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Solutions for the quasi-linear elliptic problems involving the critical Sobolev exponent

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Abstract
In this article, we study the existence and multiplicity of positive solutions for the quasi-linear elliptic problems involving critical Sobolev exponent and a Hardy term. The main tools adopted in our proofs are the concentration compactness principle and Nehari manifold.

Keywords: quasi-linear elliptic problems; Nehari manifold; positive solution; best Sobolev constant

1 Introduction

In this article, we consider the following quasi-linear elliptic problem:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = |u|^{p^*-2}u + \beta |x|^{\alpha-p}|u|^{p-2}u + \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with the smooth boundary $\partial\Omega$ such that $0 \in \Omega$. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator of u, $1 < p < N, \lambda > 0$ is a positive real number. $0 \leq \mu < \overline{\mu}$ ($\overline{\mu} = \frac{(N-p)^p}{p}$ is the best Hardy constant). $1 < q < p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. $0 < \alpha < p-1$, $0 < \beta < \beta_1$ (β_1 is the first eigenvalue that $-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = |x|^{\alpha-p}|u|^{p-2}u$ under Dirichlet boundary condition).

Definition 1.1 The function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of (1.1) if u satisfies

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla v - \mu \frac{|u|^{p-2}uv}{|x|^p} \right) dx \\ &\quad = \int_\Omega \left(|u|^{p^*-2}uv + \beta |x|^{\alpha-p}|u|^{p-2}uv + \lambda |u|^{q-2}uv \right) dx \end{aligned} \tag{1.2}$$

for all $v \in W_0^{1,p}(\Omega)$.

In this paper, we use the following norm of $W_0^{1,p}(\Omega)$:

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx \right)^{\frac{1}{p}}.$$

By the Hardy inequality (see [1, 2])

$$\int_\Omega \frac{|u|^p}{|x|^p}\,dx \le \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^p\,dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

so this norm is equivalent to $(\int_\Omega |\nabla u|^p\,dx)^{\frac{1}{p}}$, the usual norm in $W_0^{1,p}(\Omega)$.

The norm in $L^p(\Omega)$ is represented by $\|u\|_p = (\int_\Omega |u|^p\,dx)^{\frac{1}{p}}$. According to Hardy inequality, the following best Sobolev constant is well defined for $1 < p < N$, and $0 \le \mu < \overline{\mu}$:

$$S_{\mu,0} = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^p - \mu\frac{|u|^p}{|x|^p})\,dx}{(\int_\Omega |u|^{p^*}\,dx)^{\frac{p}{p^*}}}. \tag{1.3}$$

The quasi-linear problems on Hardy inequality have been studied extensively, either in the smooth bounded domain or in the whole space $\mathbb{R}^N$. More and more excellent results have been obtained, which provide us opportunities to understand the singular problems. However, compared with the semilinear case, the quasi-linear problems related to Hardy inequality are more complicated [3–16]. Abdellaoui, Felli and Peral [3] considered the extremal function which achieves the best constant $S_{\mu,0}$, and gave the properties of the extremal functions. The conclusions obtained in [3] can be applied in the problems with critical Sobolev exponent and Hardy term.

Wang, Wei and Kang [10] investigated the following problem:

$$\begin{cases} -\Delta_p u - \lambda\frac{|u|^{p-2}}{|x|^p}u = \mu f(x)|u|^{q-2}u + g(x)|u|^{p^*-2}u, & x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $1 < q < p, \mu > 0, f$ and g are non-negative functions and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. The property of the Nehari manifold was used to prove the existence of multiple positive solutions for (1.4). Furthermore, Hsu [11, 12] improved and complemented the main results obtained in [10]. Recently, Goyal and Sreenadh [13] investigated a class of singular N-Laplacian problems with exponential nonlinearities in $\mathbb{R}^N$. Very recently, Xiang [14] established the asymptotic estimates of weak solutions for p-Laplacian equation with Hardy term and critical Sobolev exponent.

We should mention that Liu, Guo and Lei [17] studied the existence and multiplicity of positive solutions of Kirchhoff equation with critical exponential nonlinearity. Inspired by [17, 18], we study the problem (1.1) on critical Sobolev exponent. Comparing with the main results obtained in [4, 6, 10–12], in this paper, on the one hand, we will analysis the effect of $\beta|x|^{\alpha-p}|u|^{p-2}u$, and the more careful estimates are needed. On the other hand, we establish an lower bound for λ_* (λ_* is defined in Theorem 1.1).

Define the energy functional associated to problem (1.1) as follows:

$$I_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{\beta}{p}\int_\Omega |u|^p|x|^{\alpha-p}\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx. \tag{1.5}$$

We obtain the following result.

Theorem 1.1 *Suppose that* $1 < q < p$, $0 < \alpha < p-1$. *Then there exists* $\lambda_* > 0$ *such that problem* (1.1) *admits at least two solutions and one of the solutions is a ground state solution for all* $\lambda \in (0, \lambda_*)$.

2 Preliminaries

Firstly, we introduce the Nehari manifold

$$\mathcal{N}_\lambda = \left\{ u \in W_0^{1,p}(\Omega)\backslash\{0\} : \left\langle I'_\lambda(u), u \right\rangle = 0 \right\}.$$

Furthermore $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\|^p - \int_\Omega |u|^{p^*}\,dx - \beta \int_\Omega |u|^p |x|^{\alpha-p}\,dx - \lambda \int_\Omega |u|^q\,dx = 0. \tag{2.1}$$

Let

$$\psi(u) := \|u\|^p - \beta \int_\Omega |u|^p |x|^{\alpha-p}\,dx - \int_\Omega |u|^{p^*}\,dx - \lambda \int_\Omega |u|^q\,dx,$$

then

$$\left\langle \psi'(u), u \right\rangle = p\|u\|^p - p\beta \int_\Omega |u|^p |x|^{\alpha-p}\,dx - p^* \int_\Omega |u|^{p^*}\,dx - q\lambda \int_\Omega |u|^q\,dx.$$

$\mathcal{N}_\lambda$ can be divided into the following three parts:

$$\begin{aligned} \mathcal{N}_\lambda^+ = \bigg\{ & u \in \mathcal{N}_\lambda : p\|u\|^p - p\beta \int_\Omega |x|^{\alpha-p}|u|^p\,dx \\ & - p^* \int_\Omega |u|^{p^*}\,dx - q\lambda \int_\Omega |u|^q\,dx > 0 \bigg\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \mathcal{N}_\lambda^0 = \bigg\{ & u \in \mathcal{N}_\lambda : p\|u\|^p - p\beta \int_\Omega |x|^{\alpha-p}|u|^p\,dx \\ & - p^* \int_\Omega |u|^{p^*}\,dx - q\lambda \int_\Omega |u|^q\,dx = 0 \bigg\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{N}_\lambda^- = \bigg\{ & u \in \mathcal{N}_\lambda : p\|u\|^p - p\beta \int_\Omega |x|^{\alpha-p}|u|^p\,dx \\ & - p^* \int_\Omega |u|^{p^*}\,dx - q\lambda \int_\Omega |u|^q\,dx < 0 \bigg\}. \end{aligned} \tag{2.4}$$

Applying the Hölder inequality and the Sobolev inequality, for all $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ we have

$$\int_\Omega |u|^q\,dx \le \left(\int_\Omega |u|^{q\cdot\frac{p^*}{q}}\,dx \right)^{\frac{q}{p^*}} \left(\int_\Omega 1\,dx \right)^{1-\frac{q}{p^*}} = |\Omega|^{\frac{p^*-q}{p^*}} \left(\int_\Omega |u|^{p^*}\,dx \right)^{\frac{q}{p^*}}. \tag{2.5}$$

Lemma 2.1 *Assume that* $\lambda \in (0, T_1)$ *with*

$$T_1 = \frac{\left(\frac{(\beta_1-\beta)(p-p^*)}{\beta_1(q-p^*)}\right)^{\frac{q-p^*}{p-p^*}} \left(\frac{q-p}{p-p^*}\right)^{\frac{q-p}{p-p^*}} S_{\mu,0}^{\frac{q-p^*}{p-p^*}}}{|\Omega|^{\frac{p^*-q}{p^*}}}.$$

Then (i) $\mathcal{N}_\lambda^\pm \neq \emptyset$, *and* (ii) $\mathcal{N}_\lambda^0 = \emptyset$.

