity has the following properties:

$$\lim_{\delta \rightarrow 0} \bar{\omega}(g; \delta) = 0$$

and

$$\bar{\omega}(g; \lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\bar{\omega}(g; \delta), \quad \lambda > 0. \tag{31}$$

For $g \in C_\theta(R_0^+)$, from equations (30) and (31)

$$\begin{aligned} |g(t) - g(x)| &\leq (1+(t-x)^2)(1+x^2)\bar{\omega}(g; |t-x|) \\ &\leq 2\left(1 + \frac{|t-x|}{\delta}\right)(1+\delta^2)\bar{\omega}(g; \delta)(1+(t-x)^2)(1+x^2). \end{aligned} \tag{32}$$

Theorem 4.3 For $f \in C_\theta^*(R_0^+)$, we have

$$\sup_{x \in R_0^+} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1+x^2)^2} \leq C\bar{\omega}\left(f; \sqrt{\frac{1}{n}}\right),$$

where C is a positive constant independent of n .

Proof By the linearity and positivity of the operators \mathcal{L}_n^ρ , we get

$$|\mathcal{L}_n^\rho(f; x) - f(x)| \leq \mathcal{L}_n^\rho(|f(t) - f(x)|; x)$$

Using equation (32) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\mathcal{L}_n^\rho(f; x) - f(x)| &\leq 2(1 + \delta^2)\bar{\omega}(f; \delta)(1 + x^2)\mathcal{L}_n^\rho\left(\left(1 + \frac{|t-x|}{\delta}\right)(1 + (t-x)^2); x\right) \\ &\leq 2(1 + \delta^2)\bar{\omega}(f; \delta)(1 + x^2)\left\{\mathcal{L}_n^\rho(1; x) + \mathcal{L}_n^\rho((t-x)^2; x)\right\} \\ &\quad + \frac{1}{\delta}\mathcal{L}_n^\rho(|t-x|; x) + \frac{1}{\delta}\mathcal{L}_n^\rho(|t-x|(t-x)^2; x)\left\} \\ &\leq 2(1 + \delta^2)\bar{\omega}(f; \delta)(1 + x^2)\left\{\mathcal{L}_n^\rho(1; x) + \mathcal{L}_n^\rho((t-x)^2; x)\right\} \\ &\quad + \frac{1}{\delta}\sqrt{\mathcal{L}_n^\rho((t-x)^2; x)} + \frac{1}{\delta}\sqrt{\mathcal{L}_n^\rho((t-x)^2; x)}\sqrt{\mathcal{L}_n^\rho((t-x)^4; x)}\left\}. \end{aligned} \tag{33}$$

Using Lemma 2.2, we obtain

$$\mathcal{L}_n^\rho((t-x)^2; x) \leq C_1 \frac{1}{n}(1+x^2) \tag{34}$$

and

$$\mathcal{L}_n^\rho((t-x)^4; x) \leq C_2 \frac{1}{n^2}(1+x^2), \tag{35}$$

for some positive constants C_1 and C_2 dependent on ρ and $A(t_1, t_2)$. Now combining equations (33)-(35) and taking $\delta = \sqrt{\frac{1}{n}}$, we have

$$\begin{aligned} |\mathcal{L}_n^\rho(f; x) - f(x)| &\leq 2\left(1 + \frac{1}{n}\right)\bar{\omega}\left(f; \sqrt{\frac{1}{n}}\right)(1+x^2)\left\{1 + C_1 \frac{1}{n}(1+x^2)\right. \\ &\quad \left.+ \sqrt{C_1}\sqrt{(1+x^2)} + \sqrt{C_1}\sqrt{(1+x^2)}\sqrt{C_2}\sqrt{(1+x^2)}\right\}. \end{aligned}$$

Hence, we get

$$\sup_{x \in R_0^+} \frac{|\mathcal{L}_n^\rho(f; x) - f(x)|}{(1+x^2)^2} \leq C\bar{\omega}\left(f; \sqrt{\frac{1}{n}}\right),$$

where $C = 2(1 + MC_1 + \sqrt{C_1} + \sqrt{C_1}\sqrt{C_2})$. This completes the proof. □

5 Quantitative Voronovskaya theorems

In the following result, we discuss a quantitative Voronovskaja type theorem by using the weighted modulus of smoothness $\bar{\omega}(f; \delta)$. Recently, many researchers [14–18] have made remarkable contributions in this area.

