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A modified two-layer iteration via a boundary point approach to generalized multivalued pseudomonotone mixed variational inequalities

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Abstract

Most mathematical models arising in stationary filtration processes as well as in the theory of soft shells can be described by single-valued or generalized multivalued pseudomonotone mixed variational inequalities with proper convex nondifferentiable functionals. Therefore, for finding the minimum norm solution of such inequalities, the current paper attempts to introduce a modified two-layer iteration via a boundary point approach and to prove its strong convergence. The results here improve and extend the corresponding recent results announced by Badriev, Zadvornov and Saddeek (Differ. Equ. 37:934-942, 2001).

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1 Introduction

Let V be a real Banach space, V^* be its dual space, $\|\cdot\|_{V^*}$ be the dual norm of the given norm $\|\cdot\|_V$, and $\langle\cdot,\cdot\rangle$ be the duality pairing between V^* and V. Let M be a nonempty closed convex subset of V. Let $C(V^*)$ be the family of nonempty compact subsets of V^*. Let H be a real Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_H$, respectively.

We denote by $\to$ and $\rightharpoonup$ strong and weak convergence, respectively. Let $A_0 : V \to V^*$ be a nonlinear single-valued mapping.

Definition 1.1 (see [2–6]) For all $u, \eta \in V$, the mapping $A_0 : V \to V^*$ is said to be as follows:

(i) pseudomonotone, if it is bounded and for every sequence $\{u_n\} \subset V$ such that

$$u_n \rightharpoonup u \in V \quad \text{and} \quad \limsup_{n\to\infty}\langle A_0 u_n, u_n - u\rangle \le 0$$

imply

$$\liminf_{n\to\infty}\langle A_0 u_n, u_n - \eta\rangle \ge \langle A_0\eta, u - \eta\rangle;$$

(ii) coercive, if there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\xi\to\infty} \rho(\xi) = +\infty$ such that

$$\langle A_0 u, u\rangle \geq \rho\bigl(\|u\|_V\bigr)\|u\|_V;$$

(iii) potential, if

$$\int_0^1 \bigl(\bigl\langle A_0\bigl(t(u+\eta)\bigr), u+\eta\bigr\rangle - \bigl\langle A_0(tu), u\bigr\rangle\bigr)\,dt = \int_0^1 \bigl\langle A_0(u+t\eta), \eta\bigr\rangle\,dt;$$

(iv) bounded Lipschitz continuous, if

$$\|A_0 u - A_0\eta\|_{V^*} \leq \mu(R)\Phi\bigl(\|u-\eta\|_V\bigr),$$

where $R = \max\{\|u\|_V, \|\eta\|_V\}$, μ is a nondecreasing function on $[0, +\infty)$, and Φ is the gauge function (i.e., it is a strictly increasing continuous function on $[0, +\infty)$ such that $\Phi(0) = 0$ and $\lim_{\xi\to\infty} \Phi(\xi) = +\infty$);

(v) uniformly monotone, if there exists a gauge Φ such that

$$\langle A_0 u - A_0\eta, u-\eta\rangle \geq \Phi\bigl(\|u-\eta\|_V\bigr)\|u-\eta\|_V;$$

(vi) inverse strongly monotone, if there exists a constant $\gamma > 0$ such that

$$\langle A_0 u - A_0\eta, u-\eta\rangle \geq \gamma\|A_0 u - A_0\eta\|_V^2.$$

If $\Phi(\xi) = \xi$ and $\mu(R) = \gamma > 0$, in (iv), the mapping A_0 is called γ-Lipschitzian mapping, and if there exists $\alpha > 0$ such that $\Phi(\xi) = \alpha\xi$, in (v), the mapping A_0 is called strongly monotone mapping. It is obvious that any inverse strongly monotone mapping is $\frac{1}{\gamma}$-Lipschitzian mapping.

The single-valued pseudomonotone mixed variational inequality problem is formulated as finding a point $u \in M$ such that

$$\langle A_0 u, \eta - u\rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u\rangle \quad \forall\eta \in M, \tag{1.1}$$

where $A_0 : V \to V^*$ is a single-valued pseudomonotone mapping, $F_1 : V \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous (but, in general, nondifferentiable) functional, and $f \in V^*$ is a given element.

Problem (1.1) is equivalent to finding $u \in V$ such that

$$0 \in A_0 u - f + \partial F_1(u), \tag{1.2}$$

where $\partial F_1(u)$ is the subdifferential of F_1, i.e.,

$$\partial F_1(u) = \bigl\{u^* \in V^* : F_1(\eta) - F_1(u) \geq \bigl\langle u^*, \eta - u\bigr\rangle\ \forall\eta \in V\bigr\}.$$

The interior of the domain of F_1 is denoted by $\mathrm{int}(D(F_1))$.

Such problems appear in many fields of physics (e.g., in hydrodynamics, elasticity or plasticity), more specifically, when describing or analyzing the steady state filtration (see, for example, [1, 7–9] and the references cited therein) and the problem of finding the equilibrium of soft shells (see, for example, [1, 7, 10–12] and the references cited therein).

The existence of at least one solution to problem (1.1) can be guaranteed by imposing pseudomonotonicity and coercivity conditions on the mapping A_0 (see, for example, [2, 3]).

If $f = 0$ and $F_1(u) = I_M(u)\ \forall u \in M$, where I_M is the indicator functional of M defined by $u \in M$ such that $I_M(u) = \begin{cases} 0, & u \in M, \\ +\infty, & \text{o.w,} \end{cases}$ then problem (1.1) is equivalent to finding $u \in M$ such that

$$\langle A_0 u, \eta - u\rangle \geq 0 \quad \forall \eta \in M, \tag{1.3}$$

which is known as the classical variational inequality problem firstly introduced and studied by Stampacchia [13]. Problem (1.3) is equivalent to the following nonlinear operator equation: find $u \in M$ such that

$$A_0 u = f. \tag{1.4}$$

A mapping $J : V \to V^*$ is called a duality mapping with gauge function Φ if, for every $u \in V$, $\langle Ju, u\rangle = \Phi(\|u\|_V)\|u\|_V$ and $\|Ju\|_{V^*} = \Phi(\|u\|_V)$. If $V = H$, then the duality mapping with the gauge function $\Phi(\xi) = \xi$ can be identified with the identity mapping of H into itself.

It is well known (see, for example, [3, 14]) that $J(0) = 0$, J is odd, single-valued, bijective and is uniformly continuous on bounded sets if V is a reflexive Banach space and V^* is uniformly convex; moreover, J^{-1} is also single-valued, bijective, and $JJ^{-1} = I_{V^*}$, $J^{-1}J = I_V$.

Therefore, we always assume that the dual space of a reflexive Banach space is uniformly convex.

Remark 1.1 (see, for example, [15]) The single-valued duality mapping J is bounded Lipschitz continuous and uniformly monotone.

In order to find a solution of problem (1.1), Badriev et al. [1] suggested the following two-layer iteration method: for an arbitrary $u_0 \in M$, define $u_{n+1} \in M$ as follows:

$$\bigl\langle J(u_{n+1} - u_n), \eta - u_{n+1}\bigr\rangle + \tau\bigl(F_1(\eta) - F_1(u_{n+1})\bigr) \geq \tau\langle f - A_0 u_n, \eta - u_{n+1}\rangle \quad \forall \eta \in M, \tag{1.5}$$

where $\tau > 0$ is an iteration parameter and $n \geq 0$.

In this way the original variational inequality problem (1.1) is thus reduced to another variational inequality problem involving the duality mapping J instead of the original pseudomonotone mapping A_0. Such a problem can then be solved by known methods (see, for example, [16, 17]).

If $V = H$, then the iteration generated by (1.5) can be written in the following form:

$$(u_{n+1} - u_n, \eta - u_{n+1}) + \tau\bigl(F_1(\eta) - F_1(u_{n+1})\bigr) \geq \tau(f - A_0 u_n, \eta - u_{n+1}) \quad \forall \eta \in M, \tag{1.6}$$

for an arbitrary $u_0 \in M$ and $\tau > 0$.

In [18], Saddeek and Ahmed considered the following two-layer iteration method for solving the nonlinear operator equation (1.4) in a Banach space V:

$$J(u_{n+1} - u_n) = \tau(f - A_0 u_n), \quad n \geq 0, \tag{1.7}$$

where u_0 is an arbitrary point in M and $\tau > 0$.

In the case when $V = H$, iteration (1.7) can be written as follows:

$$u_{n+1} = u_n - \tau(A_0 u_n - f), \quad n \geq 0, \tag{1.8}$$

for $\tau > 0$ and u_0 is an arbitrary point in M.

Saddeek and Ahmed [18] proved some weak convergence theorems of iterations (1.7) and (1.8) for approximating the solution of nonlinear equation (1.4).

Attempts to modify the two-layer iterations (1.7) and (1.8) so that strong convergence is guaranteed have recently been made.

In [19], Saddeek introduced the following modification of (1.8) in a Hilbert space H (boundary point method):

$$u_{n+1} = u_n - \tau h(u_n)(A_0 u_n - f), \quad n \geq 0, \tag{1.9}$$

where $\tau > 0$, u_0 is an arbitrary point in M, and $h : M \to [0,1]$ is a function defined by He and Zhu [20] as follows:

$$h(u) = \inf\bigl\{\alpha \in [0,1] : \alpha u \in M\bigr\} \quad \forall u \in M. \tag{1.10}$$

He obtained strong convergence results for finding the minimum norm solution of nonlinear equation (1.4).

In [20], He and Zhu have observed that, if $0 \notin M$, calculating $h(u_n)$ implies determining $h(u_n)u_n$, a boundary point of M, so iteration (1.9) is known as the boundary point method.

In [21], Saddeek extended the results of Saddeek [19] to a uniformly convex Banach space and introduced the following modification of the two-layer iteration (1.7) (boundary point method):

$$Ju_{n+1} = Ju_n - \tau h(u_n)(A_0 u_n - f), \quad n \geq 0, \tag{1.11}$$

where $\tau > 0$, u_0 is an arbitrary point in M, $\tau > 0$, and h is defined by (1.10).

In [22], Noor introduced and studied the following generalized multivalued pseudomonotone mixed variational inequality problem: find $u \in M$, $w \in A_0(u)$ such that

$$\langle w, \eta - u\rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u\rangle \quad \forall \eta \in M, \tag{1.12}$$

where $A_0 : V \to C(V^*)$ is a multivalued pseudomonotone mapping (see definition below), $F_1 : V \to \mathbb{R} \cup \{+\infty\}$ is a functional as above, and $f \in V^*$ is a given element.

Clearly, problems (1.1) and (1.3) are special cases of problem (1.12).

The set of all $u \in M$ satisfying (1.12) is denoted by $SOL(M, F_1, A_0 - f)$.

In [1], Badriev et al. obtained the following weak convergence theorems using the two-layer iteration (1.5).

Theorem 1.1 (see [1], Theorem 1) *Let V be a real reflexive Banach space with a uniformly convex dual space V^*, and let $J : V \to V^*$ be the duality mapping. Let M be a nonempty closed convex subset of V. Let $A_0 : V \to V^*$ be a pseudomonotone, coercive, potential, and bounded Lipschitz continuous mapping. Let $F_1 : V \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and γ-Lipschitzian (i.e., $| F_1(u) - F_1(\eta) | \le \gamma \|u - \eta\|_V \ \forall u, \eta \in V, \gamma > 0$) functional. Define a functional $F : V \to \mathbb{R} \cup \{+\infty\}$ by*

$$F(u) = F_0(u) + F_1(u) - \langle f, u\rangle, \qquad F_0(u) = \int_0^1 \langle A_0\bigl(t(u)\bigr), u\rangle\, dt, \quad f \in V^*. \tag{1.13}$$

Assume also that

$$0 < \tau < \min\left\{1, \frac{1}{\mu_0}\right\}, \quad \mu_0 = \mu\bigl(R_0 + \Phi^{-1}(R_1 + \gamma)\bigr), \tag{1.14}$$

where

$$R_0 = \sup_{u \in S_0} \|u\|_V, \qquad R_1 = \sup_{u \in S_0} \|A_0 u - f\|_{V^*}, \qquad S_0 = \bigl\{u \in M : F(u) \le F(u_0)\bigr\}.$$

Then the sequence $\{u_n\}$ defined by (1.5) *is bounded in V, and all of its weak limit points are solutions of problem* (1.1).

Badriev et al. [1] have remarked that, due to the reflexivity of V, the mixed variational inequality (1.1) is solvable by Theorem 1.1.

In Theorem 1.1, the assumption that V is reflexive can be dropped. Indeed, if V^* is uniformly convex, then V is uniformly smooth (and hence V is reflexive).

Theorem 1.2 (see [1], Theorem 2) *Let $V = H$ be a real Hilbert space, and let M be a nonempty closed convex subset of H. Let $A_0 : H \to H$ be a pseudomonotone, coercive, potential, and inverse strongly monotone mapping. Let $F_i : H \to \mathbb{R} \cup \{+\infty\}$, $i = 0, 1$, be the same as in Theorem* 1.1.

Then the sequence $\{u_n\}$ defined by (1.6) *with $0 < \tau < \tau_0 = 2\gamma$ converges weakly in H to a solution of problem* (1.1).

Some attempts to prove the weak convergence of the whole sequence in the framework of Banach spaces have been made by Saddeek and Ahmed [23] and Saddeek [24, 25].

Although the above mentioned theorems and all their extensions are unquestionably interesting, only weak convergence theorems are obtained unless very strong assumptions are made.

This suggests an important question: can the two-layer iteration method (1.5) be modified to prove its strong convergence to the minimum norm solution of problem (1.12).

In this paper, inspired by [20, 21], and [22], a generalized multivalued pseudomonotone mixed variational inequality is considered, and a modified two-layer iteration via a boundary point approach to find the minimum norm solution of such inequalities is introduced, and its strong convergence is proved in the framework of uniformly convex spaces. The results obtained in this paper improve and generalize the corresponding recent results announced by [1].

2 Definitions and preliminary

Definition 2.1 (see [5, 26, 27]) A multivalued mapping $A_0 : V \to C(V^*)$ is called

(i) pseudomonotone, if it is bounded and, for every sequence $\{u_n\} \subset V$, $\{w_n\} \subset A_0(u_n)$, the conditions

$$u_n \rightharpoonup u \in V \quad \text{and} \quad \limsup_{n\to\infty} \langle w_n, u_n - u\rangle \le 0$$

imply that for every $\eta \in V$ there exists $w \in A_0(u)$ such that

$$\liminf_{n\to\infty} \langle w_n, u_n - \eta\rangle \ge \langle w, u - \eta\rangle;$$

(ii) coercive, if there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\xi\to\infty} \rho(\xi) = +\infty$ such that

$$\langle w, u\rangle \ge \rho\bigl(\|u\|_V\bigr)\|u\|_V \quad \forall u \in V, w \in A_0(u);$$

(iii) potential, if

$$\int_0^1 \bigl(\langle w^1, u + \eta\rangle - \langle w^2, u\rangle\bigr)\, dt = \int_0^1 \langle w^3, \eta\rangle\, dt$$

for all $u, \eta \in V$, $w^1 \in A_0(t(u + \eta))$, $w^2 \in A_0(tu)$, $w^3 \in A_0(u + t\eta)$, $t \in [0, 1]$;

(iv) bounded Lipschitz continuous, if

$$\|w - \acute{w}\|_{V^*} \le \mu(R)\Phi\bigl(\|u - \eta\|_V\bigr)$$

for all $u, \eta \in V$, $w \in A_0(u)$, $\acute{w} \in A_0(\eta)$, where $\mu(R)$ and $\Phi(\xi)$ as above;

(v) inverse strongly monotone, if there exists a constant $\gamma > 0$ such that

$$\langle w - \acute{w}, u - \eta\rangle \ge \gamma \|w - \acute{w}\|_V^2$$

for all $u, \eta \in V$, $w \in A_0(u)$, $\acute{w} \in A_0(\eta)$.

Definition 2.1 is an extension of Definition 1.1((i)-(iv), (vi)) of single-valued mappings to multivalued mappings.

Let $G_1 : M \times V^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G_1(u, J\eta) = \|u\|_V^2 - 2\langle J\eta, u\rangle + \|J\eta\|_{V^*}^2 + 2F_1(u), \tag{2.1}$$

where $u \in M$, $\eta \in V$, $J\eta \in V^*$.

Definition 2.2 (see, for example, [28]) The mapping $\Pi_M^{F_1} : V \to C(M)$ is called generalized F_1-projection mapping if $\Pi_M^{F_1}(\eta) = \arg\min_{u\in M} G_1(u, J\eta)$, $\forall \eta \in V$.

If $V = H$ and $F_1(u) = 0$ $\forall u \in M$, then (2.1) reduces to the following simple form:

$$G_1(u, J\eta) = \|u - \eta\|_H^2, \quad \forall u \in M, \eta \in H,$$

and the generalized F_1-projection reduces to the projection Π_M from H to $C(M)$.

The following two lemmas are also useful in the sequel.

Lemma 2.1 (see [28]) *The generalized F_1-projection $\Pi_M^{F_1}(\eta)$ has the following properties*:

(i) $\Pi_M^{F_1}(\eta)$ *is a nonempty closed convex subset of M for all $\eta \in V$*;

(ii) *for all $\eta \in V$, $\bar{u} \in \Pi_M^{F_1}(\eta)$ if and only if*

$$\langle J\eta - J\bar{u}, \bar{u} - v\rangle + (F_1(v) - F_1(\bar{u})) \geq 0 \quad \forall v \in M;$$

(iii) *if V is strictly convex, then $\Pi_M^{F_1}(\eta)$ is a single-valued mapping.*

Let $G_2 : V \times V \to \mathbb{R}^+ \cup \{0\}$ be a functional defined as follows:

$$G_2(u,\eta) = \|u\|_V^2 - 2\langle J\eta, u\rangle + \|\eta\|_V^2, \quad \forall u,\eta \in V. \tag{2.2}$$

Lemma 2.2 (see [29]) *Let V be a real Banach space with a uniformly convex dual space V^*, let M be a nonempty closed convex subset of V, and let $\eta \in V$, $\bar{u} \in \Pi_M^{F_1}(\eta)$. Then*

(i) $G_2(u,\bar{u}) + G_2(\bar{u},\eta) \leq G_2(u,\eta)$ $\forall u \in M$;

(ii) *for $u,\eta \in V$, $G_2(u,\eta) = 0$ iff $u = \eta$.*

A Banach space V is said to have the Kadec-Klee property (*see, for example*, [30]) *if, for every sequence $\{u_n\}$ in V with $u_n \rightharpoonup u$ and $\|u_n\|_V \to \|u\|_V$ together imply that* $\lim_{n\to\infty} \|u_n - u\|_V = 0$.

Every Hilbert space is uniformly convex, and every uniformly convex Banach space has the Kadec-Klee property.

3 Main results

In this section, we propose a modification of the two-layer iteration method (1.5) by the boundary point method to establish strong convergence theorems of the modified iteration for finding the minimum norm solution of the following generalized pseudomonotone mixed variational inequality in uniformly convex spaces: find $u \in M$, $w \in A_0(u)$ such that

$$\big\langle h(w), \eta - u\big\rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u\rangle \quad \forall \eta \in M, \tag{3.1}$$

where A_0, F_1, f are defined as above and h is a positive constant.

3.1 The modified two-layer iteration

For an arbitrary point $u_0 \in M$, define $u_{n+1} \in M$ as follows:

$$\langle Ju_{n+1} - Ju_n, \eta - u_{n+1}\rangle + \tau\big(F_1(\eta) - F_1(u_{n+1})\big) \geq \tau\big\langle f - h(u_n)w_n, \eta - u_{n+1}\big\rangle \quad \forall \eta \in M, \tag{3.2}$$

where $\tau > 0$ is the iteration parameter, $n \geq 0$, J is the duality mapping, $w_n \in A_0(u_n)$ and h is defined by (1.10).

For $M = V$, $F_1(u) = 0$ $\forall u \in M$, and $\eta = u_{n+1} \pm z$, $z \in M$, (3.2) is equivalent to

$$Ju_{n+1} = Ju_n - \tau\big(h(u_n)w_n - f\big), \quad \forall n \geq 0, \tag{3.3}$$

where $w_n \in A_0(u_n)$, and τ, J, h are defined as above.

Observe that iteration (3.3) is a modification and generalization of iterations (1.11) and (1.9).

If $V = H$, A_0 is a single-valued mapping in (3.3) and $h(u_n) = 1$ $\forall n \geq 0$, we have iteration (1.8).

Iteration (3.3) can be considered as a modified method for solving the following operator inclusion problem: find $u \in V$ such that

$$f \in A_0 u, \quad f \in V^*. \tag{3.4}$$

For each $u \in V$, $w^2 \in A_0(tu)$, let $\tilde{F} : V \to \mathbb{R} \cup \{+\infty\}$ be a functional defined by

$$\tilde{F}(u) = \tilde{F}_0(u) + F_1(u) - \langle f, u\rangle, \qquad \tilde{F}_0(u) = \int_0^1 \langle h(u)w^2, u\rangle\, dt, \quad f \in V^*. \tag{3.5}$$

Let us assume also that

$$\begin{aligned} &\tilde{R_0} = \sup_{u \in \tilde{S_0}} \|u\|_V, \qquad \tilde{R_1} = \sup_{u \in \tilde{S_0}} \|h(u)w - f\|_{V^*}, \\ &\tilde{S_0} = \{u \in M : \tilde{F}(u) \leq \tilde{F}(u_0)\}, \end{aligned} \tag{3.6}$$

where $w \in A_0(u)$.

Let $\tilde{\mu}_0$ be a positive constant such that

$$\tilde{\mu_0} = \mu\big(2\tilde{R}_0 + \Phi^{-1}(\tilde{R}_1 + \gamma)\big). \tag{3.7}$$

Theorem 3.1 *Let V be a real uniformly convex Banach space with a uniformly convex dual space V^*, $J : V \to V^*$ be the duality mapping, and let M be a nonempty closed convex subset of V. Let $A_0 : V \to C(V^*)$ be a multivalued mapping. Suppose that A_0 is pseudomonotone, coercive, potential, and bounded Lipschitz continuous mapping. Let $F_1 : V \to \mathbb{R} \cup \{+\infty\}$ be a proper convex (not necessarily differentiable) and γ-Lipschitzian functional with $M \subset$* int$(D(F_1))$. *Let $\tilde{F}$, $\tilde{R}_0$, $\tilde{R}_1$, $\tilde{S}_0$, and $\tilde{\mu}_0$ be defined by* (3.5), (3.6), *and* (3.7). *Assume that* $0 < \tau = \min\{1, \frac{1}{\tilde{\mu}_0}\}$. *Let $\{h(u_n)\}$ be an increasing and bounded real sequence in* $[0,1]$.

Then, for an arbitrary $u_0 = u \in M$, the sequence $\{u_n\}$ defined by (3.2) *converges strongly to $\tilde{u} = \Pi^{F_1}_{SOL(M,F_1,h(w)-f)} 0$ (i.e., the minimum norm element in $SOL(M, F_1, h(w) - f)$).*

Proof Since F_1 is supposed to be convex and γ-Lipschitzian, and A_0 is coercive and bounded, it results from [1] and [2] that F_1 is weakly lower semicontinuous and $\tilde{F}$ is coercive; moreover, $\tilde{R}_0 < +\infty$ and $\tilde{R}_1 < +\infty$. Hence $\tilde{\mu}_0 < +\infty$. This means that the iterative sequence (3.2) is well defined. □

Now we divide the proof into steps.

Step 1. We prove that $\{u_n\}$ is bounded. To this end, it suffices to prove that

$$\{u_n\} \subset \tilde{S}_0, \qquad \|u_n\|_V \leq \tilde{R}_0, \quad n \geq 0. \tag{3.8}$$

Let us prove (3.8) by induction on n. For $n = 0$, we have $u_0 \in \tilde{S}_0$. Suppose now that $u_n \in \tilde{S}_0$. We will show that $u_{n+1} \in \tilde{S}_0$.

Setting $\eta = u_n$ in (3.2) and taking into account that the functional F_1 is γ-Lipschitzian and J is uniformly monotone, and the inequality $\tau \leq 1$, we obtain

$$\begin{aligned}\Phi\big(\|u_{n+1}-u_n\|_V\big)\|u_{n+1}-u_n\|_V &\leq \langle Ju_{n+1}-Ju_n, u_{n+1}-u_n\rangle \\ &\leq \tau\big[\big\langle f-h(u_n)w_n, u_{n+1}-u_n\big\rangle + F_1(u_n)-F_1(u_{n+1})\big] \\ &\leq [\tilde{R}_1+\gamma]\|u_{n+1}-u_n\|_V. \end{aligned} \tag{3.9}$$

Now, using (3.9) together with the strict monotonicity of Φ, we have

$$\|u_{n+1}-u_n\|_V \leq \Phi^{-1}(\tilde{R}_1+\gamma). \tag{3.10}$$

Furthermore, it follows from the bounded Lipschitz continuity of A_0 that, for any $t \in [0,1]$, $w_n \in A_0u_n$, $w_n^3 \in A_0(u_{n+1}+t(u_n-u_{n+1}))$

$$\begin{aligned}\big|\big\langle w_n^3-w_n, u_{n+1}-u_n\big\rangle\big| &\leq \mu(R_*)\Phi\big(\big\|(1-t)(u_{n+1}-u_n)\big\|_V\big)\|u_{n+1}-u_n\|_V \\ &\leq \mu(R_*)\Phi\big(\|u_{n+1}-u_n\|_V\big)\|u_{n+1}-u_n\|_V,\end{aligned} \tag{3.11}$$

where $R_* = \max\{\|u_{n+1}+t(u_n-u_{n+1})\|_V, \|u_n\|_V\}$.

Since

$$\big\|u_{n+1}+t(u_n-u_{n+1})\big\|_V - \|u_n\|_V \leq \big\|(1-t)(u_{n+1}-u_n)\big\|_V \leq \|u_{n+1}-u_n\|_V, \tag{3.12}$$

it follows from the definition of R_* that

$$R_* \leq 2\tilde{R}_0 + \Phi^{-1}(\tilde{R}_1+\gamma). \tag{3.13}$$

Since $\tilde{\mu}$ is an increasing function, we must have

$$\tilde{\mu}(R_*) \leq \tilde{\mu}_0. \tag{3.14}$$

Consequently, it follows from (3.11) and (3.14) that

$$-\big|\big\langle w_n^3-w_n, u_{n+1}-u_n\big\rangle\big| \geq -\tilde{\mu}_0\Phi\big(\|u_{n+1}-u_n\|_V\big)\|u_{n+1}-u_n\|_V. \tag{3.15}$$

Moreover, since A_0 is potential, we have

$$\begin{aligned}\tilde{F}(u_n)-\tilde{F}(u_{n+1}) &= \int_0^1 \big\langle h(u_n)w_n^3, u_n-u_{n+1}\big\rangle\,dt - \langle f, u_n-u_{n+1}\rangle + F_1(u_n)-F_1(u_{n+1}) \\ &= \int_0^1 \big\langle h(u_n)\big(w_n^3-w_n\big), u_n-u_{n+1}\big\rangle\,dt - \big\langle f-h(u_n)w_n, u_n-u_{n+1}\big\rangle \\ &\quad + F_1(u_n)-F_1(u_{n+1}) \\ &\geq -\int_0^1 \big|\big\langle h(u_n)\big(w_n^3-w_n\big), u_n-u_{n+1}\big\rangle\big|\,dt + \tau^{-1}\big[\tau\big\langle f-h(u_n)w_n, u_{n+1}-u_n\big\rangle \\ &\quad + F_1(u_n)-F_1(u_{n+1})\big].\end{aligned} \tag{3.16}$$

Setting $\eta = u_n$ in (3.2) and using the uniform monotonicity of J, (3.15), (3.16), it results that

$$\begin{aligned}\tilde{F}(u_n) - \tilde{F}(u_{n+1}) &\geq -\tilde{\mu}_0\Phi\big(\|u_{n+1} - u_n\|_V\big)\|u_{n+1} - u_n\|_V \\ &\quad + \tau^{-1}\big\langle J(u_{n+1}) - J(u_n), u_{n+1} - u_n\big\rangle \\ &\geq \lambda\Phi\big(\|u_{n+1} - u_n\|_V\big)\|u_{n+1} - u_n\|_V, \quad \lambda = \tau^{-1} - \tilde{\mu}_0 > 0. \end{aligned} \tag{3.17}$$

This implies that $\tilde{F}(u_{n+1}) \leq \tilde{F}(u_n) \leq \tilde{F}(u_0)$ and so $u_{n+1} \in \tilde{S}_0$. So $\{u_n\}$ is bounded.

Step 2. We prove that $\lim_{n\to\infty} \|u_{n+1} - u_n\|_V = 0$ and $\lim_{n\to\infty} \|Ju_{n+1} - Ju_n\|_{V^*} = 0$.

It follows from (3.17) that the sequence $\{\tilde{F}(u_n)\}$ is bounded and monotone, and thus we have that $\lim_{n\to\infty} \tilde{F}(u_n)$ exists. This together with (3.17) implies that

$$\lim_{n\to\infty} \lambda\Phi\big(\|u_{n+1} - u_n\|_V\big)\|u_{n+1} - u_n\|_V = 0. \tag{3.18}$$

Since Φ is continuous and strictly increasing, it follows from (3.18) that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|_V = 0. \tag{3.19}$$

Since J is bounded Lipschitz continuous, Φ is continuous and $\Phi(0) = 0$, it follows from (3.19) that

$$\lim_{n\to\infty} \|Ju_{n+1} - Ju_n\|_{V^*} = 0. \tag{3.20}$$

Step 3. We show that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup \bar{u} \in V$, $\lim_{k\to\infty} F_1(u_{n_k}) \geq F_1(\bar{u})$, and $\limsup_{k\to\infty} h(u_{n_k})\langle w_{n_k}, u_{n_k} - \bar{u}\rangle \leq 0$.

Since $\{u_n\}$ is bounded and V is reflexive, we can choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup \bar{u} \in V$ as $k \to \infty$.

This together with the weak lower semicontinuity of F_1 implies that $\lim_{k\to\infty} F_1(u_{n_k}) \geq F_1(\bar{u})$.

Since F_1 is γ-Lipschitzian, $\{h(u_n)\} \subset [0,1]$, it follows from (3.2) that, for arbitrary $\eta \in M$,

$$\begin{aligned} h(u_{n_k})\langle w_{n_k}, u_{n_k} - \eta\rangle &= h(u_{n_k})\langle w_{n_k}, u_{n_k} - u_{n_{k+1}}\rangle + h(u_{n_k})\langle w_{n_k}, u_{n_{k+1}} - \eta\rangle \\ &\leq h(u_{n_k})\langle w_{n_k}, u_{n_k} - u_{n_{k+1}}\rangle + \tau^{-1}\langle Ju_{n_{k+1}} - Ju_{n_k}, \eta - u_{n_{k+1}}\rangle \\ &\quad + \big(F_1(\eta) - F_1(u_{n_k})\big) + \big(F_1(u_{n_k}) - F_1(u_{n_{k+}})\big) \\ &\quad + \langle f, u_{n_{k+1}} - u_{n_k}\rangle + \langle f, u_{n_k} - \eta\rangle \\ &\leq \big(\|w_{n_k}\|_{V^*} + \|f\|_{V^*} + \gamma\big)\|u_{n_{k+1}} - u_{n_k}\|_V + \tau^{-1}\big[\|Ju_{n_{k+1}} - Ju_{n_k}\|_{V^*} \\ &\quad \times \|\eta - u_{n_{k+1}}\|_V\big] + \big(F_1(\eta) - F_1(u_{n_k})\big) + \langle f, u_{n_k} - \eta\rangle \\ &\leq C_\eta\big(\|Ju_{n_{k+1}} - Ju_{n_k}\|_{V^*} + \|u_{n_{k+1}} - u_{n_k}\|_V\big) \\ &\quad + \big(F_1(\eta) - F_1(u_{n_k})\big) + \langle f, u_{n_k} - \eta\rangle, \end{aligned} \tag{3.21}$$

where C_η is a positive constant depending on η.

Setting $\eta = \bar{u}$ in (3.21) and using the weak lower semicontinuity of F_1, (3.19), (3.20), we have

$$\begin{aligned}\limsup_{k\to\infty} h(u_{n_k})\langle w_{n_k}, u_{n_k} - \bar{u}\rangle &\le \limsup_{k\to\infty} C_{\bar{u}}\big(\|Ju_{n_{k+1}} - Ju_{n_k}\|_{V^*} + \|u_{n_{k+1}} - u_{n_k}\|_V\big)\\ &\quad + \limsup_{k\to\infty}\big(F_1(\bar{u}) - F_1(u_{n_k})\big) + \limsup_{k\to\infty}\langle f, u_{n_k} - \bar{u}\rangle\\ &\le 0. \end{aligned} \tag{3.22}$$

Step 4. We show that $\bar{u} \in SOL(M, F_1, h(w) - f)$.

Since $\{h(u_n)\} \subset [0,1]$ is bounded and monotone increasing, it follows that

$$\lim_{n\to\infty} h(u_n) = h > 0. \tag{3.23}$$

By (3.19)-(3.23), the lower semicontinuity of F_1 and by the pseudomonotonicity of A_0, we have

$$\begin{aligned} 0 &= \liminf_{k\to\infty} C_\eta\big(\|Ju_{n_{k+1}} - Ju_{n_k}\|_{V^*} + \|u_{n_{k+1}} - u_{n_k}\|_V\big)\\ &\ge \liminf_{k\to\infty} h(u_{n_k})\langle w_{n_k}, u_{n_k} - \eta\rangle + \liminf_{k\to\infty}\big(F_1(u_{nk}) - F_1(\eta)\big) + \liminf_{k\to\infty}\langle f, \eta - u_{n_k}\rangle\\ &\ge \big\langle h(\bar{w}), \bar{u} - \eta\big\rangle + F_1(\bar{u}) - F_1(\eta) + \langle f, \eta - \bar{u}\rangle, \end{aligned}$$

where $\bar{w} \in A_0\bar{u}$. This means that $\bar{u} \in SOL(M, F_1, h(w) - f)$.

Step 5. We prove that

$$\limsup_{k\to\infty}\big[\langle -J\bar{u}, u_{n_{k+1}} - \bar{u}\rangle + F_1(\bar{u}) - F_1(u_{n_{k+1}})\big] \le 0, \tag{3.24}$$

where $\bar{u} = \Pi^{F_1}_{SOL(M,F_1,h(w)-f)}0$.

Indeed take a subsequence $\{u_{n_{k+1}}\}$ of $\{u_n\}$ such that $u_{n_{k+1}} \rightharpoonup \bar{u}$.

Note that $\bar{u} = \Pi^{F_1}_{SOL(M,F_1,h(w)-f)}0$. Then from $\bar{u} \in SOL(M, F_1, h(w) - f)$, the weak lower semicontinuity of F_1, and Lemma 2.1(ii), the desired inequality (3.24) follows immediately.

Step 6. We show that $\lim_{n\to\infty}\|u_n - \bar{u}\|_V = 0$.

Since $u_{n_{k+1}} \rightharpoonup \bar{u}$, it follows from the weak lower semicontinuity of $\|\cdot\|_V$ that

$$\liminf_{k\to\infty}\|u_{n_{k+1}}\|_V \ge \|\bar{u}\|_V. \tag{3.25}$$

From the convexity of $D(F_1)$, F_1 and from the weak lower semicontinuity of F_1, we obtain that F_1 is subdifferentiable in $\operatorname{int}(D(F_1))$. Thus, for all $u \in D(F_1)$, there exists an element $u^* \in V^*$ such that

$$F_1(u) - F_1(\bar{u}) \ge \big\langle u^*, u - \bar{u}\big\rangle,$$

and hence

$$F_1(u_{n_{k+1}}) - F_1(\bar{u}) \ge \langle Ju_{n_k}, u_{n_{k+1}} - \bar{u}\rangle, \quad k \ge 0. \tag{3.26}$$

In view of $u_{n_{k+1}} = \Pi^{F_1}_{SOL(M,F_1,h(w)-f)} J^{-1}(Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k}))$, we have

$$G_1\bigl(u_{n_{k+1}}, Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k})\bigr) \le G_1\bigl(\bar{u}, Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k})\bigr).$$

By using (2.1) with $J\eta = Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k})$, we have

$$\begin{aligned}
&\|u_{n_{k+1}}\|_V^2 - 2\bigl\langle Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k}), u_{n_{k+1}}\bigr\rangle + \bigl\|Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k})\bigr\|_{V^*}^2 + 2F_1(u_{n_{k+1}}) \\
&\quad \le \|\bar{u}\|_V^2 - 2\bigl\langle Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k}), \bar{u}\bigr\rangle \\
&\qquad + \bigl\|Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k})\bigr\|_{V^*}^2 + 2F_1(\bar{u}) \\
&\quad = \|\bar{u}\|_V^2 - 2\bigl\langle Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k}), u_{n_{k+1}}\bigr\rangle \\
&\qquad + 2\bigl\langle Ju_{n_k} - \tau h(u_{n_k})(f - w_{n_k}), u_{n_{k+1}} - \bar{u}\bigr\rangle \\
&\qquad + \bigl\|Ju_{n_k} - \tau(u_{n_k})(f - w_{n_k})\bigr\|_{V^*}^2 + 2F_1(\bar{u}),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_{n_{k+1}}\|_V^2 &\le \|\bar{u}\|_V^2 + 2\bigl(\langle Ju_{n_k}, u_{n_{k+1}} - \bar{u}\rangle + F_1(\bar{u}) - F_1(u_{n_{k+1}})\bigr) \\
&\qquad + 2\tau h(u_{n_k})\langle w_{n_k}, u_{n_{k+1}} - \bar{u}\rangle + 2\tau h(u_{n_k})\langle f, \bar{u} - u_{n_{k+1}}\rangle.
\end{aligned} \tag{3.27}$$

Taking the $\limsup_{k\to\infty}$ on the both sides of (3.27) and using $u_{n_{k+1}} \rightharpoonup \bar{u}$, (3.22)-(3.24), and (3.26) yields

$$\limsup_{k\to\infty} \|u_{n_{k+1}}\|_V^2 \le \|\bar{u}\|_V^2,$$

which implies that

$$\limsup_{k\to\infty} \|u_{n_{k+1}}\|_V \le \|\bar{u}\|_V. \tag{3.28}$$

Combining (3.25) and (3.28), we have

$$\|\bar{u}\|_V \le \liminf_{k\to\infty} \|u_{n_{k+1}}\|_V \le \limsup_{k\to\infty} \|u_{n_{k+1}}\|_V \le \|\bar{u}\|_V. \tag{3.29}$$

This shows that

$$\lim_{k\to\infty} \|u_{n_{k+1}}\|_V = \|\bar{u}\|_V. \tag{3.30}$$

Since V is a uniformly convex Banach space, then it has the Kadec-Klee property, and so from $u_{n_{k+1}} \rightharpoonup \bar{u}$ and (3.30) we obtain

$$\lim_{k\to\infty} u_{n_{k+1}} = \bar{u}. \tag{3.31}$$

Let us now show that the whole sequence converges strongly to $\bar{u}$.

Since $\{G_2(u_{n+1},0)\}$ is bounded and nondecreasing (indeed, by Lemma 2.2(i), we have $G_2(u_{n+1},u_n)+G_2(u_{n+1},0) \leq G_2(u_n,0)$ and $G_2(u_{n+1},u_n) \geq (\|u_{n+1}\|_V - \|u_n\|_V)^2 \geq 0$), it follows that $\{G_2(u_{n+1},0)\}$ is convergent.

This together with (3.31) implies that

$$\lim_{n\to\infty} G_2(u_{n+1},0) = G_2(\bar{u},0). \tag{3.32}$$

Now, following to [31], we suppose that there exists some subsequence $\{u_{n_{j+1}}\}$ of $\{u_n\}$ such that $\lim_{j\to\infty} u_{n_{j+1}} = \hat{u}$, then by Lemma 2.2(i) we obtain

$$\begin{aligned} 0 \leq G_2(\bar{u},\hat{u}) &= \lim_{k,j\to\infty} G_2(u_{n_{k+1}},u_{n_{j+1}}) = \lim_{k,j\to\infty} G_2\bigl(u_{n_{k+1}},\Pi^{F_1}_{SOL(M,F_1,h(w)-f)}0\bigr) \\ &\leq \lim_{k,j\to\infty}\bigl[G_2(u_{n_{k+1}},0) - G_2\bigl(\Pi^{F_1}_{SOL(M,F_1,h(w)-f)}0,0\bigr)\bigr] \\ &= \lim_{k,j\to\infty}\bigl[G_2(u_{n_{k+1}},0) - G_2(u_{n_{j+1}},0)\bigr] \\ &= G_2(\bar{u},0) - G_2(\bar{u},0) = 0, \end{aligned}$$

which means that $G_2(\bar{u},\hat{u}) = 0$ and hence, by Lemma 2.2(ii), it results that $\hat{u} = \bar{u}$.

Consequently, $\lim_{n\to\infty} u_n = \bar{u}$. This completes the proof of Theorem 3.1.

Theorem 3.2 *Let $V = H$ be a real Hilbert space, and let M be a nonempty closed convex subset of H. Let $A_0 : H \to C(H