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$f_{(\lambda,\mu)}$-statistical convergence of order $\widetilde{\alpha}$ for double sequences

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Abstract

New concepts of $f_{\lambda,\mu}$-statistical convergence for double sequences of order $\widetilde{\alpha}$ and strong $f_{\lambda,\mu}$-Cesàro summability for double sequences of order $\widetilde{\alpha}$ are introduced for sequences of (complex or real) numbers. Furthermore, we give the relationship between the spaces $w^2_{\widetilde{\alpha},0}(f,\lambda,\mu)$, $w^2_{\widetilde{\alpha}}(f,\lambda,\mu)$ and $w^2_{\widetilde{\alpha},\infty}(f,\lambda,\mu)$. Then we express the properties of strong $f_{\lambda,\mu}$-Cesàro summability of order $\widetilde{\beta}$ which is related to strong $f_{\lambda,\mu}$-Cesàro summability of order $\widetilde{\alpha}$. Also, some relations between $f_{\lambda,\mu}$-statistical convergence of order $\widetilde{\alpha}$ and strong $f_{\lambda,\mu}$-Cesàro summability of order $\widetilde{\alpha}$ are given.

MSC: 40A05; 40C05; 46A45

Keywords: double sequences; statistical convergence; Cesàro summability

1 Introduction

The first idea of statistical convergence goes back to the first edition of the famous Zygmund's monograph [1]. The statistical convergence was introduced for real and complex sequences by Steinhaus [2]. Fast [3] extended the usual concept of sequential limit and called it statistical convergence. Schoenberg [4] called it as D-Convergence. The idea depends on a certain density of subsets of $\mathbb{N}$. The natural density (or asymptotic density) of a set $A \subset \mathbb{N}$ is defined by $\delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in A\}|$ if the limit exists, where $|A(n)|$ is cardinality of the set $A(n)$ (see [5]). A sequence $x = (x_k)$ of complex numbers is said to be statistically convergent to some number ℓ if $\delta(\{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\})$ has natural density zero for $\varepsilon > 0$. ℓ is necessarily unique, which is statistical limit of (x_k), and written as $S\text{-}\lim x_k = \ell$. The space of all statistically convergent sequences is denoted by S (see [5–20]).

The order of statistical convergence of a sequence of positive linear operators was given by Gadjiev and Orhan [21], and after that Çolak [22] introduced statistical convergence of order α and strong p-Cesàro summability of order α.

Statistical convergence was introduced for double sequences by Mursaleen and Edely [23]. Besides this topic was studied by many authors (such as [15, 24, 25]). For some further works in this direction, we refer to [26–30].

The concepts of convergence and statistical convergence for double sequence can be expressed as follows.

Let s^2 denote the space of all double sequences, and let ℓ_∞^2, c^2 and c_0^2 be the linear spaces of bounded, convergent and null sequences $x = (x_{jk})$ with complex terms, respectively, normed by $\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}|$, where $j, k \in \mathbb{N} = \{1, 2, \ldots\}$.

A double sequence $x = (x_{j,k})_{j,k=0}^{\infty}$ has Pringsheim limit ℓ provided that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{j,k} - \ell| < \varepsilon$ whenever $j, k > N$. In this case, we write $P\text{-}\lim x = \ell$ [31].

$x = (x_{j,k})_{j,k=0}^{\infty}$ is bounded if there exists a positive number M such that $|x_{j,k}| < M$ for all j and k, that is, $\|x\| = \sup_{j,k\geq 0} |x_{j,k}| < \infty$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double natural density of K is defined by

$$\delta_2(K) = P\text{-}\lim_{m,n} \frac{1}{mn} \left|K(m,n)\right| \quad \text{if the limit exists.}$$

A double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$ the set $\{(j,k) : j \leq m, k \leq n : |x_{jk} - \ell| \geq \varepsilon\}$ has double natural density zero [23]. In this case, one can write $st_2\text{-}\lim x = \ell$, and we denote the collection of all statistically convergent double sequences by st_2. Recently, Çolak and Altin [27] introduced double statistically convergent of order α, and they examined some inclusion relations.

The idea of a modulus function was introduced in 1953 by Nakano [32]. Later, Ruckle [33] and Maddox [34] used this concept to construct some sequence spaces. Let us remind modulus function.

