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Optimal bounds for Neuman-Sándor mean in terms of the convex combination of the logarithmic and the second Seiffert means

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Abstract

In the article, we prove that the double inequality

$$\alpha L(a,b) + (1-\alpha)T(a,b) < NS(a,b) < \beta L(a,b) + (1-\beta)T(a,b)$$

holds for $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 1/4$ and $\beta \leq 1 - \pi/[4\log(1+\sqrt{2})]$, where $NS(a,b)$, $L(a,b)$ and $T(a,b)$ denote the Neuman-Sándor, logarithmic and second Seiffert means of two positive numbers a and b, respectively.

MSC: 26E60

Keywords: Neuman-Sándor mean; logarithmic mean; the second Seiffert mean

1 Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $NS(a,b)$ [1], the second Seiffert mean $T(a,b)$ [2], and the logarithmic mean $L(a,b)$ [1] are defined by

$$NS(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

$$T(a,b) = \frac{a-b}{2\tan^{-1}[(a-b)/(a+b)]}, \tag{1.2}$$

$$L(a,b) = \frac{a-b}{\log a - \log b},$$

respectively. It can be observed that the logarithmic mean $L(a,b)$ can be rewritten as (see as [1])

$$L(a,b) = \frac{a-b}{2\tanh^{-1}[(a-b)/(a+b)]}, \tag{1.3}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$, $\tanh^{-1}(x) = \log\sqrt{(1+x)/(1-x)}$ and $\tan^{-1}(x) = \arctan(x)$, are the inverse hyperbolic sine, inverse hyperbolic tangent, and inverse tangent, respectively.

Recently, the means NS, T, L and other means have been the subject of extensive research. In particular, many remarkable inequalities for the Neuman-Sándor, second Seiffert and logarithmic means can be found in the literature [2–16].

Let $P(a,b) = (a-b)/(2\sin^{-1}[(a-b)/(a+b)])$, $S(a,b) = \sqrt{(a^2+b^2)/2}$, $A(a,b) = (a+b)/2$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ denote the first Seiffert, root-square, arithmetic, identric, geometric, and the harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that the inequality

$$S(a,b) > T(a,b) > NS(a,b) > A(a,b) > I(a,b) > P(a,b) > L(a,b) > G(a,b) > H(a,b)$$

holds for $a, b > 0$ with $a \neq b$.

In [17] and [18], the authors proved that the double inequalities

$$S(a,b)^{\alpha_3} A^{1-\alpha_3}(a,b) < NS(a,b) < S(a,b)^{\beta_3} A^{1-\beta_3}(a,b),$$
$$\alpha_4 S(a,b) + (1-\alpha_4)G(a,b) < NS(a,b) < \beta_4 S(a,b) + (1-\beta_4)G(a,b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/3$, $2(\log(2+\sqrt{2}) - \log 3)/\log 2 \leq \beta_3 \leq 1$, $\alpha_4 \leq 2/3$ and $\beta_4 \geq 1/[\sqrt{2}\log(1+\sqrt{2})]$.

In [19], it was showed that the inequality

$$P^{\alpha_2}(a,b) T^{1-\alpha_2}(a,b) < NS(a,b) < P^{\beta_2}(a,b) T^{1-\beta_2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 > 1/3$ and

$$\beta_2 \leq \log\left(\frac{4\log(1+\sqrt{2})}{\pi}\right)/\log 2 = 0.1663\ldots.$$

Let $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the Lehmer mean of two positive numbers a and b with $a \neq b$. In [10], the authors proved the double inequality

$$L_{\alpha_1}(a,b) < NS(a,b) < L_{\beta_1}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 = 1.8435\ldots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$, and $\beta_1 = 2$.

Let

$$M_p(a,b) = \begin{cases} (\frac{a^p+b^p}{2})^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

be the pth power means of two positive numbers a and b with $a \neq b$. In [20], the authors proved the sharp double inequality

$$M_{\log 2/(\log\pi - \log 2)}(a,b) < T(a,b) < M_{5/3}(a,b)$$

holds.

