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Modified forward-backward splitting midpoint method with superposition perturbations for the sum of two kinds of infinite accretive mappings and its applications

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Abstract
In a real uniformly convex and p-uniformly smooth Banach space, a modified forward-backward splitting iterative algorithm is presented, where the computational errors and the superposition of perturbed operators are considered. The iterative sequence is proved to be convergent strongly to zero point of the sum of infinite m-accretive mappings and infinite θ_i-inversely strongly accretive mappings, which is also the unique solution of one kind variational inequalities. Some new proof techniques can be found, especially, a new inequality is employed compared to some of the recent work. Moreover, the applications of the newly obtained iterative algorithm to integro-differential systems and convex minimization problems are exemplified.

Keywords: p-uniformly smooth Banach space; θ_i-inversely strongly accretive mapping; γ_i-strongly accretive mapping; μ_i-strictly pseudo-contractive mapping; perturbed operator

1 Introduction and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and X^* be its dual space. '$\rightarrow$' denotes strong convergence and $\langle x, f\rangle$ is the value of $f \in X^*$ at $x \in X$.

The function $\rho_X : [0,+\infty) \to [0,+\infty)$ is called the modulus of smoothness of X if it is defined as follows:

$$\rho_X(t) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : x, y \in X, \|x\| = 1, \|y\| \le t\right\}.$$

A Banach space X is said to be uniformly smooth if $\frac{\rho_X(t)}{t} \to 0$, as $t \to 0$. Let $p > 1$ be a real number, a Banach space X is said to be p-uniformly smooth with constant K_p if $K_p > 0$ such that $\rho_X(t) \le K_p t^p$ for $t > 0$. It is well known that every p-uniformly smooth Banach space is uniformly smooth. For $p > 1$, the generalized duality mapping $J_p : X \to 2^{X^*}$

is defined by

$$J_p x := \{f \in X^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}, \quad x \in X.$$

In particular, $J := J_2$ is called the normalized duality mapping.

For a mapping $T : D(T) \sqsubseteq X \to X$, we use $F(T)$ and $N(T)$ to denote its fixed point set and zero point set, respectively; that is, $F(T) := \{x \in D(T) : Tx = x\}$ and $N(T) = \{x \in D(T) : Tx = 0\}$. The mapping $T : D(T) \sqsubseteq X \to X$ is said to be

(1) non-expansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for } \forall x, y \in D(T);$$

(2) contraction with coefficient $k \in (0,1)$ if

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for } \forall x, y \in D(T);$$

(3) accretive [1, 2] if for all $x, y \in D(T)$, $\langle Tx - Ty, j(x - y)\rangle \geq 0$, where $j(x - y) \in J(x - y)$; m-accretive if T is accretive and $R(I + \lambda T) = X$ for $\forall \lambda > 0$;

(4) θ-inversely strongly accretive [3] if for $\theta > 0$, $\forall x, y \in D(T)$, there exists $j_p(x - y) \in J_p(x - y)$ such that

$$\langle Tx - Ty, j_p(x - y)\rangle \geq \theta\|Tx - Ty\|^p \quad \text{for } \forall x, y \in X;$$

(5) γ-strongly accretive [2, 3] if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \gamma\|x - y\|^2$$

for some $\gamma \in (0,1)$;

(6) μ-strictly pseudo-contractive [4] if for each $x, y \in X$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \mu\|x - y - (Tx - Ty)\|^2$$

for some $\mu \in (0,1)$.

If T is accretive, then for each $r > 0$, the non-expansive single-valued mapping $J_r^T : R(I + rT) \to D(T)$ defined by $J_r^T := (I + rT)^{-1}$ is called the resolvent of T [1]. Moreover, $N(T) = F(J_r^T)$.

Let D be a nonempty closed convex subset of X and Q be a mapping of X onto D. Then Q is said to be sunny [5] if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in X$ and $t \geq 0$. A mapping Q of X into X is said to be a retraction [5] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of X is said to be a sunny non-expansive retract of X [5] if there exists a sunny non-expansive retraction of X onto D and it is called a non-expansive retract of X if there exists a non-expansive retraction of X onto D.

It is a hot topic in applied mathematics to find zero points of the sum of two accretive mappings, namely, a solution of the following inclusion problem:

$$0 \in (A + B)x. \tag{1.1}$$

For example, a stationary solution to the initial value problem of the evolution equation

$$\frac{\partial u}{\partial t} + (A + B)u, \quad u(0) = u_0 \tag{1.2}$$

can be recast as (1.1). A forward-backward splitting iterative method for (1.1) means each iteration involves only A as the forward step and B as the backward step, not the sum $A + B$. The classical forward-backward splitting algorithm is given in the following way:

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, \quad n \in N. \tag{1.3}$$

Some of the related work can be seen in [6–8] and the references therein.

In 2015, Wei et al. [9] extended the related work of (1.1) from a Hilbert space to the real smooth and uniformly convex Banach space and from two accretive mappings to two finite families of accretive mappings:

$$\begin{cases} x_0 \in D, \\ y_n = Q_D[(1 - \alpha_n)(x_n + e_n)], \\ z_n = (1 - \beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i}(y_n - r_{n,i} B_i y_n)], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \in N \cup \{0\}, \end{cases} \tag{1.4}$$

where D is a nonempty, closed and convex sunny non-expansive retract of X, Q_D is the sunny non-expansive retraction of E onto D, $\{e_n\}$ is the error, A_i and B_i are m-accretive mappings and θ-inversely strongly accretive mappings, respectively, where $i = 1, 2, \ldots, N$. $T : X \to X$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f : X \to X$ is a contraction. $\sum_{m=0}^{N} a_m = 1$, $0 < a_m < 1$. The iterative sequence $\{x_n\}$ is proved to converge strongly to $p_0 \in \bigcap_{i=1}^{N} N(A_i + B_i)$, which solves the variational inequality

$$\langle (T - \eta f)p_0, J(p_0 - z) \rangle \leq 0 \tag{1.5}$$

for $\forall z \in \bigcap_{i=1}^{N} N(A_i + B_i)$ under some conditions.

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations, and it has been extensively studied by Alghamdi et al. They presented the following implicit midpoint rule for approximating the fixed point of a non-expansive mapping in a Hilbert space H in [10]:

$$x_1 \in H, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \in N, \tag{1.6}$$

where T is non-expansive from H to H. If $F(T) \neq \emptyset$, then they proved that $\{x_n\}$ converges weakly to $p_0 \in F(T)$ under some conditions.

Combining the ideas of forward-backward method and midpoint method, Wei et al. extended the study of two finite families of accretive mappings to two infinite families of accretive mappings [3] in a real q-uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in D, \\ y_n = Q_D[(1-\alpha_n)(x_n + e'_n)], \\ z_n = \delta_n y_n + \beta_n \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}[\frac{y_n+z_n}{2} - r_{n,i}B_i(\frac{y_n+z_n}{2})] + \zeta_n e''_n, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n + e'''_n, \quad n \in N \cup \{0\}, \end{cases} \tag{1.7}$$

where $\{e'_n\}$, $\{e''_n\}$ and $\{e'''_n\}$ are three error sequences, $A_i : D \to X$ and $B_i : D \to X$ are m-accretive mappings and θ_i-inversely strongly accretive mappings, respectively, where $i \in N$. $T : X \to X$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}, f : X \to X$ is a contraction, $\sum_{n=1}^{\infty} a_n = 1$, $0 < a_n < 1$, $\delta_n + \beta_n + \zeta_n \equiv 1$ for $n \in N \cup \{0\}$. The iterative sequence $\{x_n\}$ is proved to converge strongly to $p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, which solves the following variational inequality:

$$\bigl\langle (T - \eta f)p_0, J(p_0 - z) \bigr\rangle \leq 0, \quad z \in \bigcap_{i=1}^{\infty} N(A_i + B_i). \tag{1.8}$$

In 2012, Ceng et al. [11] presented the following iterative algorithm to approximate zero point of an m-accretive mapping:

$$\begin{cases} x_0 \in X, \\ y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n}^A x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)[J_{r_n}^A y_n - \lambda_n \mu_n F(J_{r_n}^A y_n)], \quad n \in N \cup \{0\}, \end{cases} \tag{1.9}$$

where $T : X \to X$ is a γ-strongly accretive and μ-strictly pseudo-contractive mapping, with $\gamma + \mu > 1$, $f : E \to E$ is a contraction and $A : X \to X$ is m-accretive. Under some assumptions, $\{x_n\}$ is proved to be convergent strongly to the unique element $p_0 \in N(A)$, which solves the following variational inequality:

$$\bigl\langle p_0 - f(p_0), J(p_0 - u) \bigr\rangle \leq 0, \quad \forall u \in N(A). \tag{1.10}$$

The mapping F in (1.9) is called a perturbed operator which only plays a role in the construction of the iterative algorithm for selecting a particular zero of A, and it is not involved in the variational inequality (1.10).

Inspired by the work mentioned above, in Section 2, we shall construct a new modified forward-backward splitting midpoint iterative algorithm to approximate the zero points of the sum of infinite m-accretive mappings and infinite θ_i-inversely strongly accretive mappings. New proof techniques can be found, the superposition of perturbed operators is considered and some restrictions on the parameters are mild compared to the existing similar works. In Section 3, we shall discuss the applications of the newly obtained iterative algorithms to integro-differential systems and the convex minimization problems.