$f : [0,\infty) \to [0,\infty)$ is called a modulus function if

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$,
3. f is increasing,
4. f is continuous from the right at 0.

Hence, f must be continuous everywhere on $[0,\infty)$. A modulus function may be bounded or unbounded. For example, $f(x) = \frac{x}{1+x}$ is bounded, but $f(x) = x^p$, $0 < p \leq 1$ is unbounded.

Aizpuru *et al.* [35] introduced and discussed the concepts of f-statistical convergence and f-statistically Cauchy sequences, a single sequence of numbers, where f is an unbounded modulus function. Bhardwaj and Dhawan [36] continued this work and defined f-statistical convergence of order α. This new idea was introduced by Borgohain and Savaş [37] under the name of 'f_λ-statistical convergence'. Aizpuru *et al.* also studied these concepts for double sequences [38]. Mursaleen [39] introduced λ-statistical convergence as an extension of (V,λ)-summability of Leindler [40] with the help of a non-decreasing sequence, $\lambda = (\lambda_n)$ being a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

λ-statistical convergence of double sequences has been expressed by Mursaleen *et al.* [41].

2 $f_{\lambda,\mu}$-double statistical convergence of order $\widetilde{\alpha}$

In this section, we introduce $f_{\lambda,\mu}$-double statistical convergence of order $\widetilde{\alpha}$ for double sequences.

Throughout this paper, we take $s,t,u,v \in (0,1]$ as otherwise indicated. We will write $\widetilde{\alpha}$ instead of (s,t) and $\widetilde{\beta}$ instead of (u,v). Also, we define the following:

$$
\begin{aligned}
&\widetilde{\alpha} \preceq \widetilde{\beta} \quad \Longleftrightarrow \quad s \leq u \quad \text{and} \quad t \leq v, \\
&\widetilde{\alpha} \prec \widetilde{\beta} \quad \Longleftrightarrow \quad s < u \quad \text{and} \quad t < v, \\
&\widetilde{\alpha} \cong \widetilde{\beta} \quad \Longleftrightarrow \quad s = u \quad \text{and} \quad t = v, \\
&\widetilde{\alpha} \in (0,1] \quad \Longleftrightarrow \quad s,t \in (0,1], \\
&\widetilde{\beta} \in (0,1] \quad \Longleftrightarrow \quad u,v \in (0,1], \\
&\widetilde{\alpha} \cong 1 \quad \text{in case } s = t = 1, \\
&\widetilde{\beta} \cong 1 \quad \text{in case } u = v = 1, \\
&\widetilde{\alpha} \succ 1 \quad \text{in case } s > 1 \quad \text{and} \quad t > 1.
\end{aligned}
$$

Furthermore, we write $S^2_{\widetilde{\alpha}}(f,\lambda,\mu)$ to denote $S^2_{(s,t)}(f,\lambda,\mu)$ and $S^2_{\widetilde{\beta}}(f,\lambda,\mu)$ to denote $S^2_{(u,v)}(f,\lambda,\mu)$ in the section below.

We begin with the following definitions.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 0$; $\mu_{n+1} \leq \mu_n + 1$, $\mu_1 = 0$ and $\widetilde{\alpha} \in (0,1]$ be given.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ *be* a two-dimensional set of positive integers and f be an unbounded modulus function. Then $\delta^{f2}_{\widetilde{\alpha}}(\lambda,\mu)$-double *density* of K is defined as

$$
\delta^{f2}_{\widetilde{\alpha}}(K) = \lim_{n,m\to\infty} \frac{1}{f(\lambda_n^s \mu_m^t)} f\big(\big|\big\{(j,k) \in I_n \times I_m : (i,j) \in K\big\}\big|\big) \quad \text{if the limit exists.}
$$

Definition 2.1 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers as above and $\widetilde{\alpha} \in (0,1]$ be given.