Theorem 5.1 *For f, f', f'' in $C_{\theta}^*(R_0^+)$ and any $x \in R_0^+$, we have*

$$\begin{aligned} & \left| n(\mathcal{L}_n^\rho(f; x) - f(x)) - f'(x) \left(\frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{A(1, 1)} \right) - \frac{f''(x)}{2!} \left[\frac{x}{n} \left(1 + \frac{1}{\rho} \right) \right. \right. \\ & \quad \left. \left. + \frac{1}{n^2 A(1, 1)} \left\{ \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right. \right. \right. \\ & \quad \left. \left. \left. + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \right\} \right] \right| \\ & = O(1) \bar{\omega} \left(f''; \frac{1}{\sqrt{n}} \right), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Proof Let $x, t \in R_0^+$, then, by Taylor’s expansion, we have

$$f(t) = f(x) + f'(x)(t - x) + \frac{f''(x)}{2!}(t - x)^2 + E(t, x),$$

where $E(t, x) = \frac{f''(\varphi) - f''(x)}{2!}(t - x)^2$ and φ lies between t and x .

Now, we get

$$\left| \mathcal{L}_n^\rho(f; x) - f(x) - f'(x) \mathcal{L}_n^\rho((t - x); x) - \frac{f''(x)}{2!} \mathcal{L}_n^\rho((t - x)^2; x) \right| \leq \mathcal{L}_n^\rho(|E(t, x)|; x).$$

Multiplying by n on both sides of the above inequality and using Lemma 2.2, we obtain

$$\begin{aligned} & \left| n(\mathcal{L}_n^\rho(f; x) - f(x)) - f'(x) \left(\frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{A(1, 1)} \right) - \frac{f''(x)}{2!} \left[\frac{x}{n} \left(1 + \frac{1}{\rho} \right) \right. \right. \\ & \quad \left. \left. + \frac{1}{n^2 A(1, 1)} \left\{ \left(1 + \frac{1}{\rho} \right) (A_{t_1}(1, 1) + A_{t_2}(1, 1)) \right. \right. \right. \\ & \quad \left. \left. \left. + A_{t_1 t_1}(1, 1) + 2A_{t_1 t_2}(1, 1) + A_{t_2 t_2}(1, 1) \right\} \right] \right| \\ & \leq n \mathcal{L}_n^\rho(|E(t, x)|; x). \tag{36} \end{aligned}$$

Using the property of weighted modulus of smoothness given by (32), we get

$$\begin{aligned} \left| \frac{f''(\varphi) - f''(x)}{2!} \right| & \leq \frac{1}{2} \bar{\omega}(f''; |\varphi - x|) (1 + (\varphi - x)^2) (1 + x^2) \\ & \leq \frac{1}{2} \bar{\omega}(f''; |t - x|) (1 + (t - x)^2) (1 + x^2) \\ & \leq \left(1 + \frac{|t - x|}{\delta} \right) (1 + \delta^2) \bar{\omega}(f''; \delta) \\ & \quad \times (1 + (t - x)^2) (1 + x^2). \end{aligned}$$

Also,

$$\left| \frac{f''(\varphi) - f''(x)}{2!} \right| \leq \begin{cases} 2(1 + \delta^2)^2(1 + x^2)\overline{\omega}(f''; \delta), & |t - x| \leq \delta, \\ 2(1 + \delta^2)^2(1 + x^2)\frac{(t-x)^2}{\delta^4}\overline{\omega}(f''; \delta), & |t - x| \geq \delta. \end{cases}$$

Now for $0 < \delta < 1$, we obtain

$$\left| \frac{f''(\varphi) - f''(x)}{2!} \right| \leq 8(1 + x^2)\overline{\omega}(f''; \delta) \left(1 + \frac{(t - x)^4}{\delta^4} \right).$$

Therefore, we get

$$|E(t, x)| \leq 8(1 + x^2)\overline{\omega}(f''; \delta) \left((t - x)^2 + \frac{(t - x)^6}{\delta^4} \right).$$