We need the following preliminaries in our paper.

Lemma 1.1 ([12]) *Let X be a real uniformly convex and p-uniformly smooth Banach space with constant K_p for some $p \in (1,2]$. Let D be a nonempty closed convex subset of X. Let $A : D \to X$ be an m-accretive mapping and $B : D \to X$ be a θ-inversely strongly accretive mapping. Then, given $s > 0$, there exists a continuous, strictly increasing and convex function $\varphi_p : R^+ \to R^+$ with $\varphi_p(0) = 0$ such that for all $x, y \in D$ with $\|x\| \leq s$ and $\|y\| \leq s$,*

$$\begin{aligned}
&\bigl\|J_r^A(I-rB)x - J_r^A(I-rB)y\bigr\|^p \\
&\quad \leq \|x-y\|^p - r\bigl(p\theta - K_p r^{p-1}\bigr)\|Bx-By\|^p \\
&\qquad - \varphi_p\bigl(\bigl\|\bigl(I-J_r^A\bigr)(I-rB)x - \bigl(I-J_r^A\bigr)(I-rB)y\bigr\|\bigr).
\end{aligned}$$

In particular, if $0 < r \leq (\frac{p\theta}{K_p})^{\frac{1}{p-1}}$, then $J_r^A(I-rB)$ is non-expansive.

Lemma 1.2 ([13]) *Let X be a real smooth Banach space and $B : X \to X$ be a μ-strictly pseudo-contractive mapping and also be a γ-strongly accretive mapping with $\mu + \gamma > 1$. Then, for any fixed number $\delta \in (0,1)$, $I - \delta B$ is a contraction with coefficient $1 - \delta(1 - \sqrt{\frac{1-\gamma}{\mu}})$.*

Lemma 1.3 ([2]) *Let X be a real Banach space and D be a nonempty closed and convex subset of X. Let $f : D \to D$ be a contraction. Then f has a unique fixed point.*

Lemma 1.4 ([14]) *Let X be a real strictly convex Banach space, and let D be a nonempty closed and convex subset of X. Let $T_m : D \to D$ be a non-expansive mapping for each $m \in N$. Let $\{a_m\}$ be a real number sequence in $(0,1)$ such that $\sum_{m=1}^{\infty} a_m = 1$. Suppose that $\bigcap_{m=1}^{\infty} F(T_m) \neq \emptyset$. Then the mapping $\sum_{m=1}^{\infty} a_m T_m$ is non-expansive and $F(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} F(T_m)$.*

Lemma 1.5 ([12]) *In a real Banach space X, for $p > 1$, the following inequality holds*:

$$\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y)\rangle, \quad \forall x, y \in X, j_p(x+y) \in J_p(x+y).$$

Lemma 1.6 ([15]) *Let X be a real Banach space, and let D be a nonempty closed and convex subset of X. Suppose $A : D \to X$ is a single-valued mapping and $B : X \to 2^X$ is m-accretive. Then*

$$F\bigl((I+rB)^{-1}(I-rA)\bigr) = N(A+B) \quad \text{for } \forall r > 0.$$

Lemma 1.7 ([16]) *Let $\{a_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_k}\}$ so that $a_{n_k} \leq a_{n_k+1}$ for all $k \in N \cup \{0\}$. For every $n > n_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{n_0 \leq k \leq n : a_k < a_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, $\max\{a_{\tau(n)}, a_n\} \leq a_{\tau(n)+1}$.

Lemma 1.8 ([17]) *For $p > 1$, the following inequality holds*:

$$ab \leq \frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}},$$

for any positive real numbers a and b.

Lemma 1.9 ([18]) *The Banach space X is uniformly smooth if and only if the duality mapping J_p is single-valued and norm-to-norm uniformly continuous on bounded subsets of X.*

2 Strong convergence theorems

Theorem 2.1 *Let X be a real uniformly convex and p-uniformly smooth Banach space with constant K_p where $p \in (1,2]$ and D be a nonempty closed and convex sunny non-expansive retract of X. Let Q_D be the sunny non-expansive retraction of X onto D. Let $f : X \to X$ be a contraction with coefficient $k \in (0,1)$, $A_i : D \to X$ be m-accretive mappings, $C_i : D \to X$ be θ_i-inversely strongly accretive mappings, $W_i : X \to X$ be μ_i-strictly pseudo-contractive mappings and γ_i-strongly accretive mappings with $\mu_i + \gamma_i > 1$ for $i \in N$. Suppose $\{\omega_i^{(1)}\}$ and $\{\omega_i^{(2)}\}$ are real number sequences in $(0,1)$ for $i \in N$. Suppose $0 < r_{n,i} \leq (\frac{p\theta_i}{K_p})^{\frac{1}{p-1}}$ for $i \in N$ and $n \in N$, $\kappa_t \in (0,1)$ for $t \in (0,1)$, $\sum_{i=1}^{\infty} \omega_i^{(1)} \|W_i\| < +\infty$, $\sum_{i=1}^{\infty} \omega_i^{(1)} = \sum_{i=1}^{\infty} \omega_i^{(2)} = 1$ and $\bigcap_{i=1}^{\infty} N(A_i + C_i) \neq \emptyset$. If, for each $t \in (0,1)$, we define $Z_t^n : X \to X$ by*

$$Z_t^n u = tf(u) + (1-t)\left(I - \kappa_t \sum_{i=1}^{\infty} \omega_i^{(1)} W_i\right)\left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u\right),$$

then Z_t^n has a fixed point u_t^n. Moreover, if $\frac{\kappa_t}{t} \to 0$, then u_t^n converges strongly to the unique solution q_0 of the following variational inequality, as $t \to 0$:

$$\langle q_0 - f(q_0), J(q_0 - u)\rangle \leq 0, \quad \forall u \in \bigcap_{i=1}^{\infty} N(A_i + C_i). \tag{2.1}$$

Proof We split the proof into five steps.

Step 1. $Z_t^n : X \to X$ is a contraction for $t \in (0,1)$, $\kappa_t \in (0,1)$ and $n \in N$.

In fact, for $\forall x, y \in X$, using Lemmas 1.1 and 1.2, we have

$$\begin{aligned}
&\|Z_t^n x - Z_t^n y\| \\
&\quad \leq t\|f(x) - f(y)\| + (1-t) \times \left\| \sum_{i=1}^{\infty} \omega_i^{(1)}(I - \kappa_t W_i)\left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D x\right) \right. \\
&\qquad \left. - \sum_{i=1}^{\infty} \omega_i^{(1)}(I - \kappa_t W_i)\left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D y\right) \right\| \\
&\quad \leq tk\|x - y\| + (1-t)\sum_{i=1}^{\infty} \omega_i^{(1)}\left[1 - \kappa_t\left(1 - \sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right]\|x - y\| \\
&\quad \leq \bigl[1 - (1-k)t\bigr]\|x - y\|,
\end{aligned}$$

which implies that Z_t^n is a contraction. By Lemma 1.3, there exists u_t^n such that $Z_t^n u_t^n = u_t^n$. That is, $u_t^n = tf(u_t^n) + (1-t)(I - \kappa_t \sum_{i=1}^{\infty} \omega_i^{(1)} W_i)(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n)$.

Step 2. If $\lim_{t\to 0}\frac{\kappa_t}{t}=0$, then $\{u_t^n\}$ is bounded for $n\in N$, $0<t\le \overline{a}$, where $\overline{a}$ is a sufficiently small positive number and u_t^n is the same as that in Step 1.

For $\forall u\in\bigcap_{i=1}^{\infty}N(A_i+C_i)$, using Lemmas 1.1, 1.2 and 1.6, we know that

$$\begin{aligned}
\|u_t^n-u\| &\le tk\|u_t^n-u\|+t\|f(u)-u\|+(1-t)\kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_iu\| \\
&\quad +(1-t)\sum_{i=1}^{\infty}\omega_i^{(1)}\left\|(I-\kappa_tW_i)\left[\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du_t^n\right.\right. \\
&\quad \left.\left.-\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du\right]\right\| \\
&\le tk\|u_t^n-u\|+t\|f(u)-u\|+(1-t)\kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|u\| \\
&\quad +(1-t)\sum_{i=1}^{\infty}\omega_i^{(1)}\left[1-\kappa_t\left(1-\sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right]\|u_t^n-u\| \\
&\le t\|f(u)-u\|+(1-t+tk)\|u_t^n-u\|+(1-t)\kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|u\|.
\end{aligned}$$

Then

$$\|u_t^n-u\|\le\frac{\|f(u)-u\|+\frac{\kappa_t}{t}\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|u\|}{1-k}.$$

Since $\lim_{t\to 0}\frac{\kappa_t}{t}=0$, then there exists a sufficiently small positive number $\overline{a}$ such that $0<\frac{\kappa_t}{t}<1$ for $0<t\le\overline{a}$. Thus $\{u_t^n\}$ is bounded for $n\in N$ and $0<t\le\overline{a}$.