(x_{jk}) is said to be $f_{\lambda,\mu}$-statistically convergent of order $\widetilde{\alpha}$ if there is a complex number ℓ such that, for every $\varepsilon > 0$,

$$
\lim_{n,m\to\infty} \frac{1}{f(\lambda_n^s \mu_m^t)} f\big(\big|\big\{(j,k) \in I_n \times I_m : |x_{jk} - \ell| \geq \varepsilon\big\}\big|\big) = 0.
$$

In this case we write $S^2_{\widetilde{\alpha}}(f,\lambda,\mu)$- $\lim_{j,k} x_{jk} = \ell$, and we denote the set of all $f_{\lambda,\mu}$-statistically convergent double sequences of order $\widetilde{\alpha}$ by $S^2_{\widetilde{\alpha}}(f,\lambda,\mu)$, where f is an unbounded modulus function.

In the case of $f(x) = x$, $\widetilde{\alpha} \cong 1$ and $\lambda_n = n$, $\mu_m = m$, $f_{\lambda,\mu}$-statistical convergence of order $\widetilde{\alpha}$ reduces to the statistical convergence of double sequences [23]. If $x = (x_{jk})$ is $f_{\lambda,\mu}$-statistically convergent of order $\widetilde{\alpha}$ to the number ℓ, then ℓ is determined uniquely. $f_{\lambda,\mu}$-double statistical convergence of order $\widetilde{\alpha}$ is well defined for $\widetilde{\alpha} \in (0,1]$ but it is not well defined for $\widetilde{\alpha} \succ 1$. For this, let us define $x = (x_{jk})$ as follows:

$$x_{jk} = \begin{cases} 1, & \text{if } j+k \text{ even}, \\ 0, & \text{if } j+k \text{ odd}. \end{cases}$$

Since $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, we have

$$\lim_{n,m\to\infty} \frac{1}{f(\lambda_n^s \mu_m^t)} f\bigl(\bigl|\bigl\{(j,k) \in I_n \times I_m : |x_{jk} - 1| \geq \varepsilon\bigr\}\bigr|\bigr) \leq \lim_{n,m\to\infty} \frac{f([|\lambda_n^s \mu_m^t|]) + 1}{f(2\lambda_n^s \mu_m^t)} = 0$$

and

$$\lim_{n,m\to\infty} \frac{1}{f(\lambda_n^s \mu_m^t)} \bigl|\bigl\{(j,k) \in I_n \times I_m : |x_{jk} - 0| \geq \varepsilon\bigr\}\bigr| \leq \lim_{n,m\to\infty} \frac{f([|\lambda_n^s \mu_m^t|]) + 1}{f(2\lambda_n^s \mu_m^t)} = 0$$

for $\widetilde{\alpha} \succ 1$, that is, $s > 1$ and $t > 1$, so that $x = (x_{jk})$ is $f_{\lambda,\mu}$-statistically convergent of order $\widetilde{\alpha}$ both to 1 and 0, i.e., $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim x_{jk} = 1$ and $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim x_{jk} = 0$. But this is impossible.

Theorem 2.2 *Let f be an unbounded modulus function and $\widetilde{\alpha} \in (0,1]$. Let $x = (x_{jk})$, $y = (y_{jk})$ be any two sequences of complex numbers. Then*

(i) *If $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim x_{jk} = \ell_0$ and $c \in \mathbb{C}$, then $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim cx_{jk} = c\ell_0$;*

(ii) *If $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim x_{jk} = \ell_o$ and $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim y_{jk} = \ell_1$, then $S_{\tilde{\alpha}}^2(f,\lambda,\mu)\text{-}\lim(x_{jk} + y_{jk}) = \ell_0 + \ell_1$.*

Theorem 2.3 *Let f be an unbounded modulus function and $\tilde{\alpha}, \tilde{\beta}$ be two real numbers such that $0 \preceq \widetilde{\alpha} \preceq \widetilde{\beta} \preceq 1$. Then $S_{\tilde{\alpha}}^2(f,\lambda,\mu) \subseteq S_{\tilde{\beta}}^2(f,\lambda,\mu)$ and strict inclusion may occur.*