Proof (i) We define a function $\Phi \in C(\mathbb{R}^+, \mathbb{R})$ by

$$\Phi(s) = \left(1 - \frac{\beta}{\beta_1}\right) s^{p-p^*} \|u\|^p - \lambda s^{q-p^*} \int_\Omega |u|^q \, dx.$$

Let $\Phi'(s) = 0$, that is,

$$\Phi'(s) = \left(1 - \frac{\beta}{\beta_1}\right)\left(p - p^*\right) s^{p-p^*-1} \|u\|^p - \lambda\left(q - p^*\right) s^{q-p^*-1} \int_\Omega |u|^q \, dx = 0.$$

We can deduce that

$$s_{\max} := s = \left[\frac{(\beta_1 - \beta)(p - p^*)\|u\|^p}{\beta_1 \lambda (q - p^*) \int_\Omega |u|^q \, dx}\right]^{\frac{1}{q-p}}.$$

It is easy to check that $\Phi'(s) > 0$ for all $0 < s < s_{\max}$ and $\Phi'(s) < 0$ for all $s > s_{\max}$. Consequently, $\Phi(s)$ attains its maximum at $s_{\max}$, that is,

$$\begin{aligned}
\Phi(s_{\max}) &= \left(1 - \frac{\beta}{\beta_1}\right)\left\{\left[\frac{(\beta_1 - \beta)(p - p^*)\|u\|^p}{\beta_1 \lambda (q - p^*) \int_\Omega |u|^q \, dx}\right]^{\frac{1}{q-p}}\right\}^{p-p^*} \|u\|^p \\
&\quad - \lambda \left\{\left[\frac{(\beta_1 - \beta)(p - p^*)\|u\|^p}{\beta_1 \lambda (q - p^*) \int_\Omega |u|^q \, dx}\right]^{\frac{1}{q-p}}\right\}^{q-p^*} \int_\Omega |u|^q \, dx \\
&= \left(\frac{(\beta_1 - \beta)(p - p^*)}{\beta_1 (q - p^*)}\right)^{\frac{q-p^*}{q-p}} \left(\frac{q - p}{p - p^*}\right) \frac{\|u\|^{\frac{p(q-p^*)}{q-p}}}{(\lambda \int_\Omega |u|^q \, dx)^{\frac{p-p^*}{q-p}}}.
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{\Phi}(s) &:= s^{p-p^*} \|u\|^p - \beta s^{p-p^*} \int_\Omega |u|^p |x|^{\alpha - p} \, dx - \lambda s^{q-p^*} \int_\Omega |u|^q \, dx \\
&\geq s^{p-p^*} \left(1 - \frac{\beta}{\beta_1}\right) \|u\|^p - \lambda s^{q-p^*} \int_\Omega |u|^q \, dx.
\end{aligned}$$

By (1.3) and (2.5), we have

$$\begin{aligned}
&\widetilde{\Phi}(s_{\max}) - \int_\Omega |u|^{p^*} \, dx \\
&\quad \geq \Phi(s_{\max}) - \int_\Omega |u|^{p^*} \, dx \\
&\quad = \left(\frac{(\beta_1 - \beta)(p - p^*)}{\beta_1 (q - p^*)}\right)^{\frac{q-p^*}{q-p}} \left(\frac{q - p}{p - p^*}\right) \frac{\|u\|^{\frac{p(q-p^*)}{q-p}}}{(\lambda \int_\Omega |u|^q \, dx)^{\frac{p-p^*}{q-p}}} - \int_\Omega |u|^{p^*} \, dx \\
&\quad > \left(\frac{(\beta_1 - \beta)(p - p^*)}{\beta_1 (q - p^*)}\right)^{\frac{q-p^*}{q-p}} \left(\frac{q - p}{p - p^*}\right) \frac{\|u\|^{\frac{p(q-p^*)}{q-p}}}{[\lambda |\Omega|^{\frac{p^*-q}{p^*}} (\int_\Omega |u|^{p^*} \, dx)^{\frac{q}{p^*}}]^{\frac{p-p^*}{q-p}}} - \int_\Omega |u|^{p^*} \, dx \\
&\quad = \left\{\left(\frac{(\beta_1 - \beta)(p - p^*)}{\beta_1 (q - p^*)}\right)^{\frac{q-p^*}{q-p}} \left(\frac{q - p}{p - p^*}\right) \frac{1}{[\lambda |\Omega|^{\frac{p^*-q}{p^*}}]^{\frac{p-p^*}{q-p}}} \left(\frac{\|u\|^p}{(\int_\Omega |u|^{p^*} \, dx)^{\frac{p}{p^*}}}\right)^{\frac{q-p^*}{q-p}} - 1\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_\Omega |u|^{p^*}\,dx \\
&\geq \left\{ \left(\frac{(\beta_1-\beta)(p-p^*)}{\beta_1(q-p^*)} \right)^{\frac{q-p^*}{q-p}} \left(\frac{q-p}{p-p^*} \right) \frac{1}{[\lambda|\Omega|^{\frac{p^*-q}{p^*}}]^{\frac{p-p^*}{q-p}}} S_{\mu,0}^{\frac{q-p^*}{q-p}} - 1 \right\} \int_\Omega |u|^{p^*}\,dx \\
&> 0,
\end{aligned}
$$

where $0 < \lambda < T_1$. Thus, there exist constants s^+ and s^- such that

$$
0 < s^+ = s^+(u) < s_{\max} < s^- = s^-(u), \quad s^+ u \in \mathcal{N}_\lambda^+ \text{ and } s^- u \in \mathcal{N}_\lambda^-.
$$

(ii) We prove that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, T_1)$. By contradiction, assume that there exists $u_0 \neq 0$ such that $u_0 \in \mathcal{N}_\lambda^0$. From (2.1), we have

$$
\|u_0\|^p - \int_\Omega |u_0|^{p^*}\,dx - \beta \int_\Omega |u_0|^p |x|^{\alpha-p}\,dx - \lambda \int_\Omega |u_0|^q\,dx = 0, \tag{2.6}
$$

combining with (2.3), we obtain

$$
(p-p^*)\|u_0\|^p = (p-p^*)\beta \int_\Omega |u_0|^p |x|^{\alpha-p}\,dx + (p^*-q)\lambda \int_\Omega |u_0|^q\,dx. \tag{2.7}
$$

Equations (2.6) and (2.7) imply that

$$
(p-q)\|u_0\|^p - (p-q)\beta \int_\Omega |u_0|^p |x|^{\alpha-p}\,dx = (p^*-q) \int_\Omega |u_0|^{p^*}\,dx,
$$

that is,

$$
\int_\Omega |u_0|^{p^*}\,dx \geq \frac{p-q}{p^*-q}\left(1 - \frac{\beta}{\beta_1}\right)\|u_0\|^p. \tag{2.8}
$$

Similarly,

$$
(p-p^*)\|u_0\|^p - (p-p^*)\beta \int_\Omega |u_0|^p |x|^{\alpha-p}\,dx = \lambda(q-p^*) \int_\Omega |u_0|^q\,dx,
$$

that is,

$$
\lambda \int_\Omega |u_0|^q\,dx \geq \frac{p-p^*}{q-p^*}\left(1 - \frac{\beta}{\beta_1}\right)\|u_0\|^p. \tag{2.9}
$$

Note that (1.3) holds for $u \in \mathcal{N}_\lambda^0 \setminus \{0\}$. Then

$$
\begin{aligned}
\Theta &:= \frac{(|\Omega|^{\frac{p^*-q}{p^*}})^{\frac{p-p^*}{q-p}}}{S_{\mu,0}^{\frac{q-p^*}{q-p}}} \frac{\|u_0\|^{\frac{p(q-p^*)}{q-p}}}{(\int_\Omega (u_0^+)^q\,dx)^{\frac{p-p^*}{q-p}}} - \int_\Omega |u_0|^{p^*}\,dx \\
&> \left[\frac{1}{S_{\mu,0}^{\frac{q-p^*}{q-p}}} \left(\frac{\|u_0\|^p}{(\int_\Omega |u_0|^{p^*}\,dx)^{\frac{p}{p^*}}} \right)^{\frac{q-p^*}{q-p}} - 1 \right] \int_\Omega |u_0|^{p^*}\,dx \geq 0.
\end{aligned}
$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}
\Theta &= \frac{(|\Omega|^{\frac{p^*-q}{p^*}})^{\frac{p-p^*}{q-p}}}{S_{\mu,0}^{\frac{q-p^*}{q-p}}}\lambda^{\frac{p-p^*}{q-p}}\frac{\|u_0\|^{\frac{p(q-p^*)}{q-p}}}{(\lambda\int_\Omega(u_0^+)^q\,dx)^{\frac{p-p^*}{q-p}}}-\int_\Omega|u_0|^{p^*}\,dx \\
&\le \frac{(|\Omega|^{\frac{p^*-q}{p^*}})^{\frac{p-p^*}{q-p}}}{S_{\mu,0}^{\frac{q-p^*}{q-p}}}\lambda^{\frac{p-p^*}{q-p}}\frac{\|u_0\|^p}{[(\frac{p-p^*}{q-p^*})(1-\frac{\beta}{\beta_1})]^{\frac{p-p^*}{q-p}}}-\left(\frac{p-q}{p^*-q}\right)\left(1-\frac{\beta}{\beta_1}\right)\|u_0\|^p \\
&= \left[\frac{(|\Omega|^{\frac{p^*-q}{p^*}})^{\frac{p-p^*}{q-p}}}{S_{\mu,0}^{\frac{q-p^*}{q-p}}}\frac{\lambda^{\frac{p-p^*}{q-p}}}{[(\frac{p-p^*}{q-p^*})(1-\frac{\beta}{\beta_1})]^{\frac{p-p^*}{q-p}}}-\left(\frac{p-q}{p^*-q}\right)\left(1-\frac{\beta}{\beta_1}\right)\right]\|u_0\|^p \\
&< 0,
\end{aligned}$$

