Zhou et al. *Journal of Inequalities and Applications* (2017) 2017:231
DOI 10.1186/s13660-017-1499-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The closure property of $\mathcal{H}$-tensors under the Hadamard product

Lixin Zhou[1,2,3*], Jianzhou Liu[1*] and Li Zhu[1*]

*Correspondence:
zhoulixin@guat.edu.cn;
liujz@xtu.edu.cn; zhuli@xtu.edu.cn
[1]School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate the closure property of $\mathcal{H}$-tensors under the Hadamard product. It is shown that the Hadamard products of Hadamard powers of strong $\mathcal{H}$-tensors are still strong $\mathcal{H}$-tensors. We then bound the minimal real eigenvalues of the comparison tensors of the Hadamard products involving strong $\mathcal{H}$-tensors. Finally, we show how to attain the bounds by characterizing these $\mathcal{H}$-tensors.

MSC: 15A18; 15A69

Keywords: $\mathcal{H}$-tensors; $\mathcal{M}$-tensors; Hadamard product; eigenvalues

1 Introduction

The study of tensors with their various applications has increasingly attracted extensive attention and interest [1–5]. A tensor can be regarded as a higher-order generalization of a matrix in linear algebra. However, unlike matrices, the problems for tensors are generally nonlinear. Hence, there is a large need to investigate tensor problems. Recently, some structured tensors such as nonnegative tensors, $\mathcal{M}$-tensors and $\mathcal{H}$-tensors have been introduced and studied well, and many interesting results for these tensors have been obtained because of their special structure properties [6–15]. These structural tensors have a wide range of applications such as spectral hypergraph theory, higher-order Markov chains, big amounts of data, polynomial optimization, magnetic resonance imaging, simulation, automatic control, and quantum entanglement problems [1, 2, 4–8, 10–18]. For example, the positive definiteness of an even-degree homogeneous polynomial form $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [19]. In [6], it is shown that the homogeneous polynomial form $f(x)$ is equivalent to the tensor product $\mathcal{A}x^m$ of an mth-order, n-dimensional supersymmetric tensor $\mathcal{A}$ and x^m, defined by the following equation (1.1) (see [4, 19]). In [16], Qi pointed out that $f(x)$ is positive definite if and only if the real supersymmetric tensor $\mathcal{A}$ is positive definite. For an even-order real supersymmetric tensor $\mathcal{A}$ of order m and dimension n, with all diagonal elements $a_{k\ldots k} > 0$, if $\mathcal{A}$ is an H-tensor, then $\mathcal{A}$ is positive definite [19]. The main aim of this paper is to study the closure property of structure properties of $\mathcal{H}$-tensors under the Hadamard product.

An mth-order n-dimensional real tensor $\mathcal{A}$ is a multidimensional array of n^m real entries of the form

$$\mathcal{A} = (a_{i_1\ldots i_m}), \quad a_{i_1\ldots i_m} \in R, 1 \le i_1, \ldots, i_m \le n.$$

The entries $a_{ii\ldots i}$ are called the diagonal entries of $\mathcal{A}$. If all its off-diagonal entries are zero, then $\mathcal{A}$ is diagonal. The identity tensor $\mathcal{I}$ is a diagonal tensor all of whose diagonal entries are 1. In the sequel, we denote by $\mathcal{R}^{(m,n)}$ the set of all mth-order n-dimensional real tensors. For a tensor $\mathcal{A} \in \mathcal{R}^{(m,n)}$ and a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, the tensor-vector multiplication $\mathcal{A}x^{m-1}$ is defined as an n-vector whose ith entries are

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n. \tag{1.1}$$

If there are a number λ and a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called the eigenvalue of $\mathcal{A}$ and x is the eigenvector of $\mathcal{A}$ associated with λ, where $x^{[m-1]}$ is the Hadamard power of x, *i.e.*, $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. Note that the definition of eigenvalues of tensors was independently introduced by Qi [16] and Lim [20]. Denote by $\varphi(\mathcal{A})$ the set of all the eigenvalues of $\mathcal{A} \in \mathcal{R}^{(m,n)}$, and denote

$$\rho(\mathcal{A}) = \max\bigl\{|\lambda| | \lambda \in \varphi(\mathcal{A})\bigr\}, \quad \tau(\mathcal{A}) = \min\bigl\{\operatorname{Re}\lambda | \lambda \in \varphi(\mathcal{A})\bigr\},$$

where $\operatorname{Re}\lambda$ is the real part of λ. It is well known that if $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is a nonnegative tensor (*i.e.*, all its entries are nonnegative), then $\rho(\mathcal{A})$ must be its eigenvalue [13, 14]; and if $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is an $\mathcal{M}$-tensor, then $\tau(\mathcal{A})$ must be its eigenvalue [15].

A tensor $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is said to be a (strong) $\mathcal{M}$-tensor if $\mathcal{A}$ can be written as $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B} \in \mathcal{R}^{(m,n)}$ is nonnegative and $s(>) \ge \rho(\mathcal{B})$. In this case, according to the proof of [15, Theorem 3.3], $\tau(\mathcal{A}) = s - \rho(\mathcal{B})$. For a tensor $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathcal{R}^{(m,n)}$, the comparison tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1\ldots i_m}) \in \mathcal{R}^{(m,n)}$ is defined as

$$m_{i_1\ldots i_m} = \begin{cases} |a_{i_1\ldots i_m}|, & \text{if } i_1 = \cdots = i_m, \\ -|a_{i_1\ldots i_m}|, & \text{otherwise,} \end{cases} \quad 1 \le i_1, \ldots, i_m \le n.$$

Definition 1.1 ([8, 11]) A tensor $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is called a (strong) $\mathcal{H}$-tensor if its comparison tensor $\mathcal{M}(\mathcal{A})$ is a (strong) $\mathcal{M}$-tensor. We denote $\sigma(\mathcal{A}) = \tau(\mathcal{M}(\mathcal{A}))$.

For a nonnegative tensor $\mathcal{A} = (a_{i_1 i_2\ldots i_m}) \in \mathcal{R}^{(m,n)}$, the matrix $R(\mathcal{A}) = (r_{ij}) \in \mathbb{R}^{n\times n}$ is called the representation of $\mathcal{A}$, where

$$r_{ij} = \sum_{\{i_2,\ldots,i_m\}\ni j} a_{ii_2\ldots i_m}, \quad i, j = 1, 2, \ldots, n.$$

Definition 1.2 ([9, 10]) A tensor $\mathcal{A} = (a_{i_1 i_2\ldots i_m}) \in \mathcal{R}^{(m,n)}$ is called weakly irreducible if the representation $R(|\mathcal{A}|)$ of $|\mathcal{A}|$ is irreducible. We denote $|\mathcal{A}| = (|a_{i_1 i_2\ldots i_m}|)$.

Many interesting properties have been provided for $\mathcal{M}$-tensors. Recall that $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is an $\mathcal{H}$-tensor if and only if $\mathcal{M}(\mathcal{A}) \in \mathcal{R}^{(m,n)}$ is an $\mathcal{M}$-tensor. So using [15, Theorem 3.4] and [8, Theorem 3], we have the following facts on $\mathcal{H}$-tensors that will be frequently used in the next sections:

(P1) If $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is an $\mathcal{H}$-tensor, then $\sigma(\mathcal{A}) = \sigma(|\mathcal{A}|)$, which is the minimal real eigenvalue of $\mathcal{M}(\mathcal{A})$. Further, let $\mathcal{M}(\mathcal{A}) = s\mathcal{I} - \mathcal{B}$ where $\mathcal{B}$ is nonnegative and $s \geq \rho(\mathcal{B})$. Then $\sigma(\mathcal{A}) = s - \rho(\mathcal{B})$.

(P2) If $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is a weakly irreducible strong $\mathcal{H}$-tensor, then $\sigma(\mathcal{A}) > 0$, and there exists an n-vector $x > 0$ such that $\mathcal{M}(\mathcal{A})x^{m-1} = \sigma(\mathcal{A})x^{[m-1]}$.

(P3) A tensor $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is a strong $\mathcal{H}$-tensor if and only if there exists an n-vector $x > 0$ such that $\mathcal{M}(\mathcal{A})x^{m-1} > 0$.