Gao [21] proved the optimal double inequality

$$I(a,b) < T(a,b) < \frac{2e}{\pi} I(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

Yang [22] proved the inequality

$$A_p^{1/(3p)}(a,b)G^{1-1/(3p)}(a,b) < L(a,b) < A_q^{1/(3q)}(a,b)G^{1-1/(3q)}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 1/\sqrt{5}$ and $0 < q \leq 1/3$. And the inequality

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b)$$

was proved by Lin in [23].

In [24], the authors present bounds for L in terms of G and A

$$G^{2/3}(a,b)A^{1/3}(a,b) < L(a,b) < \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to answer the following questions: What are the least value α and the greatest value β such that

$$\alpha L(a,b) + (1-\alpha)T(a,b) < NS(a,b) < \beta L(a,b) + (1-\beta)T(a,b)$$

holds for all $a, b > 0$ with $a \neq b$?

2 Lemmas

It is well known that, for $x \in (0,1)$,

$$\tanh^{-1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}, \tag{2.1}$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{2.2}$$

To establish our main result, we need several lemmas as follows.

Lemma 2.1 ([25]) *Let*

$$H(x) = \frac{1}{\sinh^{-1} x} - \frac{x}{\sqrt{1+x^2}(\sinh^{-1} x)^2}.$$

Then $H(x)$ is strictly increasing on $(0,1)$. Moreover, the inequality

$$H(x) < \frac{x}{3} - \frac{x^3}{9} \tag{2.3}$$

holds for any $x \in (0, 0.76)$ *and the inequality*

$$H(x) > \frac{x}{3} - \frac{17x^3}{90} \tag{2.4}$$

holds for any $x \in (0, 1)$.

Lemma 2.2 *Let* $S(x) = 1/\tanh^{-1}x - x/[(1-x^2)(\tanh^{-1}x)^2]$. *Then*

$$S(x) < -\frac{2}{3}x - \frac{1}{3}x^3 - \frac{1}{3}x^5 \tag{2.5}$$

for any $x \in (0, 1)$ *and*

$$S(x) > -\frac{2}{3}x - x^3 - \frac{x^5}{4} \tag{2.6}$$

for any $x \in (0, 0.76)$.

Proof Let

$$G(x) = (1 - x^2)(\tanh^{-1} x)^2 \left[S(x) + \frac{2}{3}x + \frac{1}{3}x^3 + \frac{1}{3}x^5 \right].$$

Then direct computation leads to

$$G(0) = 0, \tag{2.7}$$

$$G'(x) = \frac{1}{3}g(x) \tanh^{-1} x, \tag{2.8}$$

where $g(x) = (2 - 7x^6 - 3x^2)\tanh^{-1}x + 2x^5 + 2x^3 - 2x$. It follows that

$$g'(x) = \frac{1}{1 - x^2}g_1(x), \tag{2.9}$$

where $g_1(x) = (-42x^5 - 6x)(1 - x^2)\tanh^{-1}x - 17x^6 + 4x^4 + 5x^2$. Considering (2.1), we have

$$\begin{aligned} g_1(x) &< (-42x^5 - 6x)(1 - x^2)\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right) - 17x^6 + 4x^4 + 5x^2 \\ &= \frac{1}{5}(42x^{12} + 28x^{10} + 146x^8 - 291x^6 + 40x^4 - 5x^2) \\ &< x^2(216x^6 - 291x^4 + 40x^2 - 5) < 0, \end{aligned} \tag{2.10}$$

for $x \in (0, 1)$. Thus, (2.9) and (2.10) as well as $g(0) = 0$ imply $g(x) < 0$ for $x \in (0, 1)$. Therefore, combining (2.7) and (2.8), we get $G(x) < 0$ for $x \in (0, 1)$. It means inequality (2.5) holds.