for $0<\lambda<T_1$. This is a contradiction. □

Lemma 2.2 *I_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof For $u\in\mathcal{N}_\lambda$, we can deduce from (1.3) and (2.5) that

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{p}\|u\|^p-\frac{\beta}{p}\int_\Omega|u|^p|x|^{\alpha-p}\,dx-\frac{1}{p^*}\int_\Omega|u|^{p^*}\,dx-\frac{\lambda}{q}\int_\Omega|u|^q\,dx \\
&= \left(\frac{1}{p}-\frac{1}{p^*}\right)\|u\|^p-\left(\frac{1}{p}-\frac{1}{p^*}\right)\beta\int_\Omega|u|^p|x|^{\alpha-p}\,dx-\left(\frac{1}{q}-\frac{1}{p^*}\right)\lambda\int_\Omega|u|^q\,dx \\
&\ge \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(1-\frac{\beta}{\beta_1}\right)\|u\|^p-\lambda\left(\frac{1}{q}-\frac{1}{p^*}\right)|\Omega|^{\frac{p^*-q}{p^*}}S_{\mu,0}^{-\frac{q}{p}}\|u\|^q.
\end{aligned}$$

Note that $1<q<p$ and $0<\beta<\beta_1$, we see that I_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

From Lemma 2.1, we know that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty. Furthermore, taking into account Lemma 2.2, we define

$$\kappa_\lambda=\inf_{u\in\mathcal{N}_\lambda}I_\lambda(u),\qquad \kappa_\lambda^+=\inf_{u\in\mathcal{N}_\lambda^+}I_\lambda(u),\qquad \kappa_\lambda^-=\inf_{u\in\mathcal{N}_\lambda^-}I_\lambda(u).$$

Lemma 2.3 $\kappa_\lambda\le\kappa_\lambda^+<0$.

Proof For $u\in\mathcal{N}_\lambda^+$, using (2.1) and (2.2), we have

$$(p-q)\|u\|^p-(p-q)\beta\int_\Omega|u|^p|x|^{\alpha-p}\,dx>(p^*-q)\int_\Omega|u|^{p^*}\,dx$$

and

$$(p-q)\|u\|^p\left(1-\frac{\beta}{\beta_1}\right)>(p^*-q)\int_\Omega|u|^{p^*}\,dx,$$

that is,

$$\int_\Omega|u|^{p^*}\,dx<\frac{p-q}{p^*-q}\left(1-\frac{\beta}{\beta_1}\right)\|u\|^p. \tag{2.10}$$

By (2.10), we get

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p - \left(\frac{1}{p} - \frac{1}{q}\right)\beta \int_\Omega |u|^p |x|^{\alpha-p}\,dx - \left(\frac{1}{p^*} - \frac{1}{q}\right)\int_\Omega |u|^{p^*}\,dx \\ &< \left(\frac{1}{p} - \frac{1}{q}\right)\left(1 - \frac{\beta}{\beta_1}\right)\|u\|^p - \left(\frac{1}{p^*} - \frac{1}{q}\right)\left(1 - \frac{\beta}{\beta_1}\right)\left(\frac{p-q}{p^*-q}\right)\|u\|^p \\ &= \left(1 - \frac{\beta}{\beta_1}\right)(q-p)\left(\frac{1}{qp} - \frac{1}{qp^*}\right)\|u\|^p \\ &< 0. \end{aligned}$$

Therefore, we have $\kappa_\lambda \leq \kappa_\lambda^+ < 0$. □

Lemma 2.4 *For $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a differentiable function $\widehat{f} = \widehat{f}(\omega) : B(0,\varepsilon) \subset W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}^+$ such that*

$$\widehat{f}(0) = 1, \qquad \widehat{f}(\omega)(u+\omega) \in \mathcal{N}_\lambda, \quad \forall \omega \in B(0,\varepsilon).$$

Proof Define

$$\widehat{F} : \mathbb{R} \times W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$$

as follows:

$$\begin{aligned} \widehat{F}(s,\omega) &= s^{p-q}\int_\Omega \left(\left|\nabla(u+\omega)\right|^p - \mu\frac{|u+\omega|^p}{|x|^p}\right)dx - s^{p-q}\beta\int_\Omega |u+\omega|^p|x|^{\alpha-p}\,dx \\ &\quad - s^{p^*-q}\int_\Omega |u+\omega|^{p^*}\,dx - \lambda\int_\Omega |u+\omega|^q\,dx, \quad u \in \mathcal{N}_\lambda. \end{aligned}$$

It is clear that

$$\widehat{F}(1,0) = \int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - \beta\int_\Omega |u|^p|x|^{\alpha-p}\,dx - \int_\Omega |u|^{p^*}\,dx - \lambda\int_\Omega |u|^q\,dx$$

and

$$\begin{aligned} \widehat{F}_s(s,\omega) &= (p-q)s^{p-q-1}\int_\Omega \left(\left|\nabla(u+\omega)\right|^p - \mu\frac{|u+\omega|^p}{|x|^p}\right)dx \\ &\quad - (p-q)s^{p-q-1}\beta\int_\Omega |u+\omega|^p|x|^{\alpha-p}\,dx \\ &\quad - \left(p^*-q\right)s^{p^*-q-1}\int_\Omega |u+\omega|^{p^*}\,dx, \end{aligned}$$

which implies that

$$\begin{aligned} \widehat{F}_s(1,0) &= (p-q)\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - (p-q)\beta\int_\Omega |u|^p|x|^{\alpha-p}\,dx \\ &\quad - \left(p^*-q\right)\int_\Omega |u|^{p^*}\,dx. \end{aligned}$$

Lemma 2.1 tells us that $\widehat{F}_s(1,0) \neq 0$. Thus, by the implicit function theorem at the point $(0,1)$, there exist $\varepsilon > 0$, and a differentiable function

$$\widehat{f}: B(0,\varepsilon) \subset W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}^+$$

such that

$$\widehat{f}(0) = 1, \qquad \widehat{f}(\omega) > 0 \quad \text{and} \quad \widehat{f}(\omega)(u+\omega) \in \mathcal{N}_\lambda, \quad \forall \omega \in B(0,\varepsilon). \qquad \square$$

Lemma 2.5 *For $u \in \mathcal{N}_\lambda^-$, there exist $\varepsilon > 0$ and a differentiable function $\widetilde{f} = \widetilde{f}(v) : B(0,\varepsilon) \subset W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}^+$ such that*

$$\widetilde{f}(0) = 1 \quad \textit{and} \quad \widetilde{f}(v)(u+v) \in \mathcal{N}_\lambda^-, \quad \forall v \in B(0,\varepsilon).$$

Proof The proof is similar to that of Lemma 2.4, and we omit it here. $\square$

Lemma 2.6 *If $\{u_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence of I_λ, for every $\phi \in W_0^{1,p}(\Omega)$, then*

$$-\frac{|f_n'(0)|\|u_n\| + \|\phi\|}{n} \leq \langle I_\lambda'(u_n), \phi \rangle \leq \frac{|f_n'(0)|\|u_n\| + \|\phi\|}{n}. \tag{2.11}$$