Clearly, these interesting results are due to the special structures of $\mathcal{H}$-tensors. So it is natural to consider how to preserve the structure properties under certain operations. In addition, many interesting results have been obtained for the Hadamard products involving M-matrices and H-matrices [21]. It is natural to ask whether we can provide similar results for the tensor case. Motivated by these facts, the aim of this paper is to investigate the closure property of $\mathcal{H}$-tensors under the Hadamard product.

Definition 1.3 Given two tensors $\mathcal{A} = (a_{i_1\ldots i_m}), \mathcal{B} = (b_{i_1\ldots i_m}) \in \mathcal{R}^{(m,n)}$, the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \circ \mathcal{B} = (a_{i_1\ldots i_m} b_{i_1\ldots i_m}) \in \mathcal{R}^{(m,n)}$ and the rth Hadamard power of $\mathcal{A}$ is defined as $\mathcal{A}^{[r]} = (a^r_{i_1\ldots i_m}) \in \mathcal{R}^{(m,n)}$ for $r \geq 0$.

To obtain our results, we need the following two famous inequalities:

- *Hölder's inequality*: let a_i and b_i be nonnegative numbers for $i = 1, 2, \ldots, n$, and let $0 < r < 1$. Then

$$\sum_{i=1}^{n} a_i^r b_i^{1-r} \leq \left(\sum_{i=1}^{n} a_i\right)^r \left(\sum_{i=1}^{n} b_i\right)^{1-r},$$

  and the equality holds if and only if, for all $i = 1, 2, \ldots, n$, $a_i = lb_i$ for some constant l.
- *Minkowski's inequality*: let a_i be nonnegative numbers for $i = 1, 2, \ldots, n$, and let $r > 1$. Then

$$\sum_{i=1}^{n} a_i^r \leq \left(\sum_{i=1}^{n} a_i\right)^r,$$

  and the equality holds if and only if there is at most one nonzero number for $a_1, a_2, \ldots, a_n$.

The rest of the paper is organized as follows. In Section 2, we show the closure property of the Hadamard products of Hadamard powers of strong $\mathcal{H}$-tensors. In Section 3, we bound the minimal real eigenvalues of the comparison tensors of the Hadamard products involving strong $\mathcal{H}$-tensors. In Section 4, we characterize these strong $\mathcal{H}$-tensors such that the bounds can be obtained.

2 The closure property

In this section, we provide the closure property of the Hadamard products of Hadamard powers of strong $\mathcal{H}$-tensors.

Lemma 2.1 *Let $\mathcal{A},\mathcal{B}\in\mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors and let $0\le r\le 1$. Then $\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}$ is a strong $\mathcal{H}$-tensor.*

Proof Set $\mathcal{A}=(a_{i_1i_2\ldots i_m})$ and $\mathcal{B}=(b_{i_1i_2\ldots i_m})$. By (P3), there exist positive vectors $x=(x_i)\in\mathbb{R}^n$ and $y=(y_i)\in\mathbb{R}^n$ such that $\mathcal{M}(\mathcal{A})x^{m-1}>0$ and $\mathcal{M}(\mathcal{B})y^{m-1}>0$, respectively. This means that, for all $i=1,2,\ldots,n$,

$$|a_{ii\ldots i}|x_i^{m-1}>\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}$$

and

$$|b_{ii\ldots i}|y_i^{m-1}>\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}.$$

Note that $0\le r\le 1$. Thus, using the Hölder inequality, we have

$$\begin{aligned}|a_{ii\ldots i}|^r|b_{ii\ldots i}|^{1-r}\bigl(x_i^ry_i^{1-r}\bigr)^{(m-1)}&>\Biggl(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\Biggr)^r\\&\quad\times\Biggl(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}\Biggr)^{1-r}\\&\ge\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|^rx_{i_2}^r\ldots x_{i_m}^r\cdot|b_{ii_2\ldots i_m}|^{1-r}y_{i_2}^{1-r}\ldots y_{i_m}^{1-r}.\end{aligned}$$

Set $z=(x_i^ry_i^{1-r})\in\mathbb{R}^n$. Then the inequality above gives $\mathcal{M}(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})z^{m-1}>0$, from which it follows by (P3) that $\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}$ is a strong $\mathcal{H}$-tensor. The result is proved. □

Lemma 2.2 *Let $\mathcal{A}\in\mathcal{R}^{(m,n)}$ be a strong $\mathcal{H}$-tensor and let $t\ge 1$. Then $\mathcal{A}^{[t]}$ is a strong $\mathcal{H}$-tensor.*

Proof Set $\mathcal{A}=(a_{i_1i_2\ldots i_m})$. Clearly, there exists a positive vector $x=(x_i)\in\mathbb{R}^n$ such that $\mathcal{M}(\mathcal{A})x^{m-1}>0$ and so, for all $i=1,2,\ldots,n$,

$$|a_{ii\ldots i}|x_i^{m-1}>\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m},$$

from which we get, by considering $t\ge 1$ and using the Minkowski inequality,

$$\begin{aligned}|a_{ii\ldots i}|^t\bigl(x_i^t\bigr)^{(m-1)}&>\Biggl(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\Biggr)^t\\&\ge\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|^tx_{i_2}^t\ldots x_{i_m}^t.\end{aligned}$$

Set $z=(x_i^t)\in\mathbb{R}^n$. Then $\mathcal{M}(\mathcal{A}^{[t]})z^{m-1}>0$ and thus $\mathcal{A}^{[t]}$ is a strong $\mathcal{H}$-tensor by (P3). The result is proved. □

Now we are ready to present the main result of this section.

Theorem 2.3 *Let $\mathcal{A}_1,\ldots,\mathcal{A}_k \in \mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors and let $r_1,\ldots,r_k$ be positive numbers with $\sum_{i=1}^k r_i \geq 1$. Then $\mathcal{A}_1^{[r_1]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}$ is a strong $\mathcal{H}$-tensor.*

Proof Consider that $\mathcal{A} \in \mathcal{R}^{(m,n)}$ is a strong $\mathcal{H}$-tensor if and only if $|\mathcal{A}| \in \mathcal{R}^{(m,n)}$ is a strong $\mathcal{H}$-tensor. So, without loss of generality, assume that all the tensors $\mathcal{A}_i$ are nonnegative for $i = 1,2,\ldots,k$. We first use the induction on k to prove the result in the case that $\sum_{i=1}^k r_i = 1$. Clearly, the result is true for $k = 2$ by Lemma 2.1. Assume that the result is true for $k-1$. Now let

$$\mathcal{B}^{[1-r_k]} = \mathcal{A}_1^{[r_1]} \circ \cdots \circ \mathcal{A}_{k-1}^{[r_{k-1}]}.$$

Recall that each $\mathcal{A}_i$ is nonnegative. Then

$$\mathcal{B} = \mathcal{A}_1^{[\frac{r_1}{1-r_k}]} \circ \cdots \circ \mathcal{A}_{k-1}^{[\frac{r_{k-1}}{1-r_k}]}.$$

Note that $\sum_{i=1}^{k-1} \frac{r_i}{1-r_k} = 1$. Hence, using the induction assumption, we conclude that $\mathcal{B}$ is a strong tensor. Further, by Lemma 2.1, $\mathcal{B}^{[1-r_k]} \circ \mathcal{A}_k^{[r_k]}$ is a strong $\mathcal{H}$-tensor. So the result is true in the case that $\sum_{i=1}^k r_i = 1$.