Let

$$Q(x) = (1 - x^2)(\tanh^{-1} x)^2 \left[S(x) + \frac{2}{3}x + x^3 + \frac{x^5}{4} \right].$$

Direct computation leads to

$$Q(0) = 0, \tag{2.11}$$

$$Q'(x) = \frac{1}{12} q_1(x) \tanh^{-1} x, \tag{2.12}$$

where

$$q_1(x) = 6x^5 + 24x^3 - 8x + \left(8 + 12x^2 - 45x^4 - 21x^6\right) \tanh^{-1} x.$$

When $x \in (0, 0.7]$, considering (2.1) and the fact $8 + 12x^2 - 45x^4 - 21x^6 = (3 - 21x^6) + (5 + 12x^2 - 45x^4) > 0$, we can get

$$\begin{aligned} q_1(x) &> 6x^5 + 24x^3 - 8x + \left(8 + 12x^2 - 45x^4 - 21x^6\right)\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right) \\ &= -\frac{21}{5}x^{11} - 16x^9 - \frac{168}{5}x^7 - \frac{167}{5}x^5 + \frac{116}{3}x^3 \\ &> x^3\left(-\frac{269}{5}x^4 - \frac{167}{5}x^2 + \frac{116}{3}\right) > 0. \end{aligned}$$

When $x \in (0.7, 0.76)$, direct computation leads to

$$q_1(0.76) = 1.8639\ldots > 0, \tag{2.13}$$

$$q_1'(x) = q_2(x)/\left(1 - x^2\right), \tag{2.14}$$

where $q_2(x) = 92x^2 - 87x^4 - 51x^6 + (126x^7 + 54x^5 - 204x^3 + 24x) \tanh^{-1} x$. Considering (2.1) and the fact $126x^7 + 54x^5 - 204x^3 + 24x < 12x(15x^4 - 17x^2 + 2) < 0$, we can get

$$\begin{aligned} q_2(x) &< 92x^2 - 87x^4 - 51x^6 + \left(126x^7 + 54x^5 - 204x^3 + 24x\right)\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right) \\ &= \frac{126}{5}x^{12} + \frac{264}{5}x^{10} + \frac{516}{5}x^8 - \frac{301}{5}x^6 - 283x^4 + 116x^2 \\ &< 2x^4\left(91x^4 - 30x^2 - 20\right) + x^2\left(116 - 243x^2\right) < 0. \end{aligned} \tag{2.15}$$

Thus, (2.13)-(2.15) imply that

$$q_1(x) > 0 \tag{2.16}$$

holds for any $x \in (0.7, 0.76)$.

Therefore, $Q(x) > 0$ for $x \in (0, 0.76)$ follows from (2.11), (2.12) and (2.16). That means inequality (2.6) holds. □

Lemma 2.3 *Let* $T(x) = 1/\tan^{-1}x - x/[(1 + x^2)(\tan^{-1}x)^2]$. *Then*

$$T(x) < \frac{2}{3}x - \frac{1}{3}x^3 + \frac{2}{7}x^5 \tag{2.17}$$

for any $x \in (0,1)$ *and*

$$T(x) > \frac{2}{3}x - \frac{2}{5}x^3 + \frac{x^5}{7} \tag{2.18}$$

for any $x \in (0, 0.76)$.

Proof Let

$$M(x) = \left[T(x) - \frac{2}{3}x + \frac{x^3}{3} - \frac{2}{7}x^5\right](1 + x^2)\left(\tan^{-1} x\right)^2.$$

Differentiating $M(x)$, we have $M'(x) = [t(x)\tan^{-1}x]/21$, where

$$t(x) = 14x + 14x^3 - 12x^5 + \left(-42x^6 + 5x^4 - 21x^2 - 14\right)\tan^{-1}x.$$

For $x \in (0,1)$, we have $-42x^6 + 5x^4 - 21x^2 - 14 < -42x^6 - 16x^2 - 14 < 0$. Thus from (2.2), we can get

$$\begin{aligned} t(x) &< 14x + 14x^3 - 12x^5 + \left(-42x^6 + 5x^4 - 21x^2 - 14\right)\left(x - \frac{x^3}{3}\right) \\ &= 14x^9 - \frac{131}{3}x^7 - \frac{7}{3}x^3 \\ &< -\frac{89}{3}x^7 - \frac{7}{3}x^3 < 0. \end{aligned}$$

Therefore $M'(x) < 0$ for $x \in (0,1)$. Considering the fact $M(0) = 0$, we get $M(x) < 0$ for $x \in (0,1)$. So the inequality (2.17) holds.