Step 3. If $\lim_{t\to 0}\frac{\kappa_t}{t}=0$, then $u_t^n-\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du_t^n\to 0$, as $t\to 0$, for $n\in N$.

Noticing Step 2, we have

$$\begin{aligned}
&\left\|u_t^n-\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du_t^n\right\| \\
&\quad\le t\|f(u_t^n)\|+t\sum_{i=1}^{\infty}\omega_i^{(2)}\|J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du_t^n\| \\
&\qquad +(1-t)\kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}\left\|W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_Du_t^n\right)\right\| \\
&\quad\to 0,
\end{aligned}$$

as $t\to 0$.

Step 4. If the variational inequality (2.1) has solutions, the solution must be unique.

Suppose $u_0\in\bigcap_{i=1}^{\infty}N(A_i+C_i)$ and $v_0\in\bigcap_{i=1}^{\infty}N(A_i+C_i)$ are two solutions of (2.1), then

$$\langle u_0-f(u_0),J(u_0-v_0)\rangle\le 0, \tag{2.2}$$

and

$$\langle v_0-f(v_0),J(v_0-u_0)\rangle\le 0. \tag{2.3}$$

Adding up (2.2) and (2.3), we get

$$\langle u_0 - f(u_0) - v_0 + f(v_0), J(u_0 - v_0)\rangle \le 0. \tag{2.4}$$

Since

$$\begin{aligned}
&\langle u_0 - f(u_0) - v_0 + f(v_0), J(u_0 - v_0)\rangle \\
&\quad = \|u_0 - v_0\|^2 - \langle f(u_0) - f(v_0), J(u_0 - v_0)\rangle \\
&\quad \ge \|u_0 - v_0\|^2 - k\|u_0 - v_0\|^2 = (1-k)\|u_0 - v_0\|^2,
\end{aligned}$$

then (2.4) implies that $u_0 = v_0$.

Step 5. If $\lim_{t\to 0}\frac{\kappa_t}{t} = 0$, then $u_t^n \to q_0 \in \bigcap_{i=1}^\infty N(A_i + C_i)$, as $t \to 0$, which solves the variational inequality (2.1).

Assume $t_m \to 0$. Set $u_m^n := u_{t_m}^n$ and define $\mu : X \to R$ by

$$\mu(u) = \mathrm{LIM}\|u_m^n - u\|^2, \quad u \in X,$$

where LIM is the Banach limit on l^∞. Let

$$K = \Big\{x \in X : \mu(x) = \min_{x\in X}\mathrm{LIM}\|u_m^n - x\|^2\Big\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of X. Since $u_m^n - \sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n \to 0$ from Step 3, then for $u \in K$,

$$\begin{aligned}
\mu\left(\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u\right) &= \mathrm{LIM}\left\|u_m^n - \sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u\right\|^2 \\
&\le \mathrm{LIM}\|u_m^n - u\|^2 = \mu(u),
\end{aligned}$$

it follows that $\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D(K) \subset K$; that is, K is invariant under $\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D$. Since a uniformly smooth Banach space has the fixed point property for non-expansive mappings, $\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D$ has a fixed point, say q_0, in K. That is, $\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D q_0 = q_0 \in D$, which ensures from Lemmas 1.4 and 1.6 that $q_0 \in \bigcap_{i=1}^\infty N(A_i + C_i)$. Since q_0 is also a minimizer of μ over X, it follows that, for $t \in (0,1)$,

$$\begin{aligned}
0 &\le \frac{\mu(q_0 + tf(q_0) - tq_0) - \mu(q_0)}{t} \\
&= \mathrm{LIM}\frac{\|u_m^n - q_0 - tf(q_0) + tq_0\|^2 - \|u_m^n - q_0\|^2}{t} \\
&= \mathrm{LIM}\frac{\langle u_m^n - q_0 - tf(q_0) + tq_0, J(u_m^n - q_0 - tf(q_0) + tq_0)\rangle - \|u_m^n - q_0\|^2}{t} \\
&= \mathrm{LIM}\big(\langle u_m^n - q_0, J(u_m^n - q_0 - tf(q_0) + tq_0)\rangle \\
&\qquad + t\langle q_0 - f(q_0), J(u_m^n - q_0 - tf(q_0) + tq_0)\rangle - \|u_m^n - q_0\|^2\big)/t.
\end{aligned}$$

Since X is uniformly smooth, then by letting $t \to 0$, we find the two limits above can be interchanged and obtain

$$\mathrm{LIM}\langle f(q_0) - q_0, J(u_m^n - q_0)\rangle \leq 0. \tag{2.5}$$

Since $u_m^n - q_0 = t_m(f(u_m^n) - q_0) + (1 - t_m)[(I - \kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}W_i)(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n) - q_0]$, then

$$\begin{aligned}
\|u_m^n - q_0\|^2 &= \langle u_m^n - q_0, J(u_m^n - q_0)\rangle \\
&\leq t_m\langle f(u_m^n) - f(q_0), J(u_m^n - q_0)\rangle + t_m\langle f(q_0) - q_0, J(u_m^n - q_0)\rangle \\
&\quad + (1 - t_m)\left\|\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n - q_0\right\|\|u_m^n - q_0\| \\
&\quad + (1 - t_m)\kappa_{t_m}\left\|\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n\right)\right\|\|u_m^n - q_0\| \\
&\leq (1 - t_m + t_m k)\|u_m^n - q_0\|^2 + t_m\langle f(q_0) - q_0, J(u_m^n - q_0)\rangle \\
&\quad + (1 - t_m)\kappa_{t_m}\left\|\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n\right)\right\|\|u_m^n - q_0\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_m^n - q_0\|^2 \leq \frac{1}{1-k}\Bigg[&\langle f(q_0) - q_0, J(u_m^n - q_0)\rangle \\
&+ \frac{\kappa_{t_m}}{t_m}\left\|\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n\right)\right\|\|u_m^n - q_0\|\Bigg].
\end{aligned} \tag{2.6}$$

Since $\frac{\kappa_{t_m}}{t_m} \to 0$, then from (2.5), (2.6) and the result of Step 2, we have $\mathrm{LIM}\|u_m^n - q_0\|^2 \leq 0$, which implies that $\mathrm{LIM}\|u_m^n - q_0\|^2 = 0$, and then there exists a subsequence which is still denoted by $\{u_m^n\}$ such that $u_m^n \to q_0$.

Next, we shall show that q_0 solves the variational inequality (2.1).

Note that $u_m^n = t_m f(u_m^n) + (1 - t_m)(I - \kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}W_i)(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n)$, then for $\forall \nu \in \bigcap_{i=1}^{\infty}N(A_i + C_i)$,

$$\begin{aligned}
&\left\langle\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n - f(u_m^n), J(u_m^n - \nu)\right\rangle \\
&= \frac{1}{t_m}\left\langle\left(I - \kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\right)\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n\right), J(u_m^n - \nu)\right\rangle \\
&\quad - \frac{1}{t_m}\left\langle u_m^n - t_m\kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_m^n\right), J(u_m^n - \nu)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t_m}\Bigg\langle \sum_{i=1}^{\infty}\omega_i^{(1)}(I-\kappa_{t_m}W_i)\Bigg(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_C u_m^n\Bigg) \\
&\quad - \sum_{i=1}^{\infty}\omega_i^{(1)}(I-\kappa_{t_m}W_i)\Bigg(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D v\Bigg), J\big(u_m^n - v\big)\Bigg\rangle \\
&\quad - \frac{1}{t_m}\big\|u_m^n - v\big\|^2 - \frac{\kappa_{t_m}}{t_m}\Bigg\langle \sum_{i=1}^{\infty}\omega_i^{(1)}W_i v, J\big(u_m^n - v\big)\Bigg\rangle \\
&\quad + \kappa_{t_m}\Bigg\langle \sum_{i=1}^{\infty}\omega_i^{(1)}W_i\Bigg(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_m^n\Bigg), J\big(u_m^n - v\big)\Bigg\rangle \\
&\leq -\frac{1}{t_m}\Bigg\{1-\sum_{i=1}^{\infty}\omega_i^{(1)}\Bigg[1-\kappa_{t_m}\Bigg(1-\sqrt{\frac{1-\gamma_i}{\mu_i}}\Bigg)\Bigg]\Bigg\}\big\|u_m^n - v\big\|^2 \\
&\quad + \frac{\kappa_{t_m}}{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|v\|\big\|u_m^n - v\big\| \\
&\quad + \kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}\Bigg\|W_i\Bigg(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_m^n\Bigg)\Bigg\|\big\|u_m^n - v\big\| \\
&\leq \frac{\kappa_{t_m}}{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|v\|\big\|u_m^n - v\big\| \\
&\quad + \kappa_{t_m}\sum_{i=1}^{\infty}\omega_i^{(1)}\Bigg\|W_i\Bigg(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_m^n\Bigg)\Bigg\|\big\|u_m^n - v\big\| \\
&\to 0,
\end{aligned}
$$

as $t_m \to 0$. Since $x_n \to q_0$ and J is uniformly continuous on each bounded subset of X, then taking the limits on both sides of the above inequality, $\langle q_0 - f(q_0), J(q_0 - v)\rangle \leq 0$, which implies that q_0 satisfies the variational inequality (2.1).