Now consider the general case $t = \sum_{i=1}^k r_i \geq 1$. Let $l_i = r_i t^{-1}$ for all $i = 1,2,\ldots,k$. Then $\sum_{i=1}^k l_i = 1$. Thus, following the case above, we know that $\mathcal{C} = \mathcal{A}_1^{[l_1]} \circ \cdots \circ \mathcal{A}_k^{[l_k]}$ is a strong $\mathcal{H}$-tensor. Further, by considering $t \geq 1$, using Lemma 2.2 we find that $\mathcal{C}^{[t]} = \mathcal{A}_1^{[r_1]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}$ is a strong $\mathcal{H}$-tensor. The result is proved. □

Example 2.1 Let $\mathcal{A}_1 = (a_{ijkl}), \mathcal{A}_2 = (b_{ijkl}), \mathcal{A}_3 = (c_{ijkl}) \in \mathcal{R}^{(4,3)}$ be defined as follows:

$$\begin{cases} a_{1111} = 4, a_{2222} = 2, a_{3333} = 2, a_{1112} = a_{2111} = a_{1113} = a_{3111} = 1, & \text{otherwise } a_{ijkl} = 0, \\ b_{1111} = 5, b_{2222} = 3, b_{3333} = 3, b_{1112} = b_{2111} = 2, b_{1113} = b_{3111} = \frac{3}{2}, & \text{otherwise } b_{ijkl} = 0, \\ c_{1111} = 6, c_{2222} = 3, c_{3333} = 4, c_{1112} = c_{2111} = \frac{3}{2}, c_{1113} = c_{3111} = \frac{5}{2}, & \text{otherwise } c_{ijkl} = 0. \end{cases}$$

By (P3), it is ensured that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are strong $\mathcal{H}$-tensors. Set $r_1 = r_2 = r_3 = 1$ and $x = (x_1,x_2,x_3)^T = (1,2,2)^T$. Then $\mathcal{D} = \mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \mathcal{A}_3^{[r_3]} = (d_{ijkl})$, where $d_{1111} = 120$, $d_{2222} = 18$, $d_{3333} = 24$, $d_{1112} = 3$, $d_{2111} = 3$, $d_{1113} = \frac{15}{4}$, $d_{3111} = \frac{15}{4}$, otherwise $d_{ijkl} = 0$. Since

$$\begin{cases} |d_{1111}|x_1^3 = 120 \times 1 = 120 > |d_{1112}|x_1^2x_2 + |d_{1113}|x_1^2x_3 = 3 \times 1^2 \times 1 + \frac{15}{4} \times 1^2 \times 2 = \frac{21}{2}, \\ |d_{2222}|x_2^3 = 18 \times 2^3 = 144 > |d_{2111}|x_1^3 = 3 \times 1^3 = 3, \\ |d_{3333}|x_3^3 = 24 \times 2^3 = 192 > |d_{3111}|x_1^3 = \frac{15}{4} \times 1^3 = \frac{15}{4}, \end{cases}$$

we see by (P3) that $\mathcal{D}$ is a strong $\mathcal{H}$-tensor.

3 Bounding the minimal real eigenvalues

In this section, we bound the minimal real eigenvalues of the comparison tensors of the Hadamard products involving strong $\mathcal{H}$-tensors.

Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathcal{R}^{(m,n)}$ and let $\alpha \subseteq \{1,2,\ldots,n\}$ with $|\alpha| = k$, where $|\alpha|$ denotes the number of elements of α. A principal subtensor $\mathcal{A}[\alpha]$ of $\mathcal{A}$ is an mth-order k-dimensional subtensor consisting of k^m elements defined as

$$\mathcal{A}[\alpha] = (a_{i_1 i_2 \ldots i_m}), \quad \text{where } i_1, i_2, \ldots, i_m \in \alpha.$$

For a nonnegative tensor $\mathcal{B} \in \mathcal{R}^{(m,n)}$, let $\mathcal{B}[\alpha]$ be a principal subtensor with $|\alpha| < n$. Then $\rho(\mathcal{B}[\alpha]) \leq \rho(\mathcal{B})$ by [10, Lemma 2.2]. Further, if $\mathcal{B}$ is weakly irreducible, then $\rho(\mathcal{B}[\alpha]) < \rho(\mathcal{B})$ by [12, Theorem 3.3] or [11, Proposition 2.5]. Thus we immediately have the following result.

Lemma 3.1 *Let $\mathcal{A} \in \mathcal{R}^{(m,n)}$ be a strong $\mathcal{H}$-tensor and let $\mathcal{A}[\alpha]$ be a principal subtensor with $|\alpha| < n$. Then $\mathcal{A}[\alpha]$ is a strong $\mathcal{H}$-tensor and $\sigma(\mathcal{A}[\alpha]) \geq \sigma(\mathcal{A})$. Furthermore, if $\mathcal{A}$ is weakly irreducible, then $\sigma(\mathcal{A}[\alpha]) > \sigma(\mathcal{A})$.*

Proof Let $\mathcal{M}(\mathcal{A}) = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $s > \rho(\mathcal{B})$. Then $\mathcal{M}(\mathcal{A}[\alpha]) = s\mathcal{I} - \mathcal{B}[\alpha]$ and $s - \rho(\mathcal{B}[\alpha]) \geq s - \rho(\mathcal{B}) > 0$. So $\mathcal{A}[\alpha]$ is a strong $\mathcal{H}$-tensor with $\sigma(\mathcal{A}[\alpha]) \geq \sigma(\mathcal{A})$. Further, if $\mathcal{A}$ is weakly irreducible, then $\mathcal{B}$ is also weakly irreducible by Definition 1.2, so $\rho(\mathcal{B}[\alpha]) < \rho(\mathcal{B})$, which implies that $\sigma(\mathcal{A}[\alpha]) > \sigma(\mathcal{A})$. The result is proved. □

For a nonnegative tensor $\mathcal{B} \in \mathcal{R}^{(m,n)}$, by [10, Theorem 5.2], there exists a partition $\{\alpha_1, \ldots, \alpha_p\}$ of $\{1, 2, \ldots, n\}$ such that the principal subtensor $\mathcal{B}[\alpha_i]$ is weakly irreducible for $i = 1, 2, \ldots, p$. Also, $\rho(\mathcal{B}) = \rho(\mathcal{B}[\alpha_t])$ for some $1 \leq t \leq p$. Thus we immediately have the following result.

Lemma 3.2 *Let $\mathcal{A} \in \mathcal{R}^{(m,n)}$ be a strong $\mathcal{H}$-tensor. Then there exists $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $\mathcal{A}[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor with $\sigma(\mathcal{A}) = \sigma(\mathcal{A}[\alpha])$.*

Proof Let $\mathcal{M}(\mathcal{A}) = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $s > \rho(\mathcal{B})$. Assume that $\mathcal{B}[\alpha]$ is a weakly irreducible principal subtensor of $\mathcal{B}$ such that $\rho(\mathcal{B}) = \rho(\mathcal{B}[\alpha])$. Then, by Definition 1.2 and Lemma 3.1, $\mathcal{A}[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor. Moreover, $\sigma(\mathcal{A}) = s - \rho(\mathcal{B}) = s - \rho(\mathcal{B}[\alpha]) = \sigma(\mathcal{A}[\alpha])$. The result is proved. □

Lemma 3.3 ([13, Lemma 5.3]) *Let $\mathcal{B} \in \mathcal{R}^{(m,n)}$ be a nonnegative tensor and let $x = (x_i) \in \mathbb{R}^n$ be a positive vector. Then*

$$\min_{1 \leq i \leq n} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}} \leq \rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} \frac{(\mathcal{B}x^{m-1})_i}{x_i^{m-1}}.$$

Lemma 3.4 *Let $\mathcal{A} \in \mathcal{R}^{(m,n)}$ be an $\mathcal{M}$-tensor and let $\mathcal{A}z^{m-1} \geq kz^{[m-1]}$ for a positive vector $z \in \mathbb{R}^n$. Then $\tau(\mathcal{A}) \geq k$.*

Proof Let $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $s \geq \rho(\mathcal{B})$. Since $\mathcal{A}z^{m-1} \geq kz^{[m-1]}$ for $z = (z_i) \in \mathbb{R}^n > 0$, we have, for all $i = 1, 2, \ldots, n$,

$$sz_i^{m-1} - \left(\mathcal{B}z^{m-1}\right)_i \geq kz_i^{m-1},$$

from which it follows that

$$\max_{1 \leq i \leq n} \frac{(\mathcal{B}z^{m-1})_i}{z_i^{m-1}} \leq s - k.$$

So, by Lemma 3.3, $\rho(\mathcal{B}) \leq s - k$. Thus $\tau(\mathcal{A}) = s - \rho(\mathcal{B}) \geq k$. The result is proved. □

Lemma 3.5 *Let $\mathcal{A},\mathcal{B}\in\mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors, and let $0\le r\le 1$. Then*

$$\sigma\left(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}\right)\ge\sigma(\mathcal{A})^r\sigma(\mathcal{B})^{1-r}. \tag{3.1}$$