Proof It follows from Lemma 2.2 that I_λ is coercive on $\mathcal{N}_\lambda$. Using the Ekeland variational principle [19], we can find a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ satisfying

$$I_\lambda(u_n) < \kappa_\lambda + \frac{1}{n}, \qquad I_\lambda(u_n) \leq I_\lambda(w) + \frac{1}{n}\|w - u_n\| \quad \forall w \in \mathcal{N}_\lambda. \tag{2.12}$$

Without loss of generality, we can assume that $u_n \geq 0$. By Lemma 2.2, we know that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. As a consequence, there exist a subsequence (still denoted by $\{u_n\}$) and u_* in $W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_* & \text{strongly in } L^p(\Omega)\ (1 \leq p < p^*), \\ u_n(x) \to u_*(x) & \text{a.e. in } \Omega. \end{cases} \tag{2.13}$$

From Lemma 2.4, for $s > 0$ sufficiently small and $\phi \in W_0^{1,p}(\Omega)$, and set $u = u_n$, $\omega = s\phi \in W_0^{1,p}(\Omega)$, we can find that $f_n(s) = f_n(s\phi)$ such that $f_n(0) = 1$ and $f_n(s)(u_n + s\phi) \in \mathcal{N}_\lambda$. Since

$$\|u_n\|^p - \int_\Omega |u_n|^{p^*}\,dx - \beta \int_\Omega |u_n|^p |x|^{\alpha - p}\,dx - \lambda \int_\Omega |u_n|^q\,dx = 0. \tag{2.14}$$

By (2.12), we obtain

$$\begin{aligned} \frac{1}{n}\big[\big|f_n(s) - 1\big|\|u_n\| + sf_n(s)\|\phi\|\big] &\geq \frac{1}{n}\big\|f_n(s)(u_n + s\phi) - u_n\big\| \\ &\geq I_\lambda(u_n) - I_\lambda\big[f_n(s)(u_n + s\phi)\big]. \end{aligned} \tag{2.15}$$

Notice that

$$
\begin{aligned}
I_\lambda\big[f_n(s)(u_n+s\phi)\big] &= \frac{1}{p}\big\|f_n(s)(u_n+s\phi)\big\|^p - \frac{\beta}{p}\int_\Omega |x|^{\alpha-p}\big|f_n(s)(u_n+s\phi)\big|^p\,dx \\
&\quad - \frac{1}{p^*}\int_\Omega \big|f_n(s)(u_n+s\phi)\big|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega \big|f_n(s)(u_n+s\phi)\big|^q\,dx \\
&= \frac{f_n^p(s)}{p}\|u_n+s\phi\|^p - \frac{\beta}{p}f_n^p(s)\int_\Omega |x|^{\alpha-p}\big|(u_n+s\phi)\big|^p\,dx \\
&\quad - \frac{f_n^{p^*}(s)}{p^*}\int_\Omega \big|(u_n+s\phi)\big|^{p^*}\,dx - \frac{\lambda}{q}f_n^q(s)\int_\Omega \big|(u_n+s\phi)\big|^q\,dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&I_\lambda(u_n) - I_\lambda\big[f_n(s)(u_n+s\phi)\big] \\
&\quad = \frac{1}{p}\|u_n\|^p - \frac{f_n^p(s)}{p}\|u_n\|^p + \frac{f_n^{p^*}(s)}{p^*}\int_\Omega |u_n+s\phi|^{p^*}\,dx - \frac{1}{p^*}\int_\Omega |u_n+s\phi|^{p^*}\,dx \\
&\qquad + \frac{\lambda}{q}f_n^q(s)\int_\Omega |u_n+s\phi|^q\,dx - \frac{\lambda}{q}\int_\Omega |u_n+s\phi|^q\,dx + \frac{\beta}{p}f_n^p(s)\int_\Omega |x|^{\alpha-p}|u_n+s\phi|^p\,dx \\
&\qquad - \frac{\beta}{p}\int_\Omega |x|^{\alpha-p}|u_n+s\phi|^p\,dx + \frac{f_n^p(s)}{p}\|u_n\|^p - \frac{f_n^p(s)}{p}\|u_n+s\phi\|^p + \frac{1}{p^*}\int_\Omega |u_n+s\phi|^{p^*}\,dx \\
&\qquad - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\,dx + \frac{\lambda}{q}\int_\Omega |u_n+s\phi|^q\,dx - \frac{\lambda}{q}\int_\Omega |u_n|^q\,dx \\
&\qquad + \frac{\beta}{p}\int_\Omega |x|^{\alpha-p}|u_n+s\phi|^p\,dx - \frac{\beta}{p}\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx \\
&\quad = \frac{1-f_n^p(s)}{p}\|u_n\|^p + \frac{f_n^{p^*}(s)-1}{p^*}\int_\Omega |u_n+s\phi|^{p^*}\,dx + \frac{\lambda}{q}\big(f_n^q(s)-1\big)\int_\Omega |u_n+s\phi|^q\,dx \\
&\qquad + \frac{\beta}{p}\big(f_n^p(s)-1\big)\int_\Omega |x|^{\alpha-p}|u_n+s\phi|^p\,dx + \frac{f_n^p(s)}{p}\big(\|u_n\|^p - \|u_n+s\phi\|^p\big) \\
&\qquad + \frac{1}{p^*}\left(\int_\Omega |u_n+s\phi|^{p^*}\,dx - \int_\Omega |u_n|^{p^*}\,dx\right) + \frac{\lambda}{q}\int_\Omega \big(|u_n+s\phi|^q - |u_n|^q\big)\,dx \\
&\qquad + \frac{\beta}{p}\int_\Omega \big[|u_n+s\phi|^p - |u_n|^p\big]|x|^{\alpha-p}\,dx.
\end{aligned}
$$

Dividing by $s>0$ and taking the limit for $s\to 0$, combining with (2.14) and (2.15), we have

$$
\begin{aligned}
&\frac{|f_n'(0)|\|u_n\| + \|\phi\|}{n} \\
&\quad \ge -f_n'(0)\|u_n\|^p + f_n'(0)\int_\Omega |u_n|^{p^*}\,dx + \lambda f_n'(0)\int_\Omega |u_n|^q\,dx \\
&\qquad + \beta f_n'(0)\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,dx \\
&\qquad + \mu\int_\Omega \frac{|u_n|^{p-2}u_n\phi}{|x|^p}\,dx + \int_\Omega |u_n|^{p^*-1}\phi\,dx \\
&\qquad + \lambda\int_\Omega |u_n|^{q-1}\phi\,dx + \beta\int_\Omega |u_n|^{p-1}\phi|x|^{\alpha-p}\,dx
\end{aligned}
$$

$$= -f_n'(0)\left[\|u_n\|^p - \int_\Omega |u_n|^{p^*}\,dx - \lambda\int_\Omega |u_n|^q\,dx - \beta\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx\right] - \langle I_\lambda', \phi\rangle$$
$$= -\langle I_\lambda', \phi\rangle.$$

Consequently

$$-\frac{|f_n'(0)|\|u_n\| + \|\phi\|}{n} \le \langle I_\lambda', \phi\rangle \tag{2.16}$$

for every $\phi \in W_0^{1,p}(\Omega)$. Note that (2.16) holds equally for $-\phi$, we see that (2.11) holds. □

Lemma 2.7 (see [8, 10]) *Set $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}$. Assume that $1 < p < N$ and $0 \le \mu < \overline{\mu}$. Then the limiting problem*

$$\begin{cases} -\Delta_p u - \mu\frac{u^{p-1}}{|x|^p} = u^{p^*-1} \quad \text{in } \mathbb{R}^N\backslash\{0\}, \\ u > 0 \quad \text{in } \mathbb{R}^N\backslash\{0\}, \\ u \in D^{1,p}(\mathbb{R}^N) \end{cases} \tag{2.17}$$

has radially symmetric ground states

$$V_\epsilon(x) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{x}{\epsilon}\right) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{|x|}{\epsilon}\right) \quad \forall\epsilon > 0,$$

such that

$$\int_{\mathbb{R}^N}\left(\left|\nabla V_\epsilon(x)\right|^p - \mu\frac{|V_\epsilon(x)|^p}{|x|^p}\right)dx = \int_{\mathbb{R}^N}\left|V_\epsilon(x)\right|^{p^*}dx = S_{\mu,0}^{\frac{N}{p}},$$

where the function $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the above limiting problem with

$$U_{p,\mu}(1) = \left(\frac{N(\overline{\mu} - \mu)}{N - p}\right)^{\frac{1}{p^*-p}}.$$

In the following, we define $\Lambda = \frac{1}{N}S_{\mu,0}^{\frac{N}{p}}$.