Next, to prove the net $\{u_m^n\}$ converges strongly to q_0, as $t \to 0$, suppose that there is another subsequence $\{u_{t_k}^n\}$ of $\{u_t^n\}$ satisfying $u_{t_k}^n \to v_0$ as $t_k \to 0$. Denote $u_{t_k}^n$ by u_k^n. Then the result of Step 3 implies that $0 = \lim_{t_k\to 0}(u_k^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_k^n) = v_0 - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D v_0$, which ensures that $v_0 \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$ in view of Lemmas 1.4 and 1.6. Repeating the above proof, we can also know that v_0 solves the variational inequality (2.1). Thus $q_0 = v_0$ by using the result of Step 4.

Hence $u_t^n \to q_0$, as $t \to 0$, which is the unique solution of the variational inequality (2.1).

This completes the proof. □

Theorem 2.2 *Let X be a real uniformly convex and p-uniformly smooth Banach space with constant K_p where $p \in (1,2]$ and D be a nonempty closed and convex sunny non-expansive retract of X. Let Q_D be the sunny non-expansive retraction of X onto D. Let $f : X \to X$ be a contraction with coefficient $k \in (0,1)$, $A_i : D \to X$ be m-accretive mappings, $C_i : D \to X$ be θ_i-inversely strongly accretive mappings, and $W_i : X \to X$ be μ_i-strictly pseudo-contractive mappings and γ_i-strongly accretive mappings with $\mu_i + \gamma_i > 1$ for $i \in N$. Suppose $\{\omega_i^{(1)}\}$, $\{\omega_i^{(2)}\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\vartheta_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, $\{\delta_n\}$ and $\{\zeta_n\}$ are real number sequences in $(0,1)$, $\{r_{n,i}\} \subset (0,+\infty)$, $\{a_n\} \subset X$ and $\{b_n\} \subset D$ are error sequences, where $n \in N$ and $i \in N$. Suppose $\bigcap_{i=1}^{\infty} N(A_i + C_i) \neq \emptyset$. Let $\{x_n\}$ be generated by the following iterative* *algorithm*:

$$\begin{cases} x_1 \in D, \\ u_n = Q_D(\alpha_n x_n + \beta_n a_n), \\ v_n = \vartheta_n u_n + \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}) + \xi_n b_n, \\ x_{n+1} = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} W_i) \sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}), \quad n \in N. \end{cases} \tag{2.7}$$

Under the following assumptions:

(i) $\alpha_n + \beta_n \le 1$, $\vartheta_n + \nu_n + \xi_n \equiv 1$ *for* $n \in N$;

(ii) $\sum_{i=1}^{\infty} \omega_i^{(1)} = \sum_{i=1}^{\infty} \omega_i^{(2)} = 1$;

(iii) $\sum_{n=1}^{\infty} \|a_n\| < +\infty$, $\sum_{n=1}^{\infty} \|b_n\| < +\infty$, $\sum_{n=1}^{\infty}(1-\alpha_n) < +\infty$, $\sum_{n=1}^{\infty} \xi_n < +\infty$, $\lim_{n\to\infty} \sum_{i=1}^{\infty} r_{n,i} = 0$;

(iv) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = +\infty$;

(v) $1 - \alpha_n + \|a_n\| = o(\delta_n)$, $\xi_n = o(\delta_n)$, $\zeta_n = o(\xi_n)$, $\nu_n \nrightarrow 0$, *as* $n \to \infty$;

(vi) $\sum_{i=1}^{\infty} \omega_i^{(1)} \|W_i\| < +\infty$, $0 < r_{n,i} \le (\frac{p\theta_i}{K_p})^{\frac{1}{p-1}}$ *for* $i \in N$, $n \in N$,

the iterative sequence $x_n \to q_0 \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$, *which is the unique solution of the variational inequality* (2.1).

Proof We split the proof into four steps.

Step 1. $\{v_n\}$ is well defined and so is $\{x_n\}$.

For $s, t \in (0,1)$, define $H_{s,t} : D \to D$ by $H_{s,t}x := su + tH(\frac{u+x}{2}) + (1-s-t)v$, where $H : D \to D$ is non-expansive for $x \in D$ and $u, v \in D$. Then, for $\forall x, y \in D$,

$$\|H_{s,t}x - H_{s,t}y\| \le t\left\|\frac{u+x}{2} - \frac{u+y}{2}\right\| \le \frac{t}{2}\|x-y\|.$$

Thus $H_{s,t}$ is a contraction, which ensures from Lemma 1.3 that there exists $x_{s,t} \in D$ such that $H_{s,t}x_{s,t} = x_{s,t}$. That is, $x_{s,t} = su + tH(\frac{u+x_{s,t}}{2}) + (1-s-t)v$.

Since $\sum_{i=1}^{\infty} \omega_i^{(2)} = 1$ and $J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)$ is non-expansive for $n \in N$ and $i \in N$, then $\{v_n\}$ is well defined, which implies that $\{x_n\}$ is well defined.

Step 2. $\{x_n\}$ is bounded.

For $\forall p \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$, we can easily know that

$$\|u_n - p\| \le \alpha_n \|x_n - p\| + \beta_n \|a_n\| + (1-\alpha_n)\|p\|.$$

And

$$\begin{aligned} \|v_n - p\| &\le \vartheta_n \|u_n - p\| + \nu_n \left\|\frac{u_n + v_n}{2} - p\right\| + \xi_n \|b_n - p\| \\ &\le \left(\vartheta_n + \frac{\nu_n}{2}\right)\|u_n - p\| + \frac{\nu_n}{2}\|v_n - p\| + \xi_n \|b_n - p\|. \end{aligned}$$

Thus

$$\begin{aligned} \|v_n - p\| &\le \left(\frac{2\vartheta_n + \nu_n}{2 - \nu_n}\right)\|u_n - p\| + \frac{2\xi_n}{2-\nu_n}\|b_n - p\| \\ &\le \alpha_n \|x_n - p\| + \beta_n \|a_n\| + (1-\alpha_n)\|p\| + 2\|b_n\| + \frac{2\xi_n}{2-\nu_n}\|p\|. \end{aligned} \tag{2.8}$$

Using Lemma 1.2 and (2.8), we have, for $n \in N$,

$$\begin{aligned}
\|x_{n+1}-p\| &\le \delta_n \big\|f(x_n)-f(p)\big\| + \delta_n \big\|f(p)-p\big\| \\
&\quad + (1-\delta_n)\left\|\left(I-\zeta_n\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\right)\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)-p\right\| \\
&\le \delta_n k\|x_n-p\| + \delta_n\big\|f(p)-p\big\| + (1-\delta_n)\zeta_n\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|p\| \\
&\quad + (1-\delta_n)\left\|\sum_{i=1}^{\infty}\omega_i^{(1)}(I-\zeta_n W_i)\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right. \\
&\quad \left. -\sum_{i=1}^{\infty}\omega_i^{(1)}(I-\zeta_n W_i)\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)p\right\| \\
&\le \delta_n k\|x_n-p\| + \delta_n\big\|f(p)-p\big\| + (1-\delta_n)\zeta_n\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|p\| \\
&\quad + (1-\delta_n)\left[1-\zeta_n\left(1-\sum_{i=1}^{\infty}\omega_i^{(1)}\sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right] \\
&\quad \times\left[\alpha_n\|x_n-p\|+\beta_n\|a_n\|+(1-\alpha_n)\|p\|+2\|b_n\|+\frac{\xi_n}{2-\nu_n}\|p\|\right] \\
&\le \left\{(1-\delta_n)\left[1-\zeta_n\left(1-\sum_{i=1}^{\infty}\omega_i^{(1)}\sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right]+\delta_n k\right\}\|x_n-p\| + \delta_n\big\|f(p)-p\big\| \\
&\quad + (1-\delta_n)\left[1-\zeta_n\left(1-\sum_{i=1}^{\infty}\omega_i^{(1)}\sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right] \\
&\quad \times\left[\beta_n\|a_n\|+(1-\alpha_n)\|p\|+\|b_n\|+\frac{\xi_n}{2-\nu_n}\|p\|\right] \\
&\quad + (1-\delta_n)\zeta_n\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|p\|.
\end{aligned} \tag{2.9}$$

By using the inductive method, we can easily get the following result from (2.9):

$$\begin{aligned}
\|x_{n+1}-p\| &\le \max\left\{\|x_1-p\|,\frac{\sum_{i=1}^{\infty}\omega_i^{(1)}\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}\omega_i^{(1)}\sqrt{\frac{1-\gamma_i}{\mu_i}}},\frac{\|f(p)-p\|}{1-k}\right\} \\
&\quad + \sum_{k=1}^{n}(1-\delta_k)\left[1-\zeta_k\left(1-\sum_{i=1}^{\infty}\omega_i^{(1)}\sqrt{\frac{1-\gamma_i}{\mu_i}}\right)\right] \\
&\quad \times\left[\beta_k\|a_k\|+(1-\alpha_k)\|p\|+\|b_k\|+\frac{\xi_k}{2-\nu_k}\|p\|\right].
\end{aligned}$$

Therefore, from assumptions (iii) and (vi), we know that $\{x_n\}$ is bounded.