Proof The result is trivial for $r=0,1$. So let $0<r<1$. We first consider the case where $\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}$ is weakly irreducible. Obviously, both $\mathcal{A}$ and $\mathcal{B}$ must be weakly irreducible. Thus, by (P2), there exist positive eigenvectors $x=(x_i)\in\mathbb{R}^n$ and $y=(y_i)\in\mathbb{R}^n$ such that $\mathcal{M}(\mathcal{A})x^{m-1}=\sigma(\mathcal{A})x^{[m-1]}$ and $\mathcal{M}(\mathcal{B})y^{m-1}=\sigma(\mathcal{B})y^{[m-1]}$, respectively. Let $\mathcal{A}=(a_{i_1i_2\ldots i_m})$ and $\mathcal{B}=(b_{i_1i_2\ldots i_m})$. Then, for all $i=1,2,\ldots,n$,

$$\begin{cases}|a_{ii\ldots i}|x_i^{m-1}-\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}=\sigma(\mathcal{A})x_i^{m-1}>0,\\ |b_{ii\ldots i}|y_i^{m-1}-\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}=\sigma(\mathcal{B})y_i^{m-1}>0.\end{cases} \tag{3.2}$$

Set $z=(x_i^ry_i^{1-r})\in\mathbb{R}^n$. Then, by the Hölder inequality, we have, for all $i=1,2,\ldots,n$,

$$\begin{aligned}\left(\mathcal{M}\left(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}\right)z^{m-1}\right)_i&=\left(|a_{ii\ldots i}|x_i^{m-1}\right)^r\left(|b_{ii\ldots i}|y_i^{m-1}\right)^{1-r}\\&\quad-\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}\left(|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\right)^r\left(|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}\right)^{1-r}\\&\ge\left(|a_{ii\ldots i}|x_i^{m-1}\right)^r\left(|b_{ii\ldots i}|y_i^{m-1}\right)^{1-r}\\&\quad-\left(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\right)^r\\&\quad\times\left(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}\right)^{1-r}\\&\ge\left(|a_{ii\ldots i}|x_i^{m-1}-\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\right)^r\\&\quad\times\left(|b_{ii\ldots i}|y_i^{m-1}-\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)}|b_{ii_2\ldots i_m}|y_{i_2}\ldots y_{i_m}\right)^{1-r}\\&=\left(\sigma(\mathcal{A})x_i^{m-1}\right)^r\left(\sigma(\mathcal{B})y_i^{m-1}\right)^{1-r}=\sigma(\mathcal{A})^r\sigma(\mathcal{B})^{1-r}z_i^{m-1}.\end{aligned} \tag{3.3}$$

So $\mathcal{M}(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})z^{m-1}\ge\sigma(\mathcal{A})^r\sigma(\mathcal{B})^{1-r}z^{[m-1]}$ for $z>0$. Consider that $\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}$ is a strong $\mathcal{H}$-tensor by Theorem 2.3. Thus, using Lemma 3.4, we get $\sigma(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})\ge\sigma(\mathcal{A})^r\sigma(\mathcal{B})^{1-r}$.

Now we consider the general case. Recall that $\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}$ is a strong $\mathcal{H}$-tensor. By Lemma 3.2, there exists $\alpha\subseteq\{1,2,\ldots,n\}$ such that $(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})[\alpha]=(\mathcal{A}[\alpha])^{[r]}\circ(\mathcal{B}[\alpha])^{[1-r]}$ is a weakly irreducible $\mathcal{H}$-tensor with $\sigma(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})=\sigma((\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]})[\alpha])$. Note that $\mathcal{A}[\alpha]$ and $\mathcal{B}[\alpha]$ are strong $\mathcal{H}$-tensors. Thus, according to the case above, using Lemma 3.1 we get

$$\sigma\left(\mathcal{A}^{[r]}\circ\mathcal{B}^{[1-r]}\right)=\sigma\left(\left(\mathcal{A}[\alpha]\right)^{[r]}\circ\left(\mathcal{B}[\alpha]\right)^{[1-r]}\right)\ge\sigma\left(\mathcal{A}[\alpha]\right)^r\sigma\left(\mathcal{B}[\alpha]\right)^{1-r}\ge\sigma(\mathcal{A})^r\sigma(\mathcal{B})^{1-r}.$$

The result is proved. □

Lemma 3.6 *Let $\mathcal{A}\in\mathcal{R}^{(m,n)}$ be a strong $\mathcal{H}$-tensor, and let $t\ge 1$. Then $\sigma(\mathcal{A}^{[t]})\ge\sigma(\mathcal{A})^t$.*

Proof First assume that $\mathcal{A}^{[t]}$ is weakly irreducible. Obviously, $\mathcal{A}$ is weakly irreducible. Then by (P2), there exists a positive eigenvector $x = (x_i) \in \mathbb{R}^n$ such that $\mathcal{M}(\mathcal{A})x^{m-1} = \sigma(\mathcal{A})x^{[m-1]}$. Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$. Then, for all $i = 1, 2, \ldots, n$,

$$|a_{ii\ldots i}|x_i^{m-1} - \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m} = \sigma(\mathcal{A})x_i^{m-1} > 0. \tag{3.4}$$

Set $z = (x_i^t) \in \mathbb{R}^n$. Then, by the Minkowski inequality, we have, for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
\left(\mathcal{M}\left(\mathcal{A}^{[t]}\right)z^{m-1}\right)_i &= \left|a_{ii\ldots i}^t\right|\left(x_i^t\right)^{m-1} - \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} \left|a_{ii_2\ldots i_m}^t\right| x_{i_2}^t \ldots x_{i_m}^t \\
&\geq \left(|a_{ii\ldots i}|x_i^{m-1}\right)^t - \left(\sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\right)^t \\
&\geq \left(|a_{ii\ldots i}|x_i^{m-1} - \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}|x_{i_2}\ldots x_{i_m}\right)^t \\
&= \sigma(\mathcal{A})^t z_i^{m-1}.
\end{aligned} \tag{3.5}$$

So $\mathcal{M}(\mathcal{A}^{[t]})z^{m-1} \geq \sigma(\mathcal{A})^t z^{[m-1]}$ for $z > 0$. Consider that $\mathcal{A}^{[t]}$ is a strong $\mathcal{H}$-tensor by Lemma 2.2. Thus, using Lemma 3.4, we get $\sigma(\mathcal{A}^{[t]}) \geq \sigma(\mathcal{A})^t$.

Now we consider the general case. Recall that $\mathcal{A}^{[t]}$ is a strong $\mathcal{H}$-tensor. By Lemma 3.2, there exists $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $\mathcal{A}^{[t]}[\alpha] = (\mathcal{A}[\alpha])^{[t]}$ is a weakly irreducible $\mathcal{H}$-tensor with $\sigma(\mathcal{A}^{[t]}) = \sigma(\mathcal{A}^{[t]}[\alpha])$. Thus, according to the case above, using Lemma 3.1 we get

$$\sigma\left(\mathcal{A}^{[t]}\right) = \sigma\left(\left(\mathcal{A}[\alpha]\right)^{[t]}\right) \geq \sigma\left(\mathcal{A}[\alpha]\right)^t \geq \sigma(\mathcal{A})^t.$$

The result is proved. □

Our main result of this section is the following.