Now by the linearity and positivity of the operator \mathcal{L}_n^ρ and using Remark 2.3, for any $x \in R_0^+$, we obtain

$$\begin{aligned} \mathcal{L}_n^\rho(|E(t, x)|; x) &\leq 8(1 + x^2)\overline{\omega}(f''; \delta) \left\{ \mathcal{L}_n^\rho((t - x)^2; x) + \frac{1}{\delta^4} \mathcal{L}_n^\rho((t - x)^6; x) \right\} \\ &\leq 8(1 + x^2)\overline{\omega}(f''; \delta) \left\{ O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^3}\right) \right\}. \end{aligned}$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain

$$\mathcal{L}_n^\rho(|E(t, x)|; x) \leq 8(1 + x^2)\overline{\omega}(f''; n^{-\frac{1}{2}}) O\left(\frac{1}{n}\right). \tag{37}$$

Hence combining (36) and (37), we reach the required result. □

6 Grüss-Voronovskaya-type theorem

For the first time Gal and Gonska [19], studied the Grüss Voronovskaya type theorem for the Bernstein, Păltănea and Bernstein-Faber operators by means of the Grüss inequality which concerns the non-multiplicativity of these operators. For more papers in this direction we refer the reader to (cf. [20–22] etc.) Next, we study the non-multiplicativity of the positive linear operator \mathcal{L}_n^ρ .

Theorem 6.1 *For $f'(x), g'(x), f''(x), g''(x), (fg)'(x), (fg)''(x) \in C_\theta^*(R_0^+)$, we have the following equality:*

$$\lim_{n \rightarrow \infty} n \{ \mathcal{L}_n^\rho(fg; x) - \mathcal{L}_n^\rho(f; x)\mathcal{L}_n^\rho(g; x) \} = x \left(1 + \frac{1}{\rho} \right) f'(x)g'(x).$$

Proof We have

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + g''(x)f(x).$$

By making an appropriate arrangement, we get

$$\begin{aligned}
 & n \{ \mathcal{L}_n^\rho((fg); x) - \mathcal{L}_n^\rho(f; x)\mathcal{L}_n^\rho(g; x) \} \\
 &= n \left\{ \mathcal{L}_n^\rho((fg); x) - f(x)g(x) - (fg)'(x)\mathcal{L}_n^\rho(t-x; x) - \frac{(fg)''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2; x) \right. \\
 &\quad - g(x)(\mathcal{L}_n^\rho(f; x) - f(x) - f'(x)\mathcal{L}_n^\rho(t-x; x) - \frac{f''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2; x)) \\
 &\quad - \mathcal{L}_n^\rho(f; x)(\mathcal{L}_n^\rho(g; x) - g(x) - (g)'(x)\mathcal{L}_n^\rho(t-x; x) - \frac{(g)''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2; x)) \\
 &\quad + 2\frac{\mathcal{L}_n^\rho((t-x)^2; x)}{2!}f'(x)(g)'(x) + (g)''(x)\frac{\mathcal{L}_n^\rho((t-x)^2; x)}{2!}(f(x) - \mathcal{L}_n^\rho(f; x)) \\
 &\quad \left. + (g)'(x)\mathcal{L}_n^\rho(t-x; x)(f(x) - \mathcal{L}_n^\rho(f; x)) \right\}.
 \end{aligned}$$

Applying Theorem 3.1, for each $x \in R_0^+$, $\mathcal{L}_n^\rho(f; x) \rightarrow f(x)$ as $n \rightarrow \infty$ and for $f'' \in C_\theta^*(R_0^+)$, $x \in R_0^+$, by Theorem 5.1, we have

$$\lim_{n \rightarrow \infty} \left(\mathcal{L}_n^\rho(f; x) - f(x) - f'(x)\mathcal{L}_n^\rho(t-x; x) - \frac{f''(x)}{2!}\mathcal{L}_n^\rho((t-x)^2; x) \right) = 0.$$

Therefore, using Remark 2.3, we get the desired result. □

7 Rate of approximation of functions having derivative of bounded variation

In the last decade, the degree of approximation for the functions having a derivative of bounded variation has been studied by several researchers. Ispir *et al.* [23] considered the Kantorovich modification of Lupas operators based on Polya distributions and studied the rate of approximation of the functions having a derivative of bounded variation. For other significant contributions in this direction *cf.* [16, 24–27] etc. Motivated by these studies, we shall discuss the rate of approximation of functions with a derivative of bounded variation on R_0^+ for the operators \mathcal{L}_n^ρ .