Let

$$N(x) = \left[T(x) - \frac{2}{3}x + \frac{2}{5}x^3 - \frac{x^5}{7}\right](1 + x^2)\left(\tan^{-1} x\right)^2.$$

Differentiating $N(x)$, we have $N'(x) = n(x)\tan^{-1}x$, where

$$n(x) = \frac{2}{3}x + \frac{4}{5}x^3 - \frac{2}{7}x^5 - \left(x^6 - \frac{9}{7}x^4 + \frac{4}{5}x^2 + \frac{2}{3}\right)\tan^{-1} x.$$

Because of

$$\left(\frac{4}{5}x^2 - \frac{9}{7}x^4\right) + x^6 + \frac{2}{3} > 0$$

for $x \in (0, 0.76)$, it follows that

$$\begin{aligned} n(x) &> \frac{2}{3}x + \frac{4}{5}x^3 - \frac{2}{7}x^5 - \left(x^6 - \frac{9}{7}x^4 + \frac{4}{5}x^2 + \frac{2}{3}\right)x \\ &= x^5 - x^7 > 0. \end{aligned}$$

Considering the fact $N(0) = 0$, the inequality (2.18) holds. □

Lemma 2.4 *The function $f(x) = \lambda S(x) + (1-\lambda)T(x) - H(x)$ is strictly decreasing on* $(0.76, 1)$, *where* $\lambda = 1 - \pi/[4\log(1+\sqrt{2})] = 0.1089\ldots$ *and* $H(x)$, $S(x)$ *and* $T(x)$ *are defined as in Lemmas* 2.1, 2.2 *and* 2.3, *respectively.*

Proof Direct computation leads to

$$S'(x) = 2\frac{x - \tanh^{-1} x}{(1-x^2)^2(\tanh^{-1} x)^3},$$
$$S''(x) = 2\frac{\varphi(x)}{(1-x^2)^3(\tanh^{-1} x)^4},$$

where $\varphi(x) = 3(1+x^2)\tanh^{-1}x - 3x - 4x(\tanh^{-1}x)^2$. It follows that

$$\varphi'(x) = \frac{R(x)}{1-x^2},$$

where $R(x) = -4(1-x^2)(\tanh^{-1}x)^2 - (6x^3+2x)\tanh^{-1}x + 6x^2$. From (2.3), we can get

$$\begin{aligned} R(x) &< -4(1-x^2)\left(x + \frac{x^3}{3}\right)^2 - (6x^3+2x)\left(x + \frac{x^3}{3}\right) + 6x^2 \\ &= \frac{4}{9}x^8 + \frac{2}{9}x^6 - \frac{16}{3}x^4 < 0. \end{aligned}$$

Thus $\varphi(x)$ is strictly decreasing on $(0.76, 1)$. Considering the fact $\varphi(0.76) = -0.5821\ldots < 0$, we have $\varphi(x) < 0$ for any $x \in (0.76, 1)$. In other words, $S'(x)$ is strictly decreasing on $(0.76, 1)$.

Let $\phi(x) = \lambda S(x) + (1-\lambda)T(x)$. It was proved that $T'(x)$ is strictly decreasing on $(0.7, 1)$ in Lemma 5 of [26]. Thus, from the monotonicity of $S'(x)$ and $T'(x)$, we have

$$\phi'(x) < \lambda S'(0.76) + (1-\lambda)T'(0.76) = -0.0043\ldots < 0$$

for any $x \in (0.76, 1)$. That is to say, $\phi(x)$ is strictly decreasing on $(0.76, 1)$. Considering the monotonicity of $H(x)$ in Lemma 2.1, the proof is completed. □

Lemma 2.5 *We have*

$$\frac{4-11\lambda}{28}x^4 - \frac{27\lambda+13}{45}x^2 + \frac{1-4\lambda}{3} > 0$$

for $x \in (0, 0.76)$, *where* $\lambda = 1 - \pi/[4\log(1+\sqrt{2})] = 0.1089\ldots$.