Lemma 2.8 *Let $\{u_n\} \subset \mathcal{N}_\lambda^-$ be a minimizing sequence for I_λ with $\kappa_\lambda^- < \Lambda - D\lambda^{\frac{p}{p-q}}$, where*

$$D = \frac{p-q}{p}\left[\frac{p^*-q}{p^*q}|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\left(\frac{\beta_1 - \beta}{N\beta_1}\right)^{-\frac{q}{p}}\right]^{\frac{p}{p-q}}. \tag{2.18}$$

Then there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^{p^*}(\Omega)$.

Proof Since

$$I_\lambda(u_n) \to \kappa_\lambda^- \quad \text{as } n \to +\infty. \tag{2.19}$$

By Lemma 2.2, we know that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. In fact, we can deduce from (1.3) and (2.19) that

$$\begin{aligned}
&1+\kappa_\lambda^- + o\big(\|u_n\|\big) \\
&\quad\geq I_\lambda(u_n) - \frac{1}{p^*}\big\langle I'_\lambda(u_n), u_n\big\rangle \\
&\quad= \frac{1}{p}\|u_n\|^p - \frac{\beta}{p}\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u_n|^q\,dx \\
&\qquad - \frac{1}{p^*}\left(\|u_n\|^p - \int_\Omega |u_n|^{p^*}\,dx - \lambda\int_\Omega |u_n|^q\,dx - \beta\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx\right) \\
&\quad= \left(\frac{1}{p}-\frac{1}{p^*}\right)\|u_n\|^p - \left(\frac{1}{p}-\frac{1}{p^*}\right)\beta\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx \\
&\qquad + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda\int_\Omega |u_n|^q\,dx \\
&\quad\geq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(1-\frac{\beta}{\beta_1}\right)\|u_n\|^p + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda\int_\Omega |u_n|^q\,dx \\
&\quad\geq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(1-\frac{\beta}{\beta_1}\right)\|u_n\|^p \\
&\qquad + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}}\left(\int_\Omega |u_n|^{p^*}\,dx\right)^{\frac{q}{p^*}} \\
&\quad\geq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(1-\frac{\beta}{\beta_1}\right)\|u_n\|^p + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u_n\|^q,
\end{aligned}$$

where $0<\beta<\beta_1$, $1<q<p$, we see that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. We can choose a subsequence (still denoted by $\{u_n\}$) and $u\in W_0^{1,p}(\Omega)$ satisfying

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u \quad \text{strongly in } L^p(\Omega)\ (1\leq p<p^*). \\ u_n(x)\to u(x) \quad \text{a.e. in } \Omega. \end{cases} \tag{2.20}$$

In term of the concentration compactness principle, going if necessary to a subsequence, there exist an at most countable set $\mathcal{J}$, a set of points $\{x_j\}_{j\in\mathcal{J}}\subset\Omega\setminus\{0\}$, and real numbers μ_j, ν_j, $\widetilde{\chi}_0$ such that

$$\begin{aligned}
&|\nabla u_n|^p \rightharpoonup d\mu \geq |\nabla u|^p + \sum_{j\in\mathcal{J}}\mu_j\delta_{x_j} + \mu_0\delta_0, \\
&|u_n|^{p^*} \rightharpoonup d\nu = |u|^{p^*} + \sum_{j\in\mathcal{J}}\nu_j\delta_{x_j} + \nu_0\delta_0, \\
&\frac{|u_n|^p}{|x|^p} \rightharpoonup d\widetilde{\chi} = \frac{|u|^p}{|x|^p} + \widetilde{\chi}_0\delta_0,
\end{aligned}$$

where δ_{x_j} is the Dirac mass at x_j.

Let ϵ be sufficient small satisfying $0\notin B(x_j,\epsilon)$ and $B(x_j,\epsilon)\cap B(x_i,\epsilon)=\emptyset$ for $i\neq j, i,j = 1,2,\ldots,k$. Let $\psi_{\epsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0\leq\psi_{\epsilon,j}(x)\leq 1$,

$\psi_{\epsilon,j}(x) = 1$ for $x \in B(x_j, \frac{\epsilon}{2})$, $\psi_{\epsilon,j}(x) = 0$ for $x \in \Omega \backslash B(x_j, \epsilon)$ and $|\nabla \psi_{\epsilon,j}(x)| \leq \frac{4}{\epsilon}$. Note that

$$\begin{aligned}
&\langle I'_\lambda(u_n), u_n \psi_{\epsilon,j}(x)\rangle \\
&\quad = \int_\Omega |\nabla u_n|^p \psi_{\epsilon,j}(x)\,dx + \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_{\epsilon,j}(x)\,dx - \mu \int_\Omega \frac{|u_n|^p}{|x|^p} \psi_{\epsilon,j}(x)\,dx \\
&\qquad - \int_\Omega |u_n|^{p^*} \psi_{\epsilon,j}(x)\,dx - \lambda \int_\Omega |u_n|^q \psi_{\epsilon,j}(x)\,dx - \beta \int_\Omega |u_n|^p |x|^{\alpha-p} \psi_{\epsilon,j}(x)\,dx.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \psi_{\epsilon,j}(x)\,dx = \int_\Omega \psi_{\epsilon,j}(x)\,d\mu \geq \int_\Omega |\nabla u|^p \psi_{\epsilon,j}(x)\,dx + \mu_j, \\
&\lim_{n\to\infty} \int_\Omega |u_n|^{p^*} \psi_{\epsilon,j}(x)\,dx = \int_\Omega \psi_{\epsilon,j}(x)\,d\nu = \int_\Omega |u|^{p^*} \psi_{\epsilon,j}(x)\,dx + \nu_j, \\
&\lim_{\epsilon\to 0} \lim_{n\to\infty} \left| \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \psi_{\epsilon,j}(x) \right| = 0, \\
&\lim_{\epsilon\to 0} \lim_{n\to\infty} \left| \int_\Omega \frac{|u_n|^p}{|x|^p} \psi_{\epsilon,j}(x) \right| = 0.
\end{aligned}$$

By (1.3), we deduce that

$$\begin{aligned}
\left| \int_\Omega |u_n|^q \psi_{\epsilon,j}\,dx \right| &\leq \int_{B(x_j,\epsilon)} |u_n|^q\,dx \\
&\leq \left(\int_{B(x_j,\epsilon)} |u_n|^{q\frac{p^*}{q}}\,dx \right)^{\frac{q}{p^*}} \left(\int_{B(x_j,\epsilon)} dx \right)^{\frac{p^*-q}{p^*}} \\
&\leq S_{\mu,0}^{-\frac{q}{p}} \|u_n\|^q \left(\int_{B(x_j,\epsilon)} dx \right)^{\frac{p^*-q}{p^*}} \\
&\leq S_{\mu,0}^{-\frac{q}{p}} \left(\int_0^\epsilon r^{N-1}\,dr \right)^{\frac{p^*-q}{p^*}} \|u_n\|^q \\
&= \left(\frac{1}{N} \right)^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}} \epsilon^{\frac{N(p^*-q)}{p^*}} \|u_n\|^q
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_\Omega |u_n|^p |x|^{\alpha-p} \psi_{\epsilon,j}(x)\,dx \right| &\leq \left(\int_{B(x_j,\epsilon)} |u_n|^{p\frac{p^*}{p}}\,dx \right)^{\frac{p}{p^*}} \left(\int_{B(x_j,\epsilon)} |x|^{\frac{p^*(\alpha-p)}{p^*-p}}\,dx \right)^{\frac{p^*-p}{p^*}} \\
&\leq \left(\int_{B(x_j,\epsilon)} |u_n|^{p\frac{p^*}{p}}\,dx \right)^{\frac{p}{p^*}} \left(\int_{B(x_j,\epsilon)} |x-x_j|^{\frac{p^*(\alpha-p)}{p^*-p}}\,dx \right)^{\frac{p^*-p}{p^*}} \\
&\leq S_{\mu,0}^{-1} \|u_n\|^p \left(\int_0^\epsilon r^{N-1} r^{\frac{p^*(\alpha-p)}{p^*-p}}\,dr \right)^{\frac{p^*-p}{p^*}} \\
&= S_{\mu,0}^{-1} \|u_n\|^p \left(\frac{p}{N\alpha} \epsilon^{\frac{N\alpha}{p}} \right)^{\frac{p^*-p}{p^*}}.
\end{aligned}$$

Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, and $u_n \rightharpoonup u$ weakly in $L^{p^*}(\Omega)$, we conclude that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^q \psi_{\epsilon,j}(x)\,dx = 0$$

and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^p|x|^{\alpha-p} \psi_{\epsilon,j}(x)\,dx = 0.$$

By (2.11), we have

$$0 = \lim_{\epsilon\to 0}\lim_{n\to\infty}\langle I'_\lambda(u_n), u_n\psi_{\epsilon,j}(x)\rangle \geq \mu_j - \nu_j.$$

Since $S_{0,0}\nu_j^{\frac{p}{p^*}} \leq \mu_j$, we have $\mu_j = \nu_j = 0$ or $\mu_j \geq (S_{0,0})^{\frac{N}{p}}$.