Theorem 3.7 *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \in \mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors and let $r_1, r_2, \ldots, r_k$ be positive numbers such that $\sum_{i=1}^k r_i \geq 1$. Then*

$$\sigma\left(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\right) \geq \sigma(\mathcal{A}_1)^{r_1}\sigma(\mathcal{A}_2)^{r_2}\cdots\sigma(\mathcal{A}_k)^{r_k}. \tag{3.6}$$

Proof By (P1), without loss of generality, assume that all the tensors $\mathcal{A}_i$ are nonnegative for $i = 1, 2, \ldots, k$. We first use the induction on k to prove the result in the case that $\sum_{i=1}^k r_i = 1$. Obviously, the result is true for $k = 2$ by Lemma 3.5. Assume the result is true for $k - 1$. Now let

$$\mathcal{B}^{[1-r_k]} = \mathcal{A}_1^{[r_1]} \circ \cdots \circ \mathcal{A}_{k-1}^{[r_{k-1}]}.$$

Consider that each $\mathcal{A}_i$ is nonnegative. Then

$$\mathcal{B} = \mathcal{A}_1^{[\frac{r_1}{1-r_k}]} \circ \cdots \circ \mathcal{A}_{k-1}^{[\frac{r_{k-1}}{1-r_k}]}.$$

Note that $\sum_{i=1}^{k-1} \frac{r_i}{1-r_k} = 1$. Thus $\mathcal{B}$ is a strong $\mathcal{H}$-tensor by Theorem 2.3. Therefore, using the induction assumption, we get

$$\begin{aligned}\sigma\left(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\right) &= \sigma\left(\mathcal{B}^{[1-r_k]} \circ \mathcal{A}_k^{[r_k]}\right) \geq \sigma(\mathcal{B})^{1-r_k}\sigma(\mathcal{A}_k)^{r_k} \\ &\geq \left(\sigma(\mathcal{A}_1)^{\frac{r_1}{1-r_k}} \cdots \sigma(\mathcal{A}_{k-1})^{\frac{r_{k-1}}{1-r_k}}\right)^{1-r_k}\sigma(\mathcal{A}_k)^{r_k} \\ &= \sigma(\mathcal{A}_1)^{r_1} \cdots \sigma(\mathcal{A}_{k-1})^{r_{k-1}}\sigma(\mathcal{A}_k)^{r_k}. \end{aligned} \tag{3.7}$$

So the result is true in the case that $\sum_{i=1}^k r_i = 1$.

Now we consider the general case $t = \sum_{i=1}^k r_i \geq 1$. Set $l_i = r_i t^{-1}$ for $i = 1, 2, \ldots, k$. Then $\sum_{i=1}^k l_i = 1$. Thus $\mathcal{C} = \mathcal{A}_1^{[l_1]} \circ \mathcal{A}_2^{[l_2]} \circ \cdots \circ \mathcal{A}_k^{[l_k]}$ is a strong $\mathcal{H}$-tensor by Theorem 2.3. Therefore, according to the case above, using Lemma 3.6 we get

$$\begin{aligned}\sigma\left(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\right) &= \sigma\left(\mathcal{C}^{[t]}\right) \geq \sigma(\mathcal{C})^t \\ &\geq \left(\sigma(\mathcal{A}_1)^{l_1}\sigma(\mathcal{A}_2)^{l_2} \cdots \sigma(\mathcal{A}_k)^{l_k}\right)^t \\ &= \sigma(\mathcal{A}_1)^{r_1}\sigma(\mathcal{A}_2)^{r_2} \cdots \sigma(\mathcal{A}_k)^{r_k}.\end{aligned}$$

The result is proved. □

Example 3.1 Let $\mathcal{A}_1 = (a_{ijkl}), \mathcal{A}_2 = (b_{ijkl}), \mathcal{A}_3 = (c_{ijkl}) \in \mathcal{R}^{(4,2)}$ be defined as follows:

$$\begin{cases} a_{1111} = 4, a_{1112} = a_{2111} = a_{1211} = a_{1121} = 1, a_{2222} = 2, & \text{otherwise } a_{ijkl} = 0, \\ b_{1111} = 5, b_{1112} = b_{2111} = b_{1211} = b_{1121} = 1, b_{2222} = 4, & \text{otherwise } b_{ijkl} = 0, \\ c_{1111} = 6, c_{1112} = a_{2111} = c_{1211} = c_{1121} = 1, c_{2222} = 4, & \text{otherwise } c_{ijkl} = 0. \end{cases}$$

By (P3), it is assured that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are strong $\mathcal{H}$-tensors. Now set $r_1 = r_2 = r_3 = 1$. Then $\mathcal{D} = \mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \mathcal{A}_3^{[r_3]} = (d_{ijkl})$, where $d_{1111} = 120$, $d_{2222} = 32$, $d_{1112} = 1$, $d_{2111} = 1$, $d_{1211} = 1$, $d_{1121} = 1$, otherwise $d_{ijkl} = 0$. By Corollary 2 of Qi [16], we get

$$\begin{cases} \varphi[M(\mathcal{A}_1)] = \{1, 2, 2, 3.547 + 2.125i, 3.547 - 2.125i, 5.905\}, \\ \varphi[M(\mathcal{A}_2)] = \{2.422, 4, 4, 4.756 + 2.239i, 4.756 - 2.239i, 7.065\}, \\ \varphi[M(\mathcal{A}_3)] = \{3, 4, 4, 5.547 + 2.125i, 5.547 - 2.125i, 7.905\}, \\ \varphi[M(\mathcal{D})] = \{31.999, 32, 32, 119.663 + 0.585i, 119.663 - 0.585i, 120.672\}. \end{cases}$$

So $\sigma(\mathcal{D}) = 31.999 \geq \sigma(\mathcal{A}_1)\sigma(\mathcal{A}_2)\sigma(\mathcal{A}_3) = 1 \times 2.422 \times 3 = 7.266$.

4 Characterizations for the equality case

In this section, we characterize the strong $\mathcal{H}$-tensors such that the equality of (3.6) holds.

Lemma 4.1 ([12, Lemma 3.2]) *Let $\mathcal{B} \in \mathcal{R}^{(m,n)}$ be a weakly irreducible nonnegative tensor and let $\mathcal{B}z^{m-1} \leq \rho(\mathcal{B})z^{[m-1]}$ for a positive vector $z \in \mathbb{R}^n$. Then $\mathcal{B}z^{m-1} = \rho(\mathcal{B})z^{[m-1]}$.*

Using Lemma 4.1, we immediately get the following result.

Lemma 4.2 *Let $\mathcal{A} \in \mathcal{R}^{(m,n)}$ be a weakly irreducible strong $\mathcal{M}$-tensor and let $\mathcal{A}z^{m-1} \geq \tau(\mathcal{A})z^{[m-1]}$ for a positive vector $z \in \mathbb{R}^n$. Then $\mathcal{A}z^{m-1} = \tau(\mathcal{A})z^{[m-1]}$.*

Proof Let $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $s > \rho(\mathcal{B})$. Obviously, $\mathcal{B}$ is weakly irreducible. Since $\mathcal{A}z^{m-1} \geq \tau(\mathcal{A})z^{[m-1]}$ where $\tau(\mathcal{A}) = s - \rho(\mathcal{B})$, we have $\mathcal{B}z^{m-1} \leq \rho(\mathcal{B})z^{[m-1]}$ for $z > 0$. Thus, by Lemma 4.1, $\mathcal{B}z^{m-1} = \rho(\mathcal{B})z^{[m-1]}$. So $\mathcal{A}z^{m-1} = \tau(\mathcal{A})z^{[m-1]}$. The result is proved. □

For a tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathcal{R}^{(m,n)}$ and a nonsingular diagonal matrix $D = \operatorname{diag}(d_{ii}) \in \mathbb{R}^{n\times n}$, the tensor $\mathcal{C} = \mathcal{A}D^{-(m-1)} \cdot \underbrace{D \cdots D}_{m-1} = (c_{i_1 i_2 \dots i_m}) \in \mathcal{R}^{(m,n)}$ is defined as

$$c_{i_1 i_2 \dots i_m} = a_{i_1 i_2 \dots i_m} d_{i_1,i_1}^{-(m-1)} d_{i_2,i_2} \cdots d_{i_m,i_m}, \quad 1 \leq i_1, i_2, \dots, i_m \leq n.$$

It must be pointed out that $\mathcal{A}$ and $\mathcal{C}$ have the same eigenvalues [13]. In particular, if $\mathcal{A}$ and $\mathcal{C}$ are strong $\mathcal{H}$-tensors, then $\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{A})|D|^{-(m-1)} \cdot \underbrace{|D| \cdots |D|}_{m-1}$, so $\sigma(\mathcal{A}) = \sigma(\mathcal{C})$.