Let $DBV(R_0^+)$ be the subspace of $B_\theta(R_0^+)$ of all absolutely continuous functions f having a derivative f' equivalent with a function of bounded variation on every finite subinterval of R_0^+ . We observe that the functions $f \in DBV(R_0^+)$ possess a representation

$$f(x) = \int_0^x g(t) dt + f(0),$$

where $g \in BV(R_0^+)$, *i.e.*, g is a function of bounded variation on every finite subinterval of R_0^+ .

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators \mathcal{L}_n^ρ in an integral form as follows:

$$\mathcal{L}_n^\rho(f; x) = \int_0^\infty K_n^\rho(x, t)f(t) dt, \tag{38}$$

where the kernel $K_n^\rho(x, t)$ is given by

$$K_n^\rho(x, t) = \frac{e^{-nx}}{A(1, 1)} \sum_{k_1} \sum_{k_1+k_2 \geq 1} \sum_{k_2} \frac{p_{k_1, k_2}(\frac{nx}{2})}{k_1!k_2!} \frac{n\rho e^{-n\rho t} (n\rho t)^{(k_1+k_2)\rho-1}}{\Gamma(k_1+k_2)\rho} + \frac{e^{-nx}}{A(1, 1)} p_{0,0}(\frac{nx}{2}) \delta(t),$$

$\delta(t)$ being the Dirac-delta function.

Lemma 7.1 *For a fixed $x \in R_0^+$ and sufficiently large n , we have*

$$(i) \quad \xi_n^\rho(x, y) = \int_0^y K_n^\rho(x, t) dt \leq \frac{C(1 + \frac{1}{\rho})(1+x)}{n} \frac{1}{(x-y)^2}, \quad 0 \leq y < x,$$

$$(ii) \quad 1 - \xi_n^\rho(x, z) = \int_z^\infty K_n^\rho(x, t) dt \leq \frac{C(1 + \frac{1}{\rho})(1+x)}{n} \frac{1}{(z-x)^2}, \quad x < z < \infty.$$

Proof (i) Using Lemma 2.2, we get

$$\begin{aligned} \xi_n^\rho(x, y) &= \int_0^y K_n^\rho(x, t) dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n^\rho(x, t) dt \\ &\leq \mathcal{L}_n^\rho((t-x)^2; x)(x-y)^{-2} \\ &\leq \frac{C(1 + \frac{1}{\rho})(1+x)}{n} \frac{1}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar; hence the details are omitted. □

Theorem 7.2 *Let $f \in \text{DBV}(R_0^+)$. Then, for every $x \in R_0^+$ and sufficiently large n , we have*

$$\begin{aligned} &|\mathcal{L}_n^\rho(f; x) - f(x)| \\ &\leq \frac{1}{2} (f'(x+) + f'(x-)) \left(\frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{nA(1, 1)}\right) + \frac{1}{2} |f'(x+) - f'(x-)| \sqrt{\frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x)} \\ &\quad + \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \frac{(1+x)}{x^2} |f(2x) - f(x) - xf'(x+)| + \frac{x}{\sqrt{n}} \bigvee_x^{x+x/\sqrt{n}} (f'_x) \\ &\quad + \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \left(1 + \frac{1}{x}\right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+x/\sqrt{k}} f'_x + \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \left(\frac{M + |f(x)|}{x^2} + 4M\right) \\ &\quad + |f'(x+)| \sqrt{\frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x)}, \end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of $f(x)$ on $[a, b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f'(t) - f'(x+) & x < t < \infty. \end{cases} \tag{39}$$

Proof Since $\mathcal{L}_n^\rho(1; x) = 1$, using (38), for every $x \in (0, 1)$ we get

$$\begin{aligned} \mathcal{L}_n^\rho(f; x) - f(x) &= \int_0^\infty K_n^\rho(x, t)(f(t) - f(x)) dt \\ &= \int_0^\infty K_n^\rho(x, t) \int_x^t f'(u) du dt. \end{aligned} \tag{40}$$