On the other hand, let $\epsilon > 0$ be sufficiently small satisfying $x_j \notin B(0,\epsilon), \forall j \in \mathcal{J}$. Let $\psi_{\epsilon,0}(x)$ a smooth cut-off function centered at the origin such that $0 \leq \psi_{\epsilon,0}(x) \leq 1$, $\psi_{\epsilon,0}(x) = 1$ for $|x| \leq \frac{\epsilon}{2}$, $\psi_{\epsilon,0}(x) = 0$ for $|x| \geq \epsilon$ and $|\nabla\psi_{\epsilon,0}(x)| \leq \frac{4}{\epsilon}$. Hence, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\psi_{\epsilon,0}(x)\,dx = \int_\Omega \psi_{\epsilon,0}(x)\,d\mu \geq \int_\Omega |\nabla u|^p\psi_{\epsilon,0}(x)\,dx + \mu_0,\\
&\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\psi_{\epsilon,0}(x)\,dx = \int_\Omega \psi_{\epsilon,0}(x)\,d\nu = \int_\Omega |u|^{p^*}\psi_{\epsilon,0}(x)\,dx + \nu_0,\\
&\lim_{n\to\infty}\int_\Omega \frac{|u_n|^p}{|x|^p}\psi_{\epsilon,0}(x)\,dx = \int_\Omega \psi_{\epsilon,0}(x)\,d\widetilde{\chi} = \int_\Omega \frac{|u|^p}{|x|^p}\psi_{\epsilon,0}(x)\,dx + \widetilde{\chi}_0,\\
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_{\epsilon,0}(x)\,dx\right| = 0,\\
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^q\psi_{\epsilon,0}(x)\,dx = 0
\end{aligned}$$

and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^p|x|^{\alpha-p} \psi_{\epsilon,0}(x)\,dx = 0.$$

Therefore

$$0 = \lim_{\epsilon\to 0}\lim_{n\to\infty}\langle I'_\lambda(u_n), u_n\psi_{\epsilon,0}(x)\rangle \geq \mu_0 - \mu\widetilde{\chi}_0 - \nu_0.$$

Combining the definition of $S_{\mu,0}$, we get that $S_{\mu,0}\nu_0^{\frac{p}{p^*}} \leq \mu_0 - \mu\widetilde{\chi}_0 \leq \nu_0$, which implies that $\nu_0 = 0$ or $\nu_0 \geq (S_{\mu,0})^{\frac{N}{p}}$. Now, we prove that $\mu_j \geq (S_{0,0})^{\frac{N}{p}}$ and $\nu_0 \geq (S_{\mu,0})^{\frac{N}{p}}$ are not true. If not, we have

$$\begin{aligned}
\kappa_\lambda^- &= \lim_{n\to\infty}\left[I_\lambda(u_n) - \frac{1}{p^*}\langle I'_\lambda(u_n), u_n\rangle\right]\\
&\geq \lim_{n\to\infty}\left[\left(\frac{1}{p} - \frac{1}{p^*}\right)\|u_n\|^p + \left(\frac{1}{p^*} - \frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u_n\|^q\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[\frac{1}{N}\|u_n\|^p + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u_n\|^q\right] \\
&\ge \frac{1}{N}\left(\|u\|^p + \sum_{j\in\mathcal{J}}\mu_j + \mu_0 - \mu\widetilde{\chi}_0\right) + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u\|^q \\
&\ge \frac{1}{N}S_{\mu,0}^{\frac{N}{p}} + \frac{1}{N}\|u\|^p + \left(\frac{1}{p^*}-\frac{1}{q}\right)\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u\|^q \\
&= \frac{1}{N}S_{\mu,0}^{\frac{N}{p}} + \frac{1}{N}\|u\|^p - \frac{p^*-q}{p^*q}\lambda|\Omega|^{\frac{p^*-q}{p^*}} S_{\mu,0}^{-\frac{q}{p}}\|u\|^q \\
&\ge \frac{1}{N}S_{\mu,0}^{\frac{N}{p}} - D\lambda^{\frac{p}{p-q}},
\end{aligned}
$$

where D is defined in (2.18). Hence, we conclude that $\Lambda - D\lambda^{\frac{p}{p-q}} \le \kappa_\lambda^- < \Lambda - D\lambda^{\frac{p}{p-q}}$, which is a contradiction. It follows that $\nu_j = 0$ for $j \in \{0\}\cup\mathcal{J}$, which means that $\int_\Omega |u_n|^{p^*}\,dx \to \int_\Omega |u|^{p^*}\,dx$ as $n\to\infty$. The proof is completed. □

In the following, we need some estimates for the extremal function V_ϵ defined in Lemma 2.7. Given $R > 0$, let $\varphi(x) \in W_0^{1,p}(\Omega)$, $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $|x| \le R$, $\varphi(x) = 0$ for $|x| \ge 2R$. Set $v_\epsilon(x) = \varphi(x)V_\epsilon(x)$. For $1 < p < N$ and $1 < q < p^*$, we have the following estimates (see [4, 6]):

$$
\|v_\epsilon\|^p = (S_{\mu,0})^{\frac{N}{p}} + O\bigl(\epsilon^{b(\mu)p+p-N}\bigr), \tag{2.21}
$$

$$
\int_\Omega |v_\epsilon|^{p^*}\,dx = (S_{\mu,0})^{\frac{N}{p}} + O\bigl(\epsilon^{b(\mu)p^*-N}\bigr), \tag{2.22}
$$

then

$$
\int_\Omega |v_\epsilon|^q\,dx = \begin{cases} C\epsilon^{N+q(1-\frac{N}{p})} & \frac{N}{b(\mu)} < q < p, \\ C\epsilon^{N+q(1-\frac{N}{p})}|\ln\epsilon| & q = \frac{N}{b(\mu)}, \\ C\epsilon^{q(b(\mu)+1-\frac{N}{p})} & 1 < q < \frac{N}{b(\mu)}, \end{cases} \tag{2.23}
$$

where $b(\mu)$ is the zero of the function

$$
f(\xi) = (p-1)\xi^p - (N-p)\xi^{p-1} + \mu, \quad \xi \ge 0, 0 \le \mu < \overline{\mu},
$$

satisfying $0 < \frac{N-p}{p} < b(\mu) < \frac{N-p}{p-1}$.

Lemma 2.9 *There exists* $\lambda_0 > 0$ *such that*

$$
\sup_{s\ge 0} I_\lambda(sv_\epsilon) < \Lambda - D\lambda^{\frac{p}{p-q}}, \quad \textit{for } \lambda \in (0,\lambda_0),
$$

where Λ *and* D *are defined in Lemma* 2.8.

Proof For two positive constants s_0 and s_1 (independent of ϵ, λ), we show that there exists $s_\epsilon > 0$ with $0 < s_0 \le s_\epsilon \le s_1 < \infty$ such that $\sup_{s\ge0} I_\lambda(sv_\epsilon) = I_\lambda(s_\epsilon v_\epsilon)$. In fact, since $\lim_{s\to+\infty} I_\lambda(sv_\epsilon) = -\infty$, we can deduce that

$$
s_\epsilon^{p-1}\|v_\epsilon\|^p - \beta s_\epsilon^{p-1}\int_\Omega |v_\epsilon|^p|x|^{\alpha-p}\,dx - s_\epsilon^{p^*-1}\int_\Omega |v_\epsilon|^{p^*}\,dx - \lambda s_\epsilon^{q-1}\int_\Omega |v_\epsilon|^q\,dx = 0 \tag{2.24}
$$

and

$$\begin{aligned}&(p-1)s_\epsilon^{p-2}\|v_\epsilon\|^p-(p-1)\beta s_\epsilon^{p-2}\int_\Omega|v_\epsilon|^p|x|^{\alpha-p}\,dx\\&\quad-\bigl(p^*-1\bigr)s_\epsilon^{p^*-2}\int_\Omega|v_\epsilon|^{p^*}\,dx-(q-1)\lambda s_\epsilon^{q-2}\int_\Omega|v_\epsilon|^q\,dx<0.\end{aligned}\tag{2.25}$$