Step 3. There exists $q_0 \in \bigcap_{i=1}^{\infty} N(A_i+C_i)$, which solves the variational inequality (2.1).

Using Theorem 2.1, we know that there exists u_t^n such that $u_t^n = tf(u_t^n) + (1-t)(I - \kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}W_i)(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n)$ for $t \in (0,1)$. Moreover, under the assumption that $\frac{\kappa_t}{t} \to 0$, $u_t^n \to q_0 \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$, as $t \to 0$, which is the unique solution of the variational inequality (2.1).

Step 4. $x_n \to q_0$, as $n \to \infty$, where q_0 is the same as that in Step 3.

Set $C_1 := \sup\{2\|\alpha_n x_n + \beta_n a_n - q_0\|^{p-1}, 2\|q_0\|\|\alpha_n x_n + \beta_n a_n - q_0\|^{p-1} : n \in N\}$, then from Step 2 and assumption (iii), C_1 is a positive constant. Using Lemma 1.5, we have

$$\begin{aligned} \|u_n - q_0\|^p &\le \alpha_n \|x_n - q_0\|^p - p(1-\alpha_n)\langle q_0, J_p(\alpha_n x_n + \beta_n a_n - q_0)\rangle \\ &\quad + p\beta_n \langle a_n, J_p(\alpha_n x_n + \beta_n a_n - q_0)\rangle \\ &\le \alpha_n \|x_n - q_0\|^p + C_1(1-\alpha_n) + C_1\|a_n\|. \end{aligned} \tag{2.10}$$

Using Lemma 1.1, we know that

$$\begin{aligned} \|v_n - q_0\|^p &\le \vartheta_n \|u_n - q_0\|^p + \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} \left\| J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - q_0 \right\|^p \\ &\quad + \xi_n \|b_n - q_0\|^p \\ &\le \left(\vartheta_n + \frac{\nu_n}{2}\right)\|u_n - q_0\|^p + \frac{\nu_n}{2}\|v_n - q_0\|^p \\ &\quad - \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} r_{n,i}\left(\theta_i p - r_{n,i}^{p-1}K_p\right) \left\| C_i\left(\frac{u_n + v_n}{2}\right) - C_i q_0 \right\|^p \\ &\quad - \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} \varphi_p\left(\left\| \left(I - J_{r_{n,i}}^{A_i}\right)\left(\frac{u_n + v_n}{2} - r_{n,i}C_i\left(\frac{u_n + v_n}{2}\right)\right) \right.\right. \\ &\quad \left.\left. - \left(I - J_{r_{n,i}}^{A_i}\right)(q_0 - r_{n,i}C_i q_0) \right\| \right) + \xi_n \|b_n - q_0\|^p. \end{aligned}$$

Therefore,

$$\begin{aligned} \|v_n - q_0\|^p &\le \frac{2\vartheta_n + \nu_n}{2 - \nu_n}\|u_n - q_0\|^p + \frac{2\xi_n}{2 - \nu_n}\|b_n - q_0\|^p \\ &\quad - \frac{2\nu_n}{2 - \nu_n} \sum_{i=1}^{\infty} \omega_i^{(2)} r_{n,i}\left(\theta_i p - r_{n,i}^{p-1}K_p\right) \left\| C_i\left(\frac{u_n + v_n}{2}\right) - C_i q_0 \right\|^p \\ &\quad - \frac{2\nu_n}{2 - \nu_n} \sum_{i=1}^{\infty} \omega_i^{(2)} \varphi_p\left(\left\| \left(I - J_{r_{n,i}}^{A_i}\right)\left(\frac{u_n + v_n}{2} - r_{n,i}C_i\left(\frac{u_n + v_n}{2}\right)\right) \right.\right. \\ &\quad \left.\left. - \left(I - J_{r_{n,i}}^{A_i}\right)(q_0 - r_{n,i}C_i q_0) \right\| \right). \end{aligned} \tag{2.11}$$

Now, from (2.10)–(2.11) and Lemmas 1.4 and 1.5, we know that for $n \in N$,

$$\begin{aligned} &\|x_{n+1} - q_0\|^p \\ &\quad = \left\| \delta_n\left(f(x_n) - q_0\right) + (1-\delta_n)\left(\sum_{k=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - q_0\right) \right. \\ &\quad \left. - (1-\delta_n)\zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} W_i\left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right) \right\|^p \end{aligned}$$

$$
\begin{aligned}
&\le (1-\delta_n)\left\| \frac{u_n+v_n}{2} - q_0 \right\|^p \\
&\quad + p\delta_n \big\langle f(x_n) - f(q_0), J_p(x_{n+1}-q_0)\big\rangle + p\delta_n \big\langle f(q_0) - q_0, J_p(x_{n+1}-q_0)\big\rangle \\
&\quad - p(1-\delta_n)\zeta_n \left\langle \sum_{i=1}^{\infty} \omega_i^{(1)} W_i \left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i} (I - r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right), J_p(x_{n+1}-q_0)\right\rangle \\
&\le (1-\delta_n)\left(\frac{\|u_n - p_0\|^2}{2} + \frac{\|v_n - p_0\|^2}{2}\right) \\
&\quad + p\delta_n k\|x_n - q_0\|\|x_{n+1}-q_0\|^{p-1} + p\delta_n\big\langle f(q_0)-q_0, J_p(x_{n+1}-q_0)\big\rangle \\
&\quad - p(1-\delta_n)\zeta_n \left\langle \sum_{i=1}^{\infty} \omega_i^{(1)} W_i \left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i} (I - r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right), J_p(x_{n+1}-q_0)\right\rangle \\
&\le (1-\delta_n)\|u_n - p_0\|^2 + (1-\delta_n)\Bigg[\frac{\xi_n}{2-\nu_n}\|b_n - q_0\|^p \\
&\quad - \frac{\nu_n}{2-\nu_n}\sum_{i=1}^{\infty}\omega_i^{(2)} r_{n,i}\big(\theta_i p - r_{n,i}^{p-1}K_p\big)\left\| C_i\left(\frac{u_n+v_n}{2}\right) - C_i q_0\right\|^p \\
&\quad - \frac{\nu_n}{2-\nu_n}\sum_{i=1}^{\infty}\omega_i^{(2)}\varphi_p\Bigg(\Bigg\| \big(I - J_{r_{n,i}}^{A_i}\big)\left(\frac{u_n+v_n}{2} - r_{n,i}C_i\left(\frac{u_n+v_n}{2}\right)\right) \\
&\quad - \big(I - J_{r_{n,i}}^{A_i}\big)(q_0 - r_{n,i}C_i q_0)\Bigg\|\Bigg)\Bigg] \\
&\quad + pk\delta_n\|x_n - q_0\|\|x_{n+1}-q_0\|^{p-1} + p\delta_n\big\langle f(q_0)-q_0, J_p(x_{n+1}-q_0)\big\rangle \\
&\quad - p(1-\delta_n)\zeta_n \left\langle \sum_{i=1}^{\infty} \omega_i^{(1)} W_i \left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i} (I - r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right), J_p(x_{n+1}-q_0)\right\rangle \\
&\le (1-\delta_n)\|x_n - p_0\|^p + C_1\big(1-\alpha_n + \|a_n\|\big) + k\delta_n\|x_n - q_0\|^p \\
&\quad + k\delta_n\|x_{n+1}-q_0\|^p + \frac{\xi_n}{2-\nu_n}\|b_n - q_0\|^p \\
&\quad - (1-\delta_n)\frac{\nu_n}{2-\nu_n}\sum_{i=1}^{\infty}\omega_i^{(2)}\varphi_p\Bigg(\Bigg\| \big(I - J_{r_{n,i}}^{A_i}\big)\left(\frac{u_n+v_n}{2} - r_{n,i}C_i\left(\frac{u_n+v_n}{2}\right)\right) \\
&\quad - \big(I - J_{r_{n,i}}^{A_i}\big)(q_0 - r_{n,i}C_i q_0)\Bigg\|\Bigg) + p\delta_n\big\langle f(q_0)-q_0, J_p(x_{n+1}-q_0)\big\rangle \\
&\quad + 2\zeta_n\sum_{i=1}^{\infty}\omega_i^{(1)}\left\| W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\| \big\|J_p(x_{n+1}-q_0)\big\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1}-q_0\|^p \\
&\quad \le \frac{1-\delta_n(1-k)}{1-\delta_n k}\|x_n - q_0\|^p + \frac{C_1(1-\alpha_n + \|a_n\|)}{1-\delta_n k} \\
&\qquad + \frac{1}{1-\delta_n k}\left(\frac{\xi_n}{2-\nu_n}\|b_n - q_0\|^p + p\delta_n\big\langle f(q_0)-q_0, J_p(x_{n+1}-q_0)\big\rangle \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} \left\| W_i\left(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right)\right\| \left\|J_p(x_{n+1} - q_0)\right\| \Bigg) \\
&- (1 - \delta_n)\frac{v_n}{2 - v_n}\frac{1}{1 - \delta_n k}\sum_{i=1}^{\infty} \omega_i^{(2)} \varphi_p\left(\left\|\left(I - J_{r_{n,i}}^{A_i}\right)\left(\frac{u_n + v_n}{2} - r_{n,i}C_i\left(\frac{u_n + v_n}{2}\right)\right)\right.\right. \\
&- \left.\left.\left(I - J_{r_{n,i}}^{A_i}\right)(q_0 - r_{n,i}C_i q_0)\right\|\right).
\end{aligned}
$$

From Step 2, if we set $C_2 = \sup\{\sum_{i=1}^{\infty} \omega_i^{(1)} \| W_i(\sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}))\|, \|x_n - q_0\|^{p-1} : n \in N\}$, then C_2 is a positive constant.