Proof Let

$$\eta(x) = \frac{4-11\lambda}{28}x^4 - \frac{27\lambda+13}{45}x^2 + \frac{1-4\lambda}{3}.$$

Then it is easy to verify that $\eta(x)$ is decreasing on $(0, \mu)$, where

$$\mu = \sqrt{\frac{14}{15}} \times \sqrt{\frac{160\log(1+\sqrt{2}) - 27\pi}{11\pi - 28\log(1+\sqrt{2})}} = 1.3303\ldots.$$

Considering $\eta(0.76) = 0.01693\ldots > 0$, we have $\eta(x) > 0$ for $x \in (0, 0.76)$. □

3 Main results

Theorem 3.1 *The double inequality*

$$\alpha L(a,b) + (1-\alpha)T(a,b) < NS(a,b) < \beta L(a,b) + (1-\beta)T(a,b)$$

holds for any $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \geq 1/4$ *and*

$$\beta \leq 1 - \frac{\pi}{4\log(1+\sqrt{2})} = 0.1089\ldots.$$

Proof Because $NS(a,b)$, $L(a,b)$, $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we can assume that $a > b$ and $x := (a-b)/(a+b) \in (0,1)$. Let $p \in (0,1)$ and $\lambda = 1 - \pi/[4\log(1+\sqrt{2})] = 0.1089\ldots$. Then by (1.1)-(1.3), direct computation leads to

$$\frac{NS(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1} x},$$
$$\frac{L(a,b)}{A(a,b)} = \frac{x}{\tanh^{-1} x},$$
$$\frac{T(a,b)}{A(a,b)} = \frac{x}{\tan^{-1} x}.$$

Let

$$\begin{aligned} F_t(x) &= \frac{tL(a,b) + (1-t)T(a,b) - M(a,b)}{A(a,b)} \\ &= t\frac{x}{\tanh^{-1} x} + (1-t)\frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x}. \end{aligned} \tag{3.1}$$

Then it follows that

$$F_{\frac{1}{4}}(0^+) = 0, \tag{3.2}$$

$$F_\lambda(0^+) = F_\lambda(1^-) = 0. \tag{3.3}$$

Differentiating $F_t(x)$, we have

$$\begin{aligned} F_t'(x) &= t\left[\frac{1}{\tanh^{-1} x} - \frac{x}{1-x^2}\frac{1}{(\tanh^{-1} x)^2}\right] \\ &\quad + (1-t)\left[\frac{1}{\tan^{-1} x} - \frac{x}{1+x^2}\frac{1}{(\tan^{-1} x)^2}\right] \\ &\quad - \left[\frac{1}{\sinh^{-1} x} - \frac{x}{\sqrt{1+x^2}}\frac{1}{(\sinh^{-1} x)^2}\right] \\ &:= tS(x) + (1-t)T(x) - H(x), \end{aligned}$$

where $H(x)$, $S(x)$ and $T(x)$ are defined as in Lemmas 2.1-2.3, respectively.

On one hand, from inequalities (2.4), (2.5) and (2.16), we clearly see that

$$\begin{aligned} F'_{\frac{1}{4}}(x) &= \frac{1}{4}S(x) + \frac{3}{4}T(x) - H(x) \\ &< \frac{1}{4}\left(-\frac{2}{3}x - \frac{1}{3}x^3 - \frac{1}{3}x^5\right) + \frac{3}{4}\left(\frac{2}{3}x - \frac{1}{3}x^3 + \frac{2}{7}x^5\right) - \left(\frac{x}{3} - \frac{17}{90}x^3\right) \\ &= -\frac{13}{90}x^3 + \frac{11}{84}x^5 < 0 \end{aligned}$$

for any $x \in (0,1)$. It leads to

$$F_{\frac{1}{4}}(x) < F_{\frac{1}{4}}(0) = 0 \tag{3.4}$$

for any $x \in (0,1)$. Thus, from (3.1) it follows that

$$NS(a,b) > \frac{1}{4}L(a,b) + \frac{3}{4}T(a,b)$$

for all $a, b > 0$ with $a \neq b$. Considering $L(a,b) < NS(a,b) < T(a,b)$, we can get

$$NS(a,b) > \alpha L(a,b) + (1-\alpha)T(a,b) \tag{3.5}$$

for all $\alpha \geq 1/4$ and $a, b > 0$ with $a \neq b$.