Proof Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ be given such that $\tilde{\alpha} \leq \tilde{\beta}$. Since f is increasing, we have

$$\begin{aligned} &\frac{1}{f(\lambda_n^u \mu_m^v)} f\bigl(\bigl|\bigl\{(j,k) \in I_n \times I_m : |x_{jk} - \ell| \geq \varepsilon\bigr\}\bigr|\bigr) \\ &\quad \leq \frac{1}{f(\lambda_n^s \mu_m^t)} f\bigl(\bigl|\bigl\{(j,k) \in I_n \times I_m : |x_{jk} - \ell| \geq \varepsilon\bigr\}\bigr|\bigr) \end{aligned}$$

for every $\varepsilon > 0$, and this gives $S_{\tilde{\alpha}}^2(f,\lambda,\mu) \subseteq S_{\tilde{\beta}}^2(f,\lambda,\mu)$. To show that the strict inclusion may occur, consider a sequence $x = (x_{jk})$ defined by

$$x_{jk=} \begin{cases} jk, & \text{if } n - [|\lambda_n|] + 1 \leq j \leq n \text{ and } m - [|\mu_m|] + 1 \leq k \leq m, \\ 0, & \text{otherwise} \end{cases}$$

and we take $f(x) = x^p$, $(0 < p \leq 1)$ and hence $x \in S_{\tilde{\beta}}^2(f,\lambda,\mu)$ for $\tilde{\beta} \in (\frac{1}{2}, 1]$, (i.e., $\frac{1}{2} < u \leq 1$ and $\frac{1}{2} < v \leq 1$), but $x \notin S_{\tilde{\alpha}}^2(f,\lambda,\mu)$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (i.e., $0 < s \leq \frac{1}{2}$ and $0 < t \leq \frac{1}{2}$). □

The following results can be easily derived from Theorem 2.3.

Corollary 2.4 *If $x = (x_{jk})$ is $f_{\lambda,\mu}$-statistically convergent of order $\widetilde{\alpha}$ to ℓ, for some $\widetilde{\alpha}$ such that $\widetilde{\alpha} \in (0,1]$, then it is $f_{\lambda,\mu}$-statistically convergent to ℓ, and the inclusion is strict.*

Corollary 2.5 *Let $\widetilde{\alpha}, \widetilde{\beta} \in (0,1]$ be given. Then*

(i) $S^2_{\widetilde{\alpha}}(f,\lambda,\mu) = S^2_{\widetilde{\beta}}(f,\lambda,\mu)$ *if* $\widetilde{\alpha} \cong \widetilde{\beta}$.

(ii) $S^2_{\widetilde{\alpha}}(f,\lambda,\mu) = S^2(f,\lambda,\mu)$ *if* $\widetilde{\alpha} \cong 1$.

3 Strongly double Cesàro summability of order $\widetilde{\alpha}$ defined by a modulus function

In this section, we give the relationships between the spaces $w^2_{\tilde{\alpha},0}(f,\lambda,\mu), w^2_{\tilde{\alpha}}(f,\lambda,\mu)$ and $w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$.

Definition 3.1 Let f be a modulus function and $\tilde{\alpha}$ be a positive real number. We have

$$w^2_{\tilde{\alpha},0}(f,\lambda,\mu) = \left\{x = (x_{jk}) \in s^2 : \lim_{n,m\to\infty} \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk}|\right) = 0\right\},$$

$$w^2_{\tilde{\alpha}}(f,\lambda,\mu) = \left\{x = (x_{jk}) \in s^2 : \lim_{n,m\to\infty} \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk} - \ell|\right) = 0\right\},$$

$$w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu) = \left\{x = (x_{jk}) \in s^2 : \sup_{n,m} \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk}|\right) < \infty\right\}.$$

Theorem 3.2

(i) *Let f be a modulus function. For $\tilde{\alpha} \succ 0$, we have $w^2_{\tilde{\alpha},0}(f,\lambda,\mu) \subset w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$.*

(ii) *Let f be a modulus function. For $\tilde{\alpha} \succeq 1$, we have $w^2_{\tilde{\alpha}}(f,\lambda,\mu) \subset w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$.*

Proof (i) The proof of (i) is trivial.