For any $f \in \text{DBV}(R_0^+)$, from (39) we may write

$$\begin{aligned} f'(v) &= (f')_x(v) + \frac{1}{2}(f'(x+) + f'(x-)) + \frac{1}{2}(f'(x+) - f'(x-)) \operatorname{sgn}(v - x) \\ &\quad + E_x(v) \left[f'(v) - \frac{1}{2}(f'(x+) + f'(x-)) \right], \end{aligned} \tag{41}$$

where

$$E_x(u) = \begin{cases} 1, & v = x, \\ 0, & v \neq x. \end{cases}$$

We get

$$\int_0^\infty \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) E_x(v) dv \right) K_n^\rho(x, t) dt = 0. \tag{42}$$

Using Lemma 2.2 and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\int_0^\infty \left(\int_x^t \frac{1}{2}(f'(x+) + f'(x-)) dv \right) K_n^\rho(x, t) dt \\ &= \frac{1}{2}(f'(x+) + f'(x-)) \int_0^\infty (t - x) K_n^\rho(x, t) dt \\ &= \frac{1}{2}(f'(x+) + f'(x-)) \mathcal{L}_n^\rho((t - x); x) = \frac{1}{2}(f'(x+) + f'(x-)) \left(\frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{nA(1, 1)} \right) \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\int_0^\infty K_n^\rho(x, t) \left(\int_x^t \frac{1}{2}(f'(x+) - f'(x-)) \operatorname{sgn}(v - x) dv \right) dt \\ &\leq \frac{1}{2} |f'(x+) - f'(x-)| (\mathcal{L}_n^\rho((t - x)^2; x))^{1/2} \\ &\leq \frac{1}{2} |f'(x+) - f'(x-)| \sqrt{\frac{C}{n} \left(1 + \frac{1}{\rho} \right) (1 + x)}. \end{aligned} \tag{44}$$

Using Lemma 2.2 and equations (40)-(44), we obtain

$$\begin{aligned} |\mathcal{L}_n^\rho(f; x) - f(x)| &\leq \frac{1}{2} (f'(x+) + f'(x-)) \left(\frac{A_{t_1}(1, 1) + A_{t_2}(1, 1)}{nA(1, 1)} \right) \\ &\quad + \frac{1}{2} |f'(x+) - f'(x-)| \sqrt{\frac{C}{n} \left(1 + \frac{1}{\rho} \right) (1 + x)} + |I_1| + |I_2|, \end{aligned} \tag{45}$$

where

$$I_1 = \int_0^x \int_x^t ((f')_x(v) dv) K_n^\rho(x, t) dt$$

and

$$I_2 = \int_x^1 \int_x^t ((f')_x(v) dv) K_n^\rho(x, t) dt.$$

Since we know $\int_a^b d_t \xi_n^\rho(x, t) \leq 1$, for all $[a, b] \subseteq R^+$, using integration by parts and applying Lemma 7.1 and substituting $y = x - x/\sqrt{n}$, we get

$$\begin{aligned} I_1 &= \left| \int_0^x \int_x^t ((f')_x(v) dv) d_t \xi_n^\rho(x, t) \right| \\ &= \left| \int_0^x \xi_n^\rho(x, t) (f')_x(t) dt \right| \\ &\leq \int_0^y |(f')_x(t)| |\xi_n^\rho(x, t)| dt + \int_y^x |(f')_x(t)| |\xi_n^\rho(x, t)| dt \\ &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \int_0^y \bigvee_t^x ((f')_x)(x-t)^{-2} dt + \int_y^x \bigvee_t^x ((f')_x) dt \\ &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \int_0^{x-x/\sqrt{n}} \bigvee_t^x ((f')_x)(x-t)^{-2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x ((f')_x). \end{aligned}$$

Substituting $v = x/(x - t)$, we get

$$\begin{aligned} &\frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \int_0^{x-x/\sqrt{n}} (x-t)^{-2} \bigvee_t^x ((f')_x) dt \\ &= \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x)x^{-1} \int_1^{\sqrt{n}} \bigvee_{x-x/u}^x ((f')_x) dv \\ &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x)x^{-1} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-x/k}^x ((f')_x) dv \\ &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \left(1 + \frac{1}{x}\right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^x ((f')_x). \end{aligned}$$

Thus,

$$|I_1| \leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \left(1 + \frac{1}{x}\right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^x ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x ((f')_x). \tag{46}$$