On the other hand, from inequalities (2.3), (2.6) and (2.17), we have

$$\begin{aligned} F'_{\lambda}(x) &> -\lambda\left(\frac{2}{3}x + x^3 + \frac{x^5}{4}\right) + (1-\lambda)\left(\frac{2}{3}x - \frac{2}{5}x^3 + \frac{x^5}{7}\right) - \left(\frac{x}{3} - \frac{x^3}{9}\right) \\ &= x\left[\frac{4-11\lambda}{28}x^4 - \frac{27\lambda+13}{45}x^2 + \frac{1-4\lambda}{3}\right] \end{aligned}$$

for $x \in (0, 0.76)$. According to Lemma 2.5, we have

$$F'_{\lambda}(x) > 0 \tag{3.6}$$

for $x \in (0, 0.76)$. Lemma 2.4 shows that $F'_{\lambda}(x)$ is strictly decreasing on $(0.76, 1)$. This fact and $F'_{\lambda}(0.76) = 0.0713\ldots > 0$ together with $F'_{\lambda}(1^-) = -\infty$ imply that there exists $x_0 \in (0.76, 1)$ such that $F_{\lambda}(x)$ is strictly increasing on $(0, x_0]$ and strictly decreasing on $[x_0, 1)$. Equations (3.1) and (3.3) together with the piecewise monotonicity of $F_{\lambda}(x)$ lead to the conclusion that

$$NS(a,b) < \lambda L(a,b) + (1-\lambda)T(a,b)$$

for all $a, b > 0$ with $a \neq b$. Considering $L(a,b) < M(a,b) < T(a,b)$, we can get

$$NS(a,b) < \beta L(a,b) + (1-\beta)T(a,b) \tag{3.7}$$

holds for $\beta \leq \lambda$ and all $a, b > 0$ with $a \neq b$.

Finally, we prove that $L(a,b)/4 + 3T(a,b)/4$ and $\lambda L(a,b) + (1-\lambda)T(a,b)$ are the best possible lower and upper mean bound for the Neuman-Sándor mean $M(a,b)$.

For any $\epsilon_1, \epsilon_2 > 0$, let $t_1 = 1/4 - \epsilon_1$, $t_2 = \lambda + \epsilon_2$. Then one can get

$$F_{t_1}(x) = \left(\frac{1}{4} - \epsilon_1\right)\frac{x}{\tanh^{-1} x} + \left(\frac{3}{4} + \epsilon_1\right)\frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x}, \tag{3.8}$$

$$F_{t_2}(x) = (\lambda + \epsilon_2)\frac{x}{\tanh^{-1} x} + (1 - \lambda - \epsilon_2)\frac{x}{\tan^{-1} x} - \frac{x}{\sinh^{-1} x}. \tag{3.9}$$

Let $x_1 \to 0^+$ and $x_2 \to 1^-$, then the Taylor expansion leads to

$$F_{t_1}(x_1) = \frac{2}{3}\epsilon_1 x_1^2 + O\left(x_1^4\right), \tag{3.10}$$

$$F_{t_2}(x_2) = -4\epsilon_2/\pi + O(x_2 - 1). \tag{3.11}$$

Equations (3.8) and (3.10) imply that if $\alpha < 1/4$, then, for any $\epsilon_1 > 0$, there exists $\sigma_1 \in (0,1)$ such that $NS(a,b) < (1/4 - \epsilon_1)L(a,b) + (3/4 - \epsilon_1)T(a,b)$ for all a, b with $(a-b)/(a+b) \in (0,\sigma_1)$.

Equations (3.9) and (3.11) imply that if $\beta > \lambda$, then, for any $\epsilon_2 > 0$, there exists $\sigma_2 \in (0,1)$ such that $NS(a,b) > (\lambda + \epsilon_2)L(a,b) + (1 - \lambda - \epsilon_2)T(a,b)$ for all a, b with $(a-b)/(a+b) \in (1-\sigma_2, 1)$. □

4 Conclusion

In the article, we give the sharp upper and lower bounds for Neuman-Sándor mean in terms of the linear convex combination of the logarithmic and second Seiffert means.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors worked jointly. All the authors read and approved the final manuscript.

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