(ii) Let $x \in w^2_{\tilde{\alpha}}(f,\lambda,\mu)$. By the definition of modulus function (ii) and (iii), we have

$$\frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk}|\right) \leq \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk} - \ell|\right) + f\left(|\ell|\right) \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} 1,$$

and since $\tilde{\alpha} \succeq 1$ and $x \in w^2_{\tilde{\alpha}}(f,\lambda,\mu)$, we have $x \in w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$, which completes the proof. □

Theorem 3.3 *For any modulus function f and $\tilde{\alpha} \succeq 1$, we have $w^2_{\tilde{\alpha}}(\lambda,\mu) \subset w^2_{\tilde{\alpha}}(f,\lambda,\mu)$, $w^2_{\tilde{\alpha},0}(\lambda,\mu) \subset w^2_{\tilde{\alpha},0}(f,\lambda,\mu)$ and $w^2_{\tilde{\alpha},\infty}(\lambda,\mu) \subset w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$.*

Proof We give the proof only when $w^2_{\tilde{\alpha},\infty}(\lambda,\mu) \subset w^2_{\tilde{\alpha},\infty}(f,\lambda,\mu)$ and the rest of cases will follow similarly. Let $x \in w^2_{\tilde{\alpha},\infty}(\lambda,\mu)$, so that

$$\sup_{n,m} \frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} |x_{jk}| < \infty.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t < \delta$. Now we write

$$\frac{1}{(\lambda_n\mu_m)^{\tilde{\alpha}}} \sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk}|\right) = \sum_1 + \sum_2,$$

where the first summation is over $|x_{jk}| \leq \delta$ and the second is over $|x_{jk}| > \delta$. Then $\sum_1 \leq \varepsilon \cdot \frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}-1}}$ and, for $|x_{jk}| > \delta$, we use the fact that

$$|x_{jk}| < \frac{|x_{jk}|}{\delta} < 1 + \left[\left| \frac{|x_{jk}|}{\delta} \right| \right],$$

where $[|t|]$ denotes the integer part of t. Given $\varepsilon > 0$, by the definition of f, we have

$$f(|x_{jk}|) \leq \left(1 + \left[\left| \frac{|x_{jk}|}{\delta} \right| \right]\right) f(1) \leq 2f(1) \frac{|x_{jk}|}{\delta}$$

for $|x_{jk}| > \delta$ and hence $\sum_2 \leq 2f(1)\delta^{-1} \sum_{j \in J_n} \sum_{k \in I_n} |x_{jk}|$, which together with $\sum_1 \leq \varepsilon \frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}-1}}$ yields

$$\frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}}} \sum_{j \in J_n} \sum_{k \in I_n} f(|x_{jk}|) \leq \varepsilon \cdot \frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}-1}} + 2f(1)\delta^{-1} \frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}}} \sum_{j \in J_n} \sum_{k \in I_n} |x_{jk}|.$$

Since $\tilde{\alpha} \geq 1$ and $x \in w^2_{\tilde{\alpha},\infty}(\lambda, \mu)$, we have $x \in w^2_{\tilde{\alpha},\infty}(f, \lambda, \mu)$ and the proof is complete. □

Theorem 3.4 *Let f be a modulus function f and $\tilde{\alpha} \succ 0$. If* $\lim_{t \to \infty} \frac{f(t)}{t} > 0$, *then* $w^2_{\tilde{\alpha}}(f, \lambda, \mu) \subset w^2_{\tilde{\alpha}}(\lambda, \mu)$.

Proof Following the proof of Proposition 1 of Maddox [42], we have $l = \lim_{t \to \infty} \frac{f(t)}{t} = \inf\{\frac{f(t)}{t} : t > 0\}$. By the definition of l, we have $f(t) \geq lt$ for all $t \geq 0$. Since $l > 0$, we get $t \leq l^{-1} f(t)$ for all $t \geq 0$, and so

$$\frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}}} \sum_{j \in J_n} \sum_{k \in I_n} |x_{jk} - \ell| \leq l^{-1} \frac{1}{(\lambda_n \mu_m)^{\tilde{\alpha}}} \sum_{j \in J_n} \sum_{k \in I_n} f(|x_{jk} - \ell|),$$

from where it follows that $x \in w^2_{\tilde{\alpha}}(f, \lambda, \mu)$ whenever $x \in w^2_{\tilde{\alpha}}(\lambda, \mu)$. □

Theorem 3.5 *For any modulus f such that* $\lim_{t \to \infty} \frac{f(t)}{t} > 0$ *and $\tilde{\alpha} \succeq 1$. Then* $w^2_{\tilde{\alpha}}(\lambda, \mu) = w^2_{\tilde{\alpha}}(f, \lambda, \mu)$.