Lemma 4.3 *Let $\mathcal{A}, \mathcal{B} \in \mathcal{R}^{(m,n)}$ be weakly irreducible strong $\mathcal{H}$-tensors and let $0 < r < 1$. Then*

$$\sigma\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right) = \sigma(\mathcal{A})^r \sigma(\mathcal{B})^{1-r}$$

if and only if there exist $\gamma > 0$ and a positive diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that

$$|\mathcal{A}| = \gamma |\mathcal{B}| D^{-(m-1)} \cdot \underbrace{D \cdots D}_{m-1}.$$

Proof As regards sufficiency, we have $\sigma(\mathcal{A})^r \sigma(\mathcal{B})^{1-r} = \gamma^r \sigma(\mathcal{B})^r \sigma(\mathcal{B})^{1-r} = \gamma^r \sigma(\mathcal{B})$ and

$$\begin{aligned}\sigma\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right) &= \sigma\left(|\mathcal{A}|^{[r]} \circ |\mathcal{B}|^{[1-r]}\right) \\ &= \sigma\left(\gamma^r \left(|\mathcal{B}|^{[r]} \circ |\mathcal{B}|^{[1-r]}\right)\left(D^r\right)^{-(m-1)} \cdot \underbrace{D^r \cdots D^r}_{m-1}\right) = \gamma^r \sigma(\mathcal{B}),\end{aligned}$$

and thus the sufficiency is true.

Necessarily, according to the proof of Lemma 3.5, there exists $\alpha \subseteq \{1, 2, \dots, n\}$ such that $(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]})[\alpha]$ is a weakly irreducible $\mathcal{H}$-tensor and

$$\begin{aligned}\sigma\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right) &= \sigma\left(\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right)[\alpha]\right) \\ &= \sigma\left(\left(\mathcal{A}[\alpha]\right)^{[r]} \circ \left(\mathcal{B}[\alpha]\right)^{[1-r]}\right) \geq \sigma\left(\mathcal{A}[\alpha]\right)^r \sigma\left(\mathcal{B}[\alpha]\right)^{1-r}.\end{aligned}$$

Recall that $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m})$ are weakly irreducible strong $\mathcal{H}$-tensors. Thus, if $|\alpha| < n$, then, by Lemma 3.1, $\sigma(\mathcal{A}[\alpha]) > \sigma(\mathcal{A})$ and $\sigma(\mathcal{B}[\alpha]) > \sigma(\mathcal{B})$, from which it follows that $\sigma(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}) > \sigma(\mathcal{A})^r \sigma(\mathcal{B})^{1-r}$, a contradiction. So $|\alpha| = n$. Hence, $\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}$ must be weakly irreducible and thus, according to the proof of Lemma 3.5, (3.3) is true, *i.e.*,

$$\begin{aligned}\mathcal{M}\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right) z^{m-1} &\geq \sigma(\mathcal{A})^r \sigma(\mathcal{B})^{1-r} z^{[m-1]} \\ &= \sigma\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right) z^{[m-1]}, \quad 0 < z = \left(x_i^r y_i^{1-r}\right) \in \mathbb{R}^n,\end{aligned}$$

from which it follows by Lemma 4.2 that

$$\mathcal{M}\left(\mathcal{A}^{[r]} \circ \mathcal{B}^{[1-r]}\right)z^{m-1} = \sigma(\mathcal{A})^r \sigma(\mathcal{B})^{1-r} z^{[m-1]}.$$

This means that the two Hölder inequalities of (3.3) are equalities and so, for all $i = 1, 2, \ldots, n$,

$$|a_{ii_2 \ldots i_m}| x_{i_2} \ldots x_{i_m} = k_i |b_{ii_2 \ldots i_m}| y_{i_2} \ldots y_{i_m}, \quad \forall (i_2, \ldots, i_m) \neq (i, \ldots, i)$$

for some constant k_i and for some constant l_i

$$\begin{cases} \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}| x_{i_2} \ldots x_{i_m} = l_i \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |b_{ii_2\ldots i_m}| y_{i_2} \ldots y_{i_m}, \\ |a_{ii\ldots i}| x_i^{m-1} - \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |a_{ii_2\ldots i_m}| x_{i_2} \ldots x_{i_m} \\ \quad = l_i (|b_{ii\ldots i}| y_i^{m-1} - \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} |b_{ii_2\ldots i_m}| y_{i_2} \ldots y_{i_m}), \end{cases}$$

from which we get $k_i = l_i$ and

$$|a_{ii_2 \ldots i_m}| x_{i_2} \ldots x_{i_m} = k_i |b_{ii_2 \ldots i_m}| y_{i_2} \ldots y_{i_m}, \quad \forall i, i_2, \ldots, i_m.$$

By considering (3.2),

$$\sigma(\mathcal{A}) x_i^{m-1} = k_i \sigma(\mathcal{B}) y_i^{m-1} \quad \Rightarrow \quad k_i = \frac{\sigma(\mathcal{A})}{\sigma(\mathcal{B})} \frac{x_i^{m-1}}{y_i^{m-1}}.$$

Therefore we have, for all $i = 1, 2, \ldots, n$,

$$|a_{ii_2 \ldots i_m}| = |b_{ii_2 \ldots i_m}| \frac{\sigma(\mathcal{A})}{\sigma(\mathcal{B})} \frac{x_i^{m-1}}{y_i^{m-1}} \frac{y_{i_2}}{x_{i_2}} \cdots \frac{y_{i_m}}{x_{i_m}}, \quad 1 \leq i_2, \ldots, i_m \leq n. \tag{4.1}$$

Set $D = \operatorname{diag}(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}) \in \mathbb{R}^{n \times n}$ and $\gamma = \frac{\sigma(\mathcal{A})}{\sigma(\mathcal{B})}$. Then (4.1) implies that $|\mathcal{A}| = \gamma |\mathcal{B}| D^{-(m-1)} \cdot \underbrace{D \cdots D}_{m-1}$. The result is proved. □

Now we characterize strong $\mathcal{H}$-tensors such that the equality of (3.6) holds in the case that $\sum_{i=1}^k r_i = 1$.

Theorem 4.4 *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \in \mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors and let $r_1, r_2, \ldots, r_k$ be positive numbers such that $\sum_{i=1}^k r_i = 1$. Then*

$$\sigma\left(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\right) = \sigma(\mathcal{A}_1)^{r_1} \sigma(\mathcal{A}_2)^{r_2} \cdots \sigma(\mathcal{A}_k)^{r_k}$$

if and only if there exists $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $\mathcal{A}_i[\alpha]$ is weakly irreducible with $\sigma(\mathcal{A}_i[\alpha]) = \sigma(\mathcal{A}_i)$ for all $i = 1, 2, \ldots, k$ and

$$\left|\mathcal{A}_i[\alpha]\right| = \gamma_i \left|\mathcal{A}_1[\alpha]\right| D_i^{-(m-1)} \cdot \underbrace{D_i \cdots D_i}_{m-1}, \quad i = 2, \ldots, k, \tag{4.2}$$

where $\gamma_i > 0$ and $D_i \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix.

Proof As regards sufficiency, using Lemma 3.1 and Theorem 3.7, we have

$$\begin{aligned}\sigma\left(\mathcal{A}_1^{[r_1]}\circ\mathcal{A}_2^{[r_2]}\circ\cdots\circ\mathcal{A}_k^{[r_k]}\right)&\le\sigma\left(\left(\mathcal{A}_1^{[r_1]}\circ\mathcal{A}_2^{[r_2]}\circ\cdots\circ\mathcal{A}_k^{[r_k]}\right)[\alpha]\right)\\&=\sigma\left(\left|\mathcal{A}_1[\alpha]\right|^{[r_1]}\circ\left|\mathcal{A}_2[\alpha]\right|^{[r_2]}\circ\cdots\circ\left|\mathcal{A}_k[\alpha]\right|^{[r_k]}\right)\\&=\gamma_2^{r_2}\cdots\gamma_k^{r_k}\sigma\left(\left|\mathcal{A}_1[\alpha]\right|\right)\\&=\sigma\left(\mathcal{A}_1[\alpha]\right)^{r_1}\sigma\left(\mathcal{A}_2[\alpha]\right)^{r_2}\cdots\sigma\left(\mathcal{A}_k[\alpha]\right)^{r_k}\\&=\sigma(\mathcal{A}_1)^{r_1}\sigma(\mathcal{A}_2)^{r_2}\cdots\sigma(\mathcal{A}_k)^{r_k}\\&\le\sigma\left(\mathcal{A}_1^{[r_1]}\circ\mathcal{A}_2^{[r_2]}\circ\cdots\circ\mathcal{A}_k^{[r_k]}\right)\end{aligned}$$

and thus the sufficiency is true.