Let $\varepsilon_n^{(1)} = \frac{\delta_n(1-2k)}{1-\delta_n k}$, $\varepsilon_n^{(2)} = \frac{1}{\delta_n(1-2k)}[C_1(1-\alpha_n + \|a_n\|) + \frac{\xi_n}{2-v_n}\|b_n - q_0\|^p + p\delta_n\langle f(q_0) - q_0, J_p(x_{n+1} - q_0)\rangle + 2\zeta_n C_2^2]$ and $\varepsilon_n^{(3)} = (1-\delta_n)\frac{v_n}{2-v_n}\frac{1}{1-\delta_n k}\sum_{i=1}^{\infty}\omega_i^{(2)}\varphi_p(\|(I - J_{r_{n,i}}^{A_i})(\frac{u_n+v_n}{2} - r_{n,i}C_i(\frac{u_n+v_n}{2})) - (I - J_{r_{n,i}}^{A_i})(q_0 - r_{n,i}C_i q_0)\|)$.

Then

$$
\|x_{n+1} - q_0\|^p \le \left(1 - \varepsilon_n^{(1)}\right)\|x_n - q_0\|^p + \varepsilon_n^{(1)}\varepsilon_n^{(2)} - \varepsilon_n^{(3)}. \tag{2.12}
$$

Our next discussion will be divided into two cases.

Case 1. $\{\|x_n - q_0\|\}$ is decreasing.

If $\{\|x_n - q_0\|\}$ is decreasing, we know from (2.12) and assumptions (iv) and (v) that

$$
0 \le \varepsilon_n^{(3)} \le \varepsilon_n^{(1)}\left(\varepsilon_n^{(2)} - \|x_n - q_0\|^p\right) + \left(\|x_n - q_0\|^p - \|x_{n+1} - q_0\|^p\right) \to 0,
$$

which ensures that $\sum_{i=1}^{\infty}\omega_i^{(2)}\varphi_p(\|(I - J_{r_{n,i}}^{A_i})(\frac{u_n+v_n}{2} - r_{n,i}C_i(\frac{u_n+v_n}{2})) - (I - J_{r_{n,i}}^{A_i})(q_0 - r_{n,i}C_i q_0)\|) \to 0$, as $n \to +\infty$. Then, from the property of φ_p, we know that $\sum_{i=1}^{\infty}\omega_i^{(2)}\|(I - J_{r_{n,i}}^{A_i})(\frac{u_n+v_n}{2} - r_{n,i}C_i(\frac{u_n+v_n}{2})) - (I - J_{r_{n,i}}^{A_i})(q_0 - r_{n,i}C_i q_0)\| \to 0$, as $n \to +\infty$.

Note that $\lim_{n\to\infty}\sum_{i=1}^{\infty} r_{n,i} = 0$, then

$$
\begin{aligned}
&\left\|\frac{u_n + v_n}{2} - \sum_{i=1}^{\infty}\omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right\| \\
&\le \sum_{i=1}^{\infty}\omega_i^{(2)}\left\|\left(I - J_{r_{n,i}}^{A_i}\right)(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \left(I - J_{r_{n,i}}^{A_i}\right)(I - r_{n,i}C_i)q_0\right\| \\
&\quad + \sum_{i=1}^{\infty}\omega_i^{(2)} r_{n,i}\left\|C_i\left(\frac{u_n + v_n}{2}\right)\right\| + \sum_{i=1}^{\infty}\omega_i^{(2)} r_{n,i}\|C_i q_0\| \\
&\to 0,
\end{aligned}
$$

as $n \to \infty$.

Now, our purpose is to show that $\limsup_{n\to\infty}\varepsilon_n^{(2)} \le 0$, which reduces to showing that $\limsup_{n\to\infty}\langle f(q_0) - q_0, J_p(x_{n+1} - q_0)\rangle \le 0$.

Let u_t^n be the same as that in Step 3. Since $\|u_t^n\| \le \|u_t^n - q_0\| + \|q_0\|$, then $\{u_t^n\}$ is bounded, as $t \to 0$. Using Lemma 1.5 again, we have

$$\begin{aligned}
&\left\|u_t^n - \frac{u_n + v_n}{2}\right\|^p \\
&= \Bigg\|u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) \\
&\quad + \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \frac{u_n + v_n}{2}\Bigg\|^p \\
&\le \left\|u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right\|^p \\
&\quad + p\left\langle\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \frac{u_n + v_n}{2}, J_p\left(u_t^n - \frac{u_n + v_n}{2}\right)\right\rangle \\
&= \Bigg\|tf\left(u_t^n\right) + (1 - t)\left(I - \kappa_t\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\right)\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n\right) \\
&\quad - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\Bigg\|^p \\
&\quad + p\left\langle\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \frac{u_n + v_n}{2}, J_p\left(u_t^n - \frac{u_n + v_n}{2}\right)\right\rangle \\
&\le \left\|u_t^n - \frac{u_n + v_n}{2}\right\|^p + pt\Bigg\langle f\left(u_t^n\right) - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n \\
&\quad - \frac{\kappa_t}{t}(1 - t)\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n\right), \\
&\quad J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right)\Bigg\rangle \\
&\quad + p\left\langle\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \frac{u_n + v_n}{2}, J_p\left(u_t^n - \frac{u_n + v_n}{2}\right)\right\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&t\Bigg\langle\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n - f\left(u_t^n\right) \\
&\quad + \frac{\kappa_t}{t}(1 - t)\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n\right), \\
&\quad J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right)\right)\Bigg\rangle \\
&\le \left\|\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)\left(\frac{u_n + v_n}{2}\right) - \frac{u_n + v_n}{2}\right\|\left\|u_t^n - \frac{u_n + v_n}{2}\right\|^{p-1}.
\end{aligned}$$

So, $\lim_{t\to 0}\limsup_{n\to+\infty}\langle \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)Q_D u_t^n - f(u_t^n) + \frac{\kappa_t}{t}(1-t)\times \sum_{i=1}^{\infty}\omega_i^{(1)}W_i(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n), J_p(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)(\frac{u_n+v_n}{2}))\rangle \le 0$.

Since $u_t^n \to q_0$, then $\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n \to \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D q_0 = q_0$, as $t \to 0$.

Noticing that

$$\begin{aligned}
&\left\langle q_0 - f(q_0), J_p\left(q_0 - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle \\
&\quad = \left\langle q_0 - f(q_0), J_p\left(q_0 - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right. \\
&\qquad \left. - J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle \\
&\qquad + \left\langle q_0 - f(q_0), J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle \\
&\quad = \left\langle q_0 - f(q_0), J_p\left(q_0 - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right. \\
&\qquad \left. - J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle \\
&\qquad + \left\langle q_0 - f(q_0) - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n + f\left(u_t^n\right)\right. \\
&\qquad - \frac{\kappa_t}{t}(1-t)\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n\right), \\
&\qquad \left. J_p\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle \\
&\qquad + \left\langle \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n - f\left(u_t^n\right)\right. \\
&\qquad + \frac{\kappa_t}{t}(1-t)\sum_{i=1}^{\infty}\omega_i^{(1)}W_i\left(\sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)Q_D u_t^n\right), \\
&\qquad \left. J\left(u_t^n - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)\left(\frac{u_n+v_n}{2}\right)\right)\right\rangle,
\end{aligned}$$

we have $\limsup_{n\to+\infty}\langle q_0 - f(q_0), J_p(q_0 - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I-r_{n,i}C_i)(\frac{u_n+v_n}{2}))\rangle \le 0$.

From assumptions (iv) and (v) and Step 2, we know that $x_{n+1} - \sum_{i=1}^{\infty}\omega_i^{(2)}J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}) \to 0$ and then $\limsup_{n\to+\infty}\langle q_0 - f(q_0), J_p(q_0 - x_{n+1})\rangle \le 0$. Thus $\limsup_{n\to\infty}\varepsilon_n^{(2)} \le 0$.

Employing (2.12) again, we have

$$\|x_n - q_0\|^p \le \frac{\|x_n - q_0\|^p - \|x_{n+1} - q_0\|^p}{\varepsilon_n^{(1)}} + \varepsilon_n^{(2)}.$$

Assumption (iv) implies that $\liminf_{n\to\infty} \frac{\|x_n-q_0\|^p-\|x_{n+1}-q_0\|^p}{\varepsilon_n^{(1)}} = 0$. Then

$$\lim_{n\to\infty} \|x_n - q_0\|^p \le \liminf_{n\to\infty} \frac{\|x_n - q_0\|^p - \|x_{n+1} - q_0\|^p}{\varepsilon_n^{(1)}} + \limsup_{n\to\infty} \varepsilon_n^{(2)} \le 0.$$

Then the result that $x_n \to q_0$ follows.