Equations (2.24) and (2.25) imply that

$$\begin{aligned}&(p-1)s_\epsilon^{p-2}\|v_\epsilon\|^p-(p-1)\beta s_\epsilon^{p-2}\int_\Omega|v_\epsilon|^p|x|^{\alpha-p}\,dx-\bigl(p^*-1\bigr)s_\epsilon^{p^*-2}\int_\Omega|u_\epsilon|^{p^*}\,dx\\&\quad<(q-1)s_\epsilon^{p-2}\|v_\epsilon\|^p-(q-1)\beta s_\epsilon^{p-2}\int_\Omega|v_\epsilon|^p|x|^{\alpha-p}\,dx-(q-1)s_\epsilon^{p^*-2}\int_\Omega|v_\epsilon|^{p^*}\,dx.\end{aligned}$$

That is,

$$(p-q)s_\epsilon^{p-2}\|v_\epsilon\|^p-(p-q)\beta s_\epsilon^{p-2}\int_\Omega|v_\epsilon|^p|x|^{\alpha-p}\,dx<\bigl(p^*-q\bigr)s_\epsilon^{p^*-2}\int_\Omega|v_\epsilon|^{p^*}\,dx.\tag{2.26}$$

Hence, we can obtain from (2.26) that s_ϵ is bounded below. Moreover, it is clear to see from (2.24) that s_ϵ is bounded above for all $\epsilon>0$ small enough. Therefore, our claim holds.

Set

$$h(s_\epsilon v_\epsilon)=\frac{s_\epsilon^p}{p}\|v_\epsilon\|^p-\frac{s_\epsilon^{p^*}}{p^*}\int_\Omega|v_\epsilon|^{p^*}\,dx.$$

In the following, we prove that

$$h(s_\epsilon v_\epsilon)\le\Lambda+O\bigl(\epsilon^{p(b(\mu)-\frac{N}{p}+1)}\bigr).\tag{2.27}$$

Let

$$\widetilde{h}(s)=\frac{s^p}{p}\|v_\epsilon\|^p-\frac{s^{p^*}}{p^*}\int_\Omega|v_\epsilon|^{p^*}\,dx.$$

Direct computations give us that $\lim_{s\to\infty}\widetilde{h}(s)=-\infty$ and $\widetilde{h}(0)=0$. Thus $\sup_{s\ge0}\widetilde{h}(s)$ is obtained at some $S_\epsilon>0$, and

$$S_\epsilon=\left(\frac{\|v_\epsilon\|^p}{\int_\Omega|v_\epsilon|^{p^*}\,dx}\right)^{\frac{1}{p^*-p}}.$$

Since $\widetilde{h}'(s)|_{S_\epsilon}=0$, that is,

$$S_\epsilon^{p-1}\|v_\epsilon\|^p-S_\epsilon^{p^*-1}\int_\Omega|v_\epsilon|^{p^*}\,dx=0.$$

It is easy to check that $h(s)$ is increasing in $[0,S_\epsilon)$, according to (2.21) and (2.22), we have

$$\begin{aligned}h(s_\epsilon v_\epsilon)&\le\widetilde{h}(S_\epsilon)\\&=\left(\frac{1}{p}-\frac{1}{p^*}\right)\frac{(\|v_\epsilon\|^p)^{\frac{p^*}{p^*-p}}}{(\int_\Omega|u_\epsilon|^{p^*}\,dx)^{\frac{p}{p^*-p}}}\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{((S_{\mu,0})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p+p-N}))^{\frac{p^*}{p^*-p}}}{((S_{\mu,0})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p^*-N}))^{\frac{p}{p^*-p}}} \\
&\le \left(\frac{1}{p} - \frac{1}{p^*}\right) \frac{(S_{\mu,0})^{\frac{N}{p}\frac{p^*}{p^*-p}}}{(S_{\mu,0})^{\frac{N}{p}\frac{p}{p^*-p}}} + O\bigl(\epsilon^{b(\mu)p+p-N}\bigr) \\
&= \left(\frac{1}{p} - \frac{1}{p^*}\right)(S_{\mu,0})^{\frac{N}{p}} + O\bigl(\epsilon^{p(b(\mu)-\frac{N}{p}+1)}\bigr) \\
&= \Lambda + O\bigl(\epsilon^{p(b(\mu)-\frac{N}{p}+1)}\bigr).
\end{aligned} \tag{2.28}$$

Therefore, by (2.27), we have

$$\begin{aligned}
I_\lambda(s_\epsilon v_\epsilon) &= h(s_\epsilon v_\epsilon) - \frac{\beta s_\epsilon^p}{p}\int_\Omega |v_\epsilon|^p |x|^{\alpha-p}\,dx - \frac{\lambda s_\epsilon^q}{q}\int_\Omega |v_\epsilon|^q\,dx \\
&\le \Lambda + C\epsilon^{p(b(\mu)-\frac{N}{p}+1)} - \frac{\beta}{p}s_0^p \int_\Omega |v_\epsilon|^p|x|^{\alpha-p}\,dx - \frac{\lambda s_0^q}{q}\int_\Omega |v_\epsilon|^q\,dx.
\end{aligned} \tag{2.29}$$

Now, we consider the following cases:

(i) $\frac{N}{b(\mu)} < q < p$. Choose $\epsilon = \lambda^{\frac{1}{(p-q)(b(\mu)-\frac{N}{p}+1)}}$, for $\lambda < \lambda_1 := (\frac{C_1+D}{C_2})^{\frac{(p-q)(b(\mu)-\frac{N}{p}+1)}{N-qb(\mu)}}$, we have

$$\begin{aligned}
C_1\epsilon^{p(b(\mu)-\frac{N}{p}+1)} - \lambda C_2 \epsilon^{N+q(1-\frac{N}{p})} &= C_1\lambda^{\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{N+q(1-\frac{N}{p})}{(p-q)(b(\mu)-\frac{N}{p}+1)}} \\
&= C_1\lambda^{\frac{p}{p-q}} - C_2 \lambda^{\frac{N+q(1-\frac{N}{p})}{(p-q)(b(\mu)-\frac{N}{p}+1)}+1} \\
&= \lambda^{\frac{p}{p-q}}\bigl(C_1 - C_2\lambda^{\frac{N-qb(\mu)}{(p-q)(b(\mu)-\frac{N}{p}+1)}}\bigr) \\
&< -D\lambda^{\frac{p}{p-q}}.
\end{aligned}$$

(ii) $q = \frac{N}{b(\mu)}$. We still choose $\epsilon = \lambda^{\frac{1}{(p-q)(b(\mu)-\frac{N}{p}+1)}}$, for $\lambda < \lambda_2 := e^{-(\frac{C_1+D}{C_3})}$, we have

$$\begin{aligned}
C_1\epsilon^{p(b(\mu)-\frac{N}{p}+1)} - \lambda C_2 \epsilon^{N+q(1-\frac{N}{p})}|\ln\epsilon| &= C_1\lambda^{\frac{p}{p-q}} - \lambda C_3 \lambda^{\frac{N+q(1-\frac{N}{p})}{(p-q)(b(\mu)-\frac{N}{p}+1)}}|\ln\lambda| \\
&= C_1\lambda^{\frac{p}{p-q}} - C_3 \lambda^{\frac{N+q(1-\frac{N}{p})}{(p-q)(b(\mu)-\frac{N}{p}+1)}+1}|\ln\lambda| \\
&< \lambda^{\frac{p}{p-q}}\bigl(C_1 - C_3|\ln\lambda|\bigr) \\
&< -D\lambda^{\frac{p}{p-q}},
\end{aligned}$$

where $C_3 = \frac{C_2}{(p-q)(b(\mu)-\frac{N}{p}+1)}$.