Necessarily, by (P1), without loss of generality, assume that $\mathcal{A}_i$ is nonnegative for all $i=1,2,\dots,k$. Note that $\mathcal{C}=\mathcal{A}_1^{[r_1]}\circ\mathcal{A}_2^{[r_2]}\circ\cdots\circ\mathcal{A}_k^{[r_k]}$ is a strong $\mathcal{H}$-tensor by Theorem 2.3. Thus by Lemma 3.2, there exists $\alpha\subseteq\{1,2,\dots,n\}$ such that $\mathcal{C}[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor with $\sigma(\mathcal{C})=\sigma(\mathcal{C}[\alpha])$. Consider that $\mathcal{C}[\alpha]=(\mathcal{A}_1[\alpha])^{[r_1]}\circ\cdots\circ(\mathcal{A}_k[\alpha])^{[r_k]}$. Thus $\mathcal{A}_i[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor for $i=1,2,\dots,k$. Denote $\mathcal{B}^{[1-r_k]}=(\mathcal{A}_1[\alpha])^{[r_1]}\circ\cdots\circ(\mathcal{A}_{k-1}[\alpha])^{[r_{k-1}]}$, which is weakly irreducible. Then $\mathcal{B}=(\mathcal{A}_1[\alpha])^{[\frac{r_1}{1-r_k}]}\circ\cdots\circ(\mathcal{A}_{k-1}[\alpha])^{[\frac{r_{k-1}}{1-r_k}]}$ is a weakly irreducible strong $\mathcal{H}$-tensor. Hence, by Theorem 3.7 and Lemma 3.1, we have

$$\begin{aligned}\sigma(\mathcal{C})&=\sigma\left(\mathcal{B}^{[1-r_k]}\circ\left(\mathcal{A}_k[\alpha]\right)^{[r_k]}\right)\ge\sigma(\mathcal{B})^{1-r_k}\sigma\left(\mathcal{A}_k[\alpha]\right)^{r_k}\\&\ge\left(\sigma\left(\mathcal{A}_1[\alpha]\right)^{\frac{r_1}{1-r_k}}\cdots\sigma\left(\mathcal{A}_{k-1}[\alpha]\right)^{\frac{r_{k-1}}{1-r_k}}\right)^{1-r_k}\sigma\left(\mathcal{A}_k[\alpha]\right)^{r_k}\\&=\sigma\left(\mathcal{A}_1[\alpha]\right)^{r_1}\cdots\sigma\left(\mathcal{A}_{k-1}[\alpha]\right)^{r_{k-1}}\sigma\left(\mathcal{A}_k[\alpha]\right)^{r_k}\\&\ge\sigma(\mathcal{A}_1)^{r_1}\cdots\sigma(\mathcal{A}_{k-1})^{r_{k-1}}\sigma(\mathcal{A}_k)^{r_k}=\sigma(\mathcal{C}).\end{aligned}\tag{4.3}$$

Thus $\sigma(\mathcal{A}_i[\alpha])=\sigma(\mathcal{A}_i)$ for all $i=1,2,\dots,k$. Thus according to the observation that

$$\sigma\left(\left(\mathcal{A}_1[\alpha]\right)^{[r_1]}\circ\cdots\circ\left(\mathcal{A}_k[\alpha]\right)^{[r_k]}\right)=\sigma\left(\mathcal{A}_1[\alpha]\right)^{r_1}\cdots\sigma\left(\mathcal{A}_{k-1}[\alpha]\right)^{r_{k-1}}\sigma\left(\mathcal{A}_k[\alpha]\right)^{r_k},$$

where each $\mathcal{A}_i[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor, we use the induction on k to prove that (4.2) is true. Clearly, (4.2) is true for $k=2$ by Lemma 4.3. Assume that (4.2) is true for $k-1$. Now by (4.3) we have the following statements:

- $\sigma(\mathcal{B}^{[1-r_k]}\circ(\mathcal{A}_k[\alpha])^{[r_k]})=\sigma(\mathcal{B})^{(1-r_k)}\sigma(\mathcal{A}_k[\alpha])^{r_k}$ and so, by Lemma 4.3, there exist $\gamma_k'>0$ and a positive diagonal matrix $D_k'\in\mathbb{R}^{n\times n}$ such that

$$\left|\mathcal{A}_k[\alpha]\right|=\gamma_k'|\mathcal{B}|\left(D_k'\right)^{-(m-1)}\cdot\underbrace{D_k'\cdots D_k'}_{m-1}.\tag{4.4}$$

- $\sigma(\mathcal{B})=\sigma(\mathcal{A}_1[\alpha])^{\frac{r_1}{1-r_k}}\cdots\sigma(\mathcal{A}_{k-1}[\alpha])^{\frac{r_{k-1}}{1-r_k}}$ and thus, by the induction assumption, we find that, for all $i=2,\dots,k-1$, there exist $\gamma_i>0$ and a positive diagonal matrix $D_i\in\mathbb{R}^{n\times n}$ such that

$$\left|\mathcal{A}_i[\alpha]\right|=\gamma_i\left|\mathcal{A}_1[\alpha]\right|D_i^{-(m-1)}\cdot\underbrace{D_i\cdots D_i}_{m-1}.\tag{4.5}$$

- Using (4.4) and (4.5), we derive that there exist $\gamma_k > 0$ and a positive diagonal matrix $D_k \in \mathbb{R}^{n\times n}$ such that

$$\bigl|\mathcal{A}_k[\alpha]\bigr| = \gamma_k \bigl|\mathcal{A}_1[\alpha]\bigr| D_k^{-(m-1)} \cdot \underbrace{D_k \cdots D_k}_{m-1}.$$

Thus the result is proved. □

Next we characterize strong $\mathcal{H}$-tensors such that the equality of (3.6) holds in the case that $\sum_{i=1}^k r_i > 1$.

Lemma 4.5 *Let $\mathcal{A} \in \mathcal{R}^{(m,n)}$ be a weakly irreducible strong $\mathcal{H}$-tensor and let $t > 1$. Then $\sigma(\mathcal{A}^{[t]}) = \sigma(\mathcal{A})^t$ if and only if $n = 1$.*

Proof The sufficiency is trivial. Necessarily, $\mathcal{A}^{[t]}$ is obviously a weakly irreducible strong $\mathcal{H}$-tensor and thus, according to the proof of Lemma 3.6, (3.5) is true, *i.e.*,

$$\mathcal{M}\bigl(\mathcal{A}^{[t]}\bigr)z^{m-1} \geq \sigma(\mathcal{A})^t z^{[m-1]} = \sigma\bigl(\mathcal{A}^{[t]}\bigr)z^{[m-1]}, \quad 0 < z = \bigl(x_i^t\bigr) \in \mathbb{R}^n,$$

from which it follows by Lemma 4.2 that

$$\mathcal{M}\bigl(\mathcal{A}^{[t]}\bigr)z^{m-1} = \sigma(\mathcal{A})^t z^{[m-1]}.$$

This means that the two Minkowski inequalities of (3.5) are equalities, and so, for all $i = 1, \dots, n$, there is at most one nonzero element for the elements

$$|a_{ii_2\dots i_m}| x_{i_2} \dots x_{i_m}, \quad \forall (i_2, \dots, i_m) \neq (i, \dots, i),$$

and there is at most one nonzero element for the two elements

$$\sum_{(i_2,\dots,i_m)\neq(i,\dots,i)} |a_{ii_2\dots i_m}| x_{i_2} \dots x_{i_m}, \qquad |a_{ii\dots i}| x_i^{m-1} - \sum_{(i_2,\dots,i_m)\neq(i,\dots,i)} |a_{ii_2\dots i_m}| x_{i_2} \dots x_{i_m}.$$

So, because of (3.4), we have, for all $i = 1, \dots, n$,

$$a_{ii_2\dots i_m} = 0, \quad \forall (i_2, \dots, i_m) \neq (i, \dots, i),$$

by considering the fact that $x_{i_2} \dots x_{i_m} > 0$, which means that $\mathcal{A}$ is diagonal. Recall that $\mathcal{A}$ is weakly irreducible. So, $n = 1$. The result is proved. □

Theorem 4.6 *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k \in \mathcal{R}^{(m,n)}$ be strong $\mathcal{H}$-tensors and let $r_1, r_2, \dots, r_k$ be positive numbers such that $\sum_{i=1}^k r_i > 1$. Then*

$$\sigma\bigl(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\bigr) = \sigma(\mathcal{A}_1)^{r_1} \sigma(\mathcal{A}_2)^{r_2} \dots \sigma(\mathcal{A}_k)^{r_k}$$

if and only if there exists $\alpha \subseteq \{1, 2, \dots, n\}$ with $|\alpha| = 1$ such that $\sigma(\mathcal{A}_i[\alpha]) = \sigma(\mathcal{A}_i)$ for all $i = 1, 2, \dots, k$.