Case 2. If $\{\|x_n - q_0\|\}$ is not eventually decreasing, then we can find a subsequence $\{\|x_{n_k} - q_0\|\}$ so that $\|x_{n_k} - q_0\| \le \|x_{n_{k+1}} - q_0\|$ for all $k \ge 1$. From Lemma 1.7, we can define a subsequence $\{\|x_{\tau(n)} - q_0\|\}$ so that $\max\{\|x_{\tau(n)} - q_0\|, \|x_n - q_0\|\} \le \|x_{\tau(n)+1} - q_0\|$ for all $n > n_1$. This enables us to deduce that (similar to Case 1)

$$0 \le \varepsilon_{\tau(n)}^{(3)} \le \varepsilon_{\tau(n)}^{(1)}\bigl(\varepsilon_{\tau(n)}^{(2)} - \|x_{\tau(n)} - q_0\|^p\bigr) + \bigl(\|x_{\tau(n)} - q_0\|^p - \|x_{\tau(n)+1} - q_0\|^p\bigr) \to 0,$$

and then copying Case 1, we have $\lim_{n\to\infty} \|x_{\tau(n)} - q_0\| = 0$. Thus $0 \le \|x_n - q_0\| \le \|x_{\tau(n)+1} - q_0\| \to 0$, as $n \to \infty$.

This completes the proof. □

Remark 2.3 Theorem 2.2 is reasonable if we suppose $X = D = (-\infty, +\infty)$, take $f(x) = \frac{x}{4}$, $A_i x = C_i x = \frac{x}{2^i}$, $W_i x = \frac{x}{2^{i+1}}$, $\theta_i = 2^i$, $\omega_i^{(1)} = \omega_i^{(2)} = \frac{1}{2^i}$, $\alpha_n = 1 - \frac{1}{n^2}$, $\beta_n = \frac{1}{n^3}$, $\vartheta_n = \delta_n = \frac{1}{n}$, $\xi_n = \zeta_n = a_n = b_n = \frac{1}{n^2}$, $\gamma_i = \frac{1}{2^{i+2}}$, $\mu_i = \frac{2^{i+1} - \frac{3}{2} + \frac{1}{2^{i+1}}}{2^{i+1} - 1}$, $r_{n,i} = \frac{1}{2^{n+i}}$ for $n \in N$ and $i \in N$.

Remark 2.4 Our differences from the main references are:

(i) the normalized duality mapping $J : E \to E^*$ is no longer required to be weakly sequentially continuous at zero as that in [9];

(ii) the parameter $\{r_{n,i}\}$ in the resolvent $J_{r_{n,i}}^{A_i}$ does not need satisfying the condition '$\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ and $r_{n,i} \ge \varepsilon > 0$ for $i \in N$ and some $\varepsilon > 0$' as that in [3] or [9];

(iii) Lemma 1.7 plays an important role in the proof of strong convergence of the iterative sequence, which leads to different restrictions on the parameters and different proof techniques compared to the already existing similar works.

3 Applications

3.1 Integro-differential systems

In Section 3.1, we shall investigate the following nonlinear integro-differential systems involving the generalized p_i-Laplacian, which have been studied in [3]:

$$\begin{cases} \frac{\partial u^{(i)}(x,t)}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u^{(i)}|^2)^{\frac{p_i-2}{2}} \nabla u^{(i)}] + \varepsilon |u^{(i)}|^{r_i-2} u^{(i)} \\ \quad + g(x, u^{(i)}, \nabla u^{(i)}) + a \frac{\partial}{\partial t} \int_\Omega u^{(i)}\, dx = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u^{(i)}|^2)^{\frac{p_i-2}{2}} \nabla u^{(i)} \rangle \in \beta_x(u^{(i)}), \quad (x,t) \in \Gamma \times (0,T), \\ u^{(i)}(x,0) = u^{(i)}(x,T), \quad x \in \Omega, i \in N, \end{cases} \tag{3.1}$$

where Ω is a bounded conical domain of a Euclidean space R^N ($N \ge 1$), Γ is the boundary of Ω with $\Gamma \in C^1$ and ϑ denotes the exterior normal derivative to Γ. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner-product and the Euclidean norm in R^N, respectively. T is a positive constant. $\nabla u^{(i)} = (\frac{\partial u^{(i)}}{\partial x_1}, \frac{\partial u^{(i)}}{\partial x_2}, \dots, \frac{\partial u^{(i)}}{\partial x_N})$ and $x = (x_1, x_2, \dots, x_N) \in \Omega$. β_x is the subdifferential of

φ_x, where $\varphi_x = \varphi(x,\cdot) : R \to R$ for $x \in \Gamma$. a and ε are non-expansive constants, $0 \le C(x,t) \in \bigcap_{i=1}^{\infty} V_i := \bigcap_{i=1}^{\infty} L^{p_i}(0,T;W^{1,p_i}(\Omega))$, $f(x,t) \in \bigcap_{i=1}^{\infty} W_i := \bigcap_{i=1}^{\infty} L^{\max\{p_i,p_i'\}}(0,T;L^{\max\{p_i,p_i'\}}(\Omega))$ and $g : \Omega \times R^{N+1} \to R$ are given functions.

Just like [3], we need the following assumptions to discuss (3.1).

Assumption 1 $\{p_i\}_{i=1}^{\infty}$ is a real number sequence with $\frac{2N}{N+1} < p_i < +\infty$, $\{\theta_i\}_{i=1}^{\infty}$ is any real number sequence in $(0,1]$ and $\{r_i\}_{i=1}^{\infty}$ is a real number sequence satisfying $\frac{2N}{N+1} < r_i \le \min\{p_i,p_i'\} < +\infty$. $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ and $\frac{1}{r_i} + \frac{1}{r_i'} = 1$ for $i \in N$.

Assumption 2 Green's formula is available.

Assumption 3 For each $x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : R \to R$ is a proper, convex and lower-semicontinuous function and $\varphi_x(0) = 0$.

Assumption 4 $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Assumption 5 Suppose that $g : \Omega \times R^{N+1} \to R$ satisfies the following conditions:

(a) Carathéodory's conditions;

(b) Growth condition.

$$\left|g(x,r_1,\dots,r_{N+1})\right|^{\max\{p_i,p_i'\}} \le \left|h_i(x,t)\right|^{p_i} + b_i|r_1|^{p_i},$$

where $(r_1,r_2,\dots,r_{N+1}) \in R^{N+1}$, $h_i(x,t) \in W_i$ and b_i is a positive constant for $i \in N$;

(c) Monotone condition. g is monotone in the following sense:

$$\bigl(g(x,r_1,\dots,r_{N+1}) - g(x,t_1,\dots,t_{N+1})\bigr) \ge (r_1 - t_1)$$

for all $x \in \Omega$ and $(r_1,\dots,r_{N+1}),(t_1,\dots,t_{N+1}) \in R^{N+1}$.

Assumption 6 For $i \in N$, let V_i^* denote the dual space of V_i. The norm in V_i, $\|\cdot\|_{V_i}$, is defined by

$$\left\|u(x,t)\right\|_{V_i} = \left(\int_0^T \left\|u(x,t)\right\|_{W^{1,p_i}(\Omega)}^{p_i}\,dt\right)^{\frac{1}{p_i}}, \quad u(x,t) \in V_i.$$

Definition 3.1 ([3]) For $i \in N$, define the operator $B_i : V_i \to V_i^*$ by

$$\langle w, B_i u\rangle = \int_0^T \int_\Omega \bigl\langle \bigl(C(x,t) + |\nabla u|^2\bigr)^{\frac{p_i-2}{2}} \nabla u, \nabla w\bigr\rangle\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^{r_i-2} u w\,dx\,dt$$

for $u, w \in V_i$.

Definition 3.2 ([3]) For $i \in N$, define the function $\Phi_i : V_i \to R$ by

$$\Phi_i(u) = \int_0^T \int_\Gamma \varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt$$

for $u(x,t) \in V_i$.

Definition 3.3 ([3]) For $i \in N$, define $S_i : D(S_i) = \{u(x,t) \in V_i : \frac{\partial u}{\partial t} \in V_i^*, u(x,0) = u(x,T)\} \to V_i^*$ by

$$S_i u = \frac{\partial u}{\partial t} + a\frac{\partial}{\partial t}\int_\Omega u\,dx.$$

Lemma 3.4 ([3]) *For $i \in N$, define a mapping $A_i : W_i \to 2^{W_i}$ as follows*:

$$D(A_i) = \{u \in W_i | \text{there exists an } f \in W_i \text{ such that } f \in B_i u + \partial\Phi_i(u) + S_i u\},$$

where $\partial\Phi_i : V_i \to V_i^*$ is the subdifferential of Φ_i. For $u \in D(A_i)$, we set $A_i u = \{f \in W_i | f \in B_i u + \partial\Phi_i(u) + S_i u\}$. Then $A_i : W_i \to 2^{W_i}$ is m-accretive, where $i \in N$.