Again, using integration by parts, applying the Cauchy-Schwarz inequality, Lemma 7.1 and substituting $z = x + (1 - x)/\sqrt{n}$, we get

$$\begin{aligned}
 |I_2| &= \left| \int_x^\infty \int_x^t ((f')_x(v) dv) K_n^\rho(x, t) dt \right| \\
 &= \left| \int_x^{2x} \int_x^t ((f')_x(v) dv) d_t(1 - \xi_n^\rho(x, t)) + \int_{2x}^\infty \int_x^t ((f')_x(v) dv) K_n^\rho(x, t) dt \right| \\
 &= \left| \left[\int_x^t ((f')_x(v) dv) (1 - \xi_n^\rho(x, t)) \right]_x^{2x} \right| + \left| \int_x^{2x} (f')_x(t) (1 - \xi_n^\rho(x, t)) dt \right| \\
 &\quad + \left| \int_{2x}^\infty \int_x^t ((f'(v) - f'(x+)) dv) K_n^\rho(x, t) dt \right| \\
 &\leq \left| \int_x^{2x} ((f')_x(v) dv) (1 - \xi_n^\rho(x, 2x)) \right| + \left| \int_x^{2x} (f')_x(t) (1 - \xi_n^\rho(x, t)) dt \right| \\
 &\quad + \left| \int_{2x}^\infty f(t) K_n^\rho(x, t) dt \right| + |f(x)| \left| \int_{2x}^\infty K_n^\rho(x, t) dt \right| \\
 &\quad + |f'(x+)| \left| \int_{2x}^\infty ((t - x)) K_n^\rho(x, t) dt \right| \\
 &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \frac{(1+x)}{x^2} \left| \int_x^{2x} ((f'(v) - f(x+)) dv) \right| + \left| \int_x^{x+x/\sqrt{n}} f'_x(t) dt \right| \\
 &\quad + \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \left| \int_{x+x/\sqrt{n}}^{2x} (t-x)^{-2} f'_x(t) dt \right| + \left| \int_{2x}^\infty f(t) K_n^\rho(x, t) dt \right| \\
 &\quad + |f(x)| \left| \int_{2x}^\infty K_n^\rho(x, t) dt \right| + |f'(x+)| \left(\int_{2x}^\infty (t-x)^2 K_n^\rho(x, t) dt \right)^{1/2}. \tag{47}
 \end{aligned}$$

By substituting $t = x + \frac{x}{u}$ and proceeding in a similar way to I_1 , we get

$$\begin{aligned}
 |I_2| &\leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \frac{(1+x)}{x^2} |f(2x) - f(x) - xf'(x+)| + \frac{x}{\sqrt{n}} \bigvee_x^{x+x/\sqrt{n}} (f'_x) \\
 &\quad + \frac{C}{n} \left(1 + \frac{1}{\rho}\right) \left(1 + \frac{1}{x}\right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+x/\sqrt{k}} f'_x + \int_{2x}^\infty M(1+t^2) K_n^\rho(x, t) dt \\
 &\quad + |f(x)| \left| \int_{2x}^\infty K_n^\rho(x, t) dt \right| + |f'(x+)| \sqrt{\frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x)}. \tag{48}
 \end{aligned}$$

Now for $t \geq 2x$, we may write $t \leq 2(t - x)$ and $x \leq t - x$. Now using Lemma 7.1, we obtain

$$\begin{aligned}
 &\int_{2x}^\infty M(1+t^2) K_n^\rho(x, t) dt + |f(x)| \int_{2x}^\infty K_n^\rho(x, t) dt \\
 &\leq \frac{M}{x^2} \int_{2x}^\infty (t-x)^2 K_n^\rho(x, t) dt + 4M \int_{2x}^\infty (t-x)^2 K_n^\rho(x, t) dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{|f(x)|}{x^2} \int_{2x}^{\infty} (t-x)^2 K_n^\rho(x,t) dt \\
& \leq \frac{C}{n} \left(1 + \frac{1}{\rho}\right) (1+x) \left(\frac{M + |f(x)|}{x^2} + 4M\right). \tag{49}
\end{aligned}$$

Collecting the estimates (45)-(49), we get the required result. \square

8 Conclusion

A link between Szász-Durrmeyer type operators and multiple Appell polynomials has been established. The quantitative Voronovskaya type theorem and the Grüss-Voronovskaya type theorem have been proved. A local approximation result and the weighted approximation theorem have been discussed besides the approximation of functions whose derivatives are locally of bounded variation.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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