Proof As regards sufficiency, by considering $|\alpha| = 1$, using Lemma 3.1 and Theorem 3.7, we have

$$\begin{aligned}\sigma\big(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\big) &\le \sigma\big(\big(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\big)[\alpha]\big)\\ &= \sigma\big(\mathcal{A}_1[\alpha]\big)^{r_1}\sigma\big(\mathcal{A}_2[\alpha]\big)^{r_2}\cdots\sigma\big(\mathcal{A}_k[\alpha]\big)^{r_k}\\ &= \sigma(\mathcal{A}_1)^{r_1}\sigma(\mathcal{A}_2)^{r_2}\cdots\sigma(\mathcal{A}_k)^{r_k}\\ &\le \sigma\big(\mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}\big),\end{aligned}$$

and thus the sufficiency is true.

Without loss of generality, assume that $\mathcal{A}_i$ is nonnegative for all $i = 1, 2, \ldots, k$. Note that $\mathcal{C} = \mathcal{A}_1^{[r_1]} \circ \mathcal{A}_2^{[r_2]} \circ \cdots \circ \mathcal{A}_k^{[r_k]}$ is a strong $\mathcal{H}$-tensor by Theorem 2.3. Thus, by Lemma 3.2, there exists $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $\mathcal{C}[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor with $\sigma(\mathcal{C}) = \sigma(\mathcal{C}[\alpha])$. Set $t = \sum_{i=1}^k r_i$ and $l_i = r_i t^{-1}$ for $i = 1, 2, \ldots, k$. Denote $\mathcal{B} = \mathcal{A}_1^{[l_1]} \circ \mathcal{A}_2^{[l_2]} \circ \cdots \circ \mathcal{A}_k^{[l_k]}$. Then $\mathcal{B}[\alpha]$ is a weakly irreducible strong $\mathcal{H}$-tensor. Hence, by using Lemma 3.6, Theorem 3.7 and Lemma 3.1,

$$\begin{aligned}\sigma(\mathcal{C}) = \sigma\big(\mathcal{C}[\alpha]\big) = \sigma\big(\big(\mathcal{B}[\alpha]\big)^{[t]}\big) &\ge \sigma\big(\mathcal{B}[\alpha]\big)^t\\ &\ge \big(\sigma\big(\mathcal{A}_1[\alpha]\big)^{l_1}\sigma\big(\mathcal{A}_2[\alpha]\big)^{l_2}\cdots\sigma\big(\mathcal{A}_k[\alpha]\big)^{l_k}\big)^t\\ &= \sigma\big(\mathcal{A}_1[\alpha]\big)^{r_1}\sigma\big(\mathcal{A}_2[\alpha]\big)^{r_2}\cdots\sigma\big(\mathcal{A}_k[\alpha]\big)^{r_k}\\ &\ge \sigma(\mathcal{A}_1)^{r_1}\sigma(\mathcal{A}_2)^{r_2}\cdots\sigma(\mathcal{A}_k)^{r_k} = \sigma(\mathcal{C}),\end{aligned}$$

from which it follows that $\sigma(\mathcal{A}_i[\alpha]) = \sigma(\mathcal{A}_i)$ for all $i = 1, 2, \ldots, k$ and $\sigma((\mathcal{B}[\alpha])^{[t]}) = \sigma(\mathcal{B}[\alpha])^t$, which implies by Lemma 4.5 that $|\alpha| = 1$. The result is proved. □

5 Conclusions

In this paper, we investigate the closure property of $\mathcal{H}$-tensors under the Hadamard product. It is shown that the Hadamard products of Hadamard powers of strong $\mathcal{H}$-tensors are still strong $\mathcal{H}$-tensors. We then bound the minimal real eigenvalues of the comparison tensors of the Hadamard products involving strong $\mathcal{H}$-tensors. Finally, we show how to attain the bounds by characterizing these $\mathcal{H}$-tensors.

Acknowledgements
The authors would like to thank the anonymous referee for his/her valuable comments. The work was supported in part by the National Natural Science Foundation of China (11571292) and the Guangxi Municipality Project for the Basic Ability Enhancement of Young and Middle-Aged Teachers (KY2016YB532).

Competing interests
All the authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, P.R. China. [2]Faculty of Science, Guilin University of Aerospace Technology, Guilin, Guangxi 541004, P.R. China. [3]Guangxi Aviation Logistics Research Center, Guilin University of Aerospace Technology, Guilin, Guangxi 541004, P.R. China.

Received: 15 June 2017 Accepted: 1 September 2017 Published online: 20 September 2017

References

1. Ding, W, Wei, Y: Generalized tensor eigenvalue problems. SIAM J. Matrix Anal. Appl. **36**, 1073-1099 (2015)
2. Kolda, T, Bader, B: Tensor decompositions and applications. SIAM Rev. **51**, 455-500 (2009)
3. Kolda, T, Mayo, J: Shifted power method for computing tensor eigenpairs. SIAM J. Matrix Anal. Appl. **31**, 1095-1124 (2011)
4. Ng, M, Qi, L, Zhou, G: Finding the largest eigenvalue of a nonnegative tensor. SIAM J. Matrix Anal. Appl. **31**, 1096-1107 (2009)
5. Qi, L: Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. **439**, 228-238 (2013)
6. Chang, K-C, Pearson, K, Zhang, T: Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. **6**, 507-520 (2008)
7. Chang, K-C, Pearson, K, Zhang, T: Primitivity, the convergence of the NQZ method, and the largest eigenvalue for nonnegative tensors. SIAM J. Matrix Anal. Appl. **32**, 806-819 (2011)
8. Ding, W, Qi, L, Wei, Y: $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors. Linear Algebra Appl. **439**, 3264-3278 (2013)
9. Friedland, S, Gaubert, S, Han, L: Perron-Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl. **438**, 738-749 (2013)
10. Hu, S, Huang, Z, Qi, L: Strictly nonnegative tensors and nonnegative tensor partition. Sci. China Math. **57**, 181-195 (2014)
11. Kannan, M, Shaked-Monderer, N, Berman, A: Some properties of strong $\mathcal{H}$-tensors and general $\mathcal{H}$-tensors. Linear Algebra Appl. **476**, 42-55 (2015)
12. Kannan, M, Shaked-Monderer, N, Berman, A: On weakly irreducible nonnegative tensors and interval hull of some classes of tensors. arXiv:1409.3003
13. Yang, Y, Yang, Q: Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. **31**, 2517-2530 (2010)
14. Yang, Q, Yang, Y: Further results for Perron-Frobenius theorem for nonnegative tensors II. SIAM J. Matrix Anal. Appl. **32**, 1236-1250 (2011)
15. Zhang, L, Qi, L, Zhou, G: *M*-tensors and some applications. SIAM J. Matrix Anal. Appl. **35**, 437-452 (2014)
16. Qi, L: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**, 1302-1324 (2005)
17. Qi, L, Luo, Z: Tensor Analysis: Spectral Theory and Special Tensors. Society for Industrial and Applied Mathematics, Philadelphia (2017)
18. Wei, Y, Ding, W: Theory and Computation of Tensors: Multi-dimensional Arrays. Academic Press, San Diego (2016)
19. Li, C, Wang, F, Zhao, J, Zhu, Y, Li, Y: Criterions for the positive definiteness of real supersymmetric tensors. J. Comput. Appl. Math. **255**, 1-14 (2014)
20. Lim, L-H: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the First IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, Puerto Vallarta, Mexico, pp. 129-132 (2005)
21. Horn, R, Johnson, C: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1994)