(iii) $1 < q < \frac{N}{b(\mu)}$. Put $\epsilon^{p(b(\mu)-\frac{N}{p}+1)} \le \lambda^{\frac{p}{p-q}}$, for $\lambda < \lambda_3 := (\frac{C_2-D}{C_1})^{\frac{p-q}{pq-p}}$ with $C_2 > D$, we have

$$\begin{aligned}
C_1\epsilon^{p(b(\mu)-\frac{N}{p}+1)} - \lambda C_2 \epsilon^{q(b(\mu)+1-\frac{N}{p})} &:= C_1\lambda^{\frac{pq}{p-q}} - \lambda C_2 \lambda^{\frac{q}{p-q}} \\
&= \lambda^{\frac{p}{p-q}}\bigl(C_1\lambda^{\frac{pq-p}{p-q}} - C_2\bigr) \\
&< -D\lambda^{\frac{p}{p-q}}.
\end{aligned}$$

Consequently, for $\lambda < \lambda_0 := \min\{\lambda_1, \lambda_2, \lambda_3\}$, we deduce that

$$I_\lambda(s_\epsilon v_\epsilon) < \Lambda - D\lambda^{\frac{p}{p-q}}. \qquad \square$$

3 Proof of main result

We can find a constant $\delta > 0$ such that $\Lambda - D\lambda^{\frac{p}{p-q}} > 0$ for $\lambda < \delta$. Let $\lambda_* = \min\{T_1, \delta, \lambda_0\}$. For $\lambda \in (0, \lambda_*)$, Lemmas 2.1-2.4, 2.6 and 2.8 hold.

Let $\{u_n\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of I_λ. It is easy to see that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_\lambda & \text{strongly in } L^s(\Omega)\ (1 \le s < p^*), \\ u_n(x) \to u_\lambda(x) & \text{a.e. in } \Omega, \end{cases} \tag{3.1}$$

as $n \to \infty$.

Firstly, by Lemma 2.4, we can know that $f_n'(0)$ is bounded with respect to $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.11), we deduce that

$$\begin{aligned} &\int_\Omega |\nabla u_*|^{p-2}\nabla u_* \cdot \nabla\phi - \mu\int_\Omega \frac{|u_*|^{p-2}u_*}{|x|^p}\phi \\ &\quad = \int_\Omega |u_*|^{p^*-1}\phi + \lambda\int_\Omega |u_*|^{q-1}\phi + \beta\int_\Omega |u_*|^{p-1}|x|^{\alpha-p}\phi \end{aligned} \tag{3.2}$$

for all $\phi \in W_0^{1,p}(\Omega)$. Equation (3.2) implies that u_λ is a solution of (1.1). We claim that $u_\lambda \not\equiv 0$. If not, $u_\lambda = 0$, since $u_n \in \mathcal{N}_\lambda$, we have

$$\|u_n\|^p - \int_\Omega |u_n|^{p^*} - \beta\int_\Omega |u_n|^p|x|^{\alpha-p} - \lambda\int_\Omega |u_n|^q = 0.$$

Note that

$$\lim_{n\to\infty}\int_\Omega |u_n|^p|x|^{\alpha-p}\,dx = 0, \qquad \lim_{n\to\infty}\int_\Omega |u_n|^q\,dx = 0.$$

Put $\lim_{n\to\infty}\|u_n\| = m$, we conclude that $m \ge S_{\mu,0}^{\frac{p^*}{p(p^*-p)}}$. By Lemma 2.8, we obtain

$$\begin{aligned} \kappa_\lambda &= \lim_{n\to\infty} I_\lambda(u_n) \\ &= \lim_{n\to\infty}\left[\frac{1}{p}\|u_n\|^p - \frac{\beta}{p}\int_\Omega |u_n|^p|x|^{\alpha-p} - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u_n|^q\,dx\right] \\ &\ge \lim_{n\to\infty}\left(\frac{1}{p} - \frac{1}{p^*}\right)\|u_n\|^p \\ &\ge \frac{p^*-p}{pp^*}S_{\mu,0}^{\frac{p^*}{p^*-p}} \\ &= \frac{1}{N}S_{\mu,0}^{\frac{N}{p}}, \end{aligned}$$

which contradicts with $\kappa_\lambda < \Lambda - D\lambda^{\frac{p}{p-q}}$ (from Lemma 2.9).

Secondly, we prove that $u_\lambda \in \mathcal{N}_\lambda^+$. Suppose that this is not true, *i.e.*, $u_\lambda \in \mathcal{N}_\lambda^-$. From Lemma 2.1, we can find positive numbers s^+ and s^- with $s^+ < s_{\max} < s^- = 1$ such that $s^+u_\lambda \in \mathcal{N}_\lambda^+$, $s^-u_\lambda \in \mathcal{N}_\lambda^-$ and

$$\kappa_\lambda < I_\lambda\big(s^+u_\lambda\big) < I_\lambda\big(s^-u_\lambda\big) = I_\lambda(u_\lambda) = \kappa_\lambda,$$

which is a contradiction. Hence $u_\lambda \in \mathcal{N}_\lambda^+$. Furthermore, combining with Lemma 2.3, we can obtain

$$I_\lambda(u_\lambda) = \kappa_\lambda^+ = \kappa_\lambda < 0.$$

Therefore, we see that u_λ is a non-negative ground state solution of problem (1.1).

In the following, we prove that problem (1.1) has a second solution v_λ with $v_\lambda \in \mathcal{N}_\lambda^-$. Since I_λ is coercive on $\mathcal{N}_\lambda^-$, according to the Ekeland variational principle and Lemma 2.9, there exists a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ of I_λ such that

(i) $I_\lambda(v_n) < \kappa_\lambda^- + \frac{1}{n}$;

(ii) $I_\lambda(u) \geq I_\lambda(v_n) - \frac{1}{n}\|u - v_n\|$ for all $u \in \mathcal{N}_\lambda^-$.

Note that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$, there exist a subsequence (still denoted by $\{v_n\}$) and $v_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_\lambda & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v_\lambda & \text{strongly in } L^s(\Omega)\ (1 \leq s < p^*), \\ v_n(x) \to v_\lambda(x) & \text{a.e. in } \Omega, \end{cases} \tag{3.3}$$

as $n \to \infty$.

Similar to the above discussion, we can deduce that $v_n \to v_\lambda$ in $W_0^{1,p}(\Omega)$ and v_λ is a non-negative solution of (1.1). Thirdly, we show that $v_\lambda \neq 0$ in Ω. According to $v_n \in \mathcal{N}_\lambda^-$, we obtain

$$\begin{aligned} (p-q)\|v_n\|^p &= \big(p^* - q\big)\int_\Omega |v_n|^{p^*}\,dx + (p-q)\beta\int_\Omega |v_n|^p|x|^{\alpha-p}\,dx \\ &< \big(p^*-q\big)S_{\mu,0}^{-\frac{p^*}{p}}\|v_n\|^{p^*} + (p-q)\frac{\beta}{\beta_1}\|v_n\|^p, \end{aligned}$$

hence

$$\|v_n\| > \left[\frac{(p-q)(1-\frac{\beta}{\beta_1})S_{\mu,0}^{\frac{p^*}{p}}}{p^*-q}\right]^{\frac{1}{p^*-p}}, \quad \forall v_n \in \mathcal{N}_\lambda^-, \tag{3.4}$$

together with $v_n \to v_\lambda$ in $W_0^{1,p}(\Omega)$ means that $v_\lambda \not\equiv 0$.

Lastly, we show that $v_\lambda \in \mathcal{N}_\lambda^-$. We only need to prove that $\mathcal{N}_\lambda^-$ is closed. In fact, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, it follows from Lemmas 2.8 and 2.9 that

$$\lim_{n\to\infty}\int_\Omega |v_n|^{p^*}\,dx = \int_\Omega |v_\lambda|^{p^*}\,dx.$$

In addition

$$(p-q)\|v_n\|^p - (p^*-q)\int_\Omega |v_n|^{p^*}\,dx - (p-q)\beta\int_\Omega |v_n|^p|x|^{\alpha-p}\,dx < 0.$$

Thus

$$(p-q)\|v_\lambda\|^p - (p^*-q)\int_\Omega |v_\lambda|^{p^*}\,dx - (p-q)\beta\int_\Omega |v_\lambda|^p|x|^{\alpha-p}\,dx \le 0,$$

which means that $v_\lambda \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$. Combining with Lemma 2.1 and $v_\lambda \not\equiv 0$, we see that $\mathcal{N}_\lambda^-$ is closed. Note that $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, we know that u_λ and v_λ are different.

4 Conclusions

In this paper, we study the existence and multiplicity of positive solutions for the quasilinear elliptic problem which consists of critical Sobolev exponent and a Hardy term.

The main conclusions of this work:

(1) Adding a linear perturbation in the nonlinear term of elliptic equation.

(2) The main challenge of this study is the lack of compactness of the embedding $W_0^{1,p} \hookrightarrow L^{p^*}$. We overcome it by the concentration compactness principle.

(3) We apply the Ekeland variational principle to obtain a minimizing sequence with good properties.

5 Discussion

In the future, a natural question is whether the multiplicity of positive solutions for (1.1) can be established with negative exponent $\frac{1}{u^\gamma}$ $(0 < \gamma < 1)$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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