4 Relation between $f_{\lambda,\mu}$-statistical convergence of order $\widetilde{\alpha}$ and strongly double Cesàro summability of order $\widetilde{\alpha}$ defined by a modulus function

In this section, we give the relationship between the strong $f_{\lambda,\mu}$-Cesàro summability of order $\widetilde{\alpha}$ and $f_{\lambda,\mu}$-statistical convergence of order $\widetilde{\beta}$.

Lemma 4.1 *Let f be an unbounded function such that there is a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$* [42].

Theorem 4.2 *Let $0 \prec \tilde{\alpha} \preceq \tilde{\beta} \preceq 1$ and f be an unbounded modulus function such that there is a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$ and* $\lim_{t \to \infty} \frac{f(t)}{t} > 0$. *If a sequence $x = (x_{jk})$ is strongly $f_{\lambda,\mu}$-Cesàro summable of order $\tilde{\alpha}$ with respect to f to ℓ, then it is $f_{\lambda,\mu}$-statistically convergent of order $\tilde{\beta}$ to ℓ.*

Proof For any sequence $x = (x_{jk})$ and $\varepsilon > 0$, using the definition of modulus function (ii) and (iii), we have

$$\begin{aligned}\sum_{j\in J_n}\sum_{k\in I_n} f\left(|x_{jk}-L|\right) &\geq f\left(\sum_{j\in J_n}\sum_{k\in I_n}|x_{jk}-\ell|\right) \geq f\left(\left|\left\{(j,k)\in I_n\times I_m : |x_{jk}-\ell|\geq\varepsilon\right\}\right|\varepsilon\right) \\ &\geq cf\left(\left|\left\{(j,k)\in I_n\times I_m : |x_{jk}-\ell|\geq\varepsilon\right\}\right| f(\varepsilon)\right)\end{aligned}$$

and since $\tilde{\alpha} \preceq \tilde{\beta}$

$$\begin{aligned}&\frac{1}{n^s m^t}\sum_{j=1}^{n}\sum_{k=1}^{m} f\left(|x_{jk}-\ell|\right) \\ &\quad\geq \frac{1}{n^s m^t} cf\left(\left|\left\{(j,k)\in I_n\times I_m : |x_{jk}-\ell|\geq\varepsilon\right\}\right| f(\varepsilon)\right) \\ &\quad\geq \frac{1}{n^u m^v} cf\left(\left|\left\{(j,k)\in I_n\times I_m : |x_{jk}-\ell|\geq\varepsilon\right\}\right| f(\varepsilon)\right) \\ &\quad= \frac{1}{n^u m^v f(n^u m^v)} cf\left(\left|\left\{(j,k)\in I_n\times I_m : |x_{jk}-\ell|\geq\varepsilon\right\}\right| f(\varepsilon)\right) f\left(n^u m^v\right),\end{aligned}$$

where, using the fact that $\lim_{t\to\infty}\frac{f(t)}{t} > 0$ and $x \in w_{\tilde{\alpha}}^2(f,\lambda,\mu)$, it follows that $x \in S_{\tilde{\beta}}^2(\lambda,\mu)$ and the proof is complete. □

If we take $\widetilde{\beta} \cong \widetilde{\alpha}$ in Theorem 4.2, we have the following.

Corollary 4.3 *Let f be an unbounded modulus function $f(xy) \geq cf(x)f(y)$, where c is a positive constant for all $x \geq 0$, $y \geq 0$ and $\lim_{t\to\infty}\frac{f(t)}{t} > 0$ and $\tilde{\alpha} \in (0,1]$. If a sequence is strongly $f_{\lambda,\mu}$-Cesàro summable of order $\tilde{\alpha}$ with respect to f to ℓ, then it is $f_{\lambda,\mu}$-statistically convergent of order $\tilde{\alpha}$ to ℓ.*

5 Conclusions

In this study, we define $f_{\lambda,\mu}$-statistical convergence for double sequences of order $\widetilde{\alpha}$, where f is an unbounded modulus function. Besides this we also study strong $f_{\lambda,\mu}$-Cesàro summability for double sequences of order $\widetilde{\alpha}$ and give inclusion relations. These results are the generalizations of the studies by Meenakshi *et al.* [43].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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