Lemma 3.5 ([3]) *Define $C_i : D(C_i) = L^{\max\{p_i,p_i'\}}(0,T;W^{1,\max\{p_i,p_i'\}}(\Omega)) \subset W_i \to W_i$ by*

$$(C_i u)(x,t) = g(x,u,\nabla u) - f(x,t)$$

for $\forall u(x,t) \in D(C_i)$ and $f(x,t)$ is the same as that in (3.1), *where $i \in N$. Then $C_i : D(C_i) \subset W_i \to W_i$ is continuous and strongly accretive. If we further assume that $g(x,r_1,\ldots,r_{N+1}) \equiv r_1$, then C_i is θ_i-inversely strongly accretive, where $i \in N$.*

Lemma 3.6 ([3]) *For $f(x,t) \in \bigcap_{i=1}^\infty W_i$, integro-differential systems* (3.1) *have a unique solution $u^{(i)}(x,t) \in W_i$ for $i \in N$.*

Lemma 3.7 ([3]) *If $\varepsilon \equiv 0$, $g(x,r_1,\ldots,r_{N+1}) \equiv r_1$ and $f(x,t) \equiv k$, where k is a constant, then $u(x,t) \equiv k$ is the unique solution of integro-differential systems* (3.1). *Moreover, $\{u(x,t) \in \bigcap_{i=1}^\infty W_i | u(x,t) \equiv k \text{ satisfying } (3.1)\} = \bigcap_{i=1}^\infty N(A_i + C_i)$.*

Remark 3.8 ([3]) Set $p := \inf_{i\in N}(\min\{p_i,p_i'\})$ and $q := \sup_{i\in N}(\max\{p_i,p_i'\})$.
Let $X := L^{\min\{p,p'\}}(0,T;L^{\min\{p,p'\}}(\Omega))$, where $\frac{1}{p} + \frac{1}{p'} = 1$.
Let $D := L^{\max\{q,q'\}}(0,T;W^{1,\max\{q,q'\}}(\Omega))$, where $\frac{1}{q} + \frac{1}{q'} = 1$.
Then $X = L^p(0,T;L^p(\Omega))$, $D = L^q(0,T;W^{1,q}(\Omega))$ and $D \subset W_i \subset X$, $\forall i \in N$.

Theorem 3.9 *Let D and X be the same as those in Remark* 3.8. *Suppose A_i and C_i are the same as those in Lemmas* 3.4 *and* 3.5, *respectively. Let $f : X \to X$ be a fixed contractive mapping with coefficient $k \in (0,1)$ and $W_i : X \to X$ be μ_i-strictly pseudo-contractive mappings and γ_i-strongly accretive mappings with $\mu_i + \gamma_i > 1$ for $i \in N$. Suppose that $\{\omega_i^{(1)}\}$, $\{\omega_i^{(2)}\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\vartheta_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{r_{n,i}\}$, $\{a_n\} \subset X$ and $\{b_n\} \subset D$ satisfy the same conditions as those in Theorem* 2.2, *where $n \in N$ and $i \in N$. Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} x_1 \in D, \\ u_n = Q_D(\alpha_n x_n + \beta_n a_n), \\ v_n = \vartheta_n u_n + \nu_n \sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}) + \xi_n b_n, \\ x_{n+1} = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^\infty \omega_i^{(1)} W_i)\sum_{i=1}^\infty \omega_i^{(2)} J_{r_{n,i}}^{A_i}(I - r_{n,i}C_i)(\frac{u_n+v_n}{2}), \quad n \in N. \end{cases} \tag{3.2}$$

If, in integro-differential systems (3.1), *$\varepsilon \equiv 0$, $g(x,r_1,\ldots,r_{N+1}) \equiv r_1$ and $f(x,t) \equiv k$, then under the following assumptions in Theorem* 2.2, *the iterative sequence $x_n \to q_0 \in$*

$\bigcap_{i=1}^{\infty} N(A_i + C_i)$, *which is the unique solution of integro-differential systems* (3.1) *and which satisfies the following variational inequality*: *for* $\forall y \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$,

$$\big\langle (I-f)q_0(x,t), J\big(q_0(x,t)-y\big)\big\rangle \le 0.$$

3.2 Convex minimization problems

Let H be a real Hilbert space. Suppose $h_i : H \to (-\infty, +\infty)$ are proper convex, lower-semicontinuous and nonsmooth functions [2], suppose $g_i : H \to (-\infty, +\infty)$ are convex and smooth functions for $i \in N$. We use ∇g_i to denote the gradient of g_i and ∂h_i the subdifferential of h_i for $i \in N$.

The convex minimization problems are to find $x^* \in H$ such that

$$h_i\big(x^*\big) + g_i\big(x^*\big) \le h_i(x) + g_i(x), \quad i \in N, \tag{3.3}$$

for $\forall x \in H$.

By Fermats' rule, (3.3) is equivalent to finding $x^* \in H$ such that

$$0 \in \partial h_i\big(x^*\big) + \nabla g_i\big(x^*\big), \quad i \in N. \tag{3.4}$$

Theorem 3.10 *Let H be a real Hilbert space and D be the nonempty closed convex sunny non-expansive retract of H. Let Q_D be the sunny non-expansive retraction of H onto D. Let $f : H \to H$ be a contraction with coefficient $k \in (0,1)$. Let $h_i : H \to (-\infty,+\infty)$ be proper convex, lower-semicontinuous and nonsmooth functions and $g_i : H \to (-\infty,+\infty)$ be convex and smooth functions for $i \in N$. Let $W_i : H \to H$ be μ_i-strictly pseudo-contractive mappings and γ_i-strongly accretive mappings with $\mu_i + \gamma_i > 1$ for $i \in N$. Suppose $\{\omega_i^{(1)}\}$, $\{\omega_i^{(2)}\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\vartheta_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{r_{n,i}\} \subset (0,+\infty)$, $\{a_n\} \subset H$ and $\{b_n\} \subset D$ satisfy the same conditions as those in Theorem* 2.2, *where $n \in N$ and $i \in N$. Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} x_1 \in D, \\ u_n = Q_D(\alpha_n x_n + \beta_n a_n), \\ v_n = \vartheta_n u_n + \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{\partial h_i}(I - r_{n,i}\nabla g_i)(\frac{u_n+v_n}{2}) + \xi_n b_n, \\ x_{n+1} = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} W_i) \sum_{i=1}^{\infty} \omega_i^{(2)} J_{r_{n,i}}^{\partial h_i}(I - r_{n,i}\nabla g_i)(\frac{u_n+v_n}{2}), \quad n \in N. \end{cases} \tag{3.5}$$

If, further, suppose ∇g_i is $\frac{1}{\theta_i}$-Lipschitz continuous and $h_i + g_i$ attains a minimizer, then $\{x_n\}$ converges strongly to the minimizer of $h_i + g_i$ for $i \in N$.

Proof It follows from [2] that ∂h_i is m-accretive. From [19], since ∇g_i is $\frac{1}{\theta_i}$-Lipschitz continuous, then ∇g_i is θ_i-inversely strongly accretive. Thus Theorem 2.2 ensures the result.

This completes the proof. □

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Competing interests
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References
1. Barbu, V: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff, Leyden (1976)
2. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitz-Type Mappings with Applications. Springer, Berlin (2008)
3. Wei, L, Agarwal, RP: A new iterative algorithm for the sum of infinite m-accretive mappings and infinite μ_i-inversely strongly accretive mappings and its applications to integro-differential systems. Fixed Point Theory Appl. **2016**, 7 (2016)
4. Browder, FE, Petryshn, WV: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**, 197-228 (1967)
5. Takahashi, W: Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces. Taiwan. J. Math. **12**(8), 1883-1910 (2008)
6. Lions, PL, Mercier, B: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964-979 (1979)
7. Combettes, PL, Wajs, VR: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4**, 1168-1200 (2005)
8. Tseng, P: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**(2), 431-446 (1998)
9. Wei, L, Duan, LL: A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach space and its connection with capillarity equation. Fixed Point Theory Appl. **2015**, 25 (2015)
10. Alghamdi, MA, Alghamdi, MA, Shahzad, N, Xu, H: The implicit midpoint rule for nonexpansive mappings. Fixed Point Theory Appl. **2014**, 96 (2014)
11. Ceng, LC, Ansari, QH, Schaible, S, Yao, JC: Hybrid viscosity approximation method for zeros of m-accretive operators in Banach spaces. Numer. Funct. Anal. Optim. **33**(2), 142-165 (2012)
12. Lopez, G, Martin-Marquez, V, Wang, FH, Xu, HK: Forward-backward splitting methods for accretive operators in Banach spaces. Abstr. Appl. Anal. **2012**, Article ID 109236 (2012)
13. Ceng, LC, Ansari, QH, Yao, YC: Mann-type steepest descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces. Numer. Funct. Anal. Optim. **29**(9-10), 987-1033 (2008)
14. Bruck, RE: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)
15. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: On a strongly nonexpansive sequence in Hilbert spaces. J. Nonlinear Convex Anal. **8**, 471-489 (2007)
16. Mainge, PE: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **66**, 899-912 (2008)
17. Mitrinovic, DS: Analytic Inequalities. Springer, New York (1970)
18. Cioranescu, I: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
19. Baillon, JB, Haddad, G: Quelques proprietes des operateurs angle-bornes et cycliquement monotones. Isr. J. Math. **26**, 137-150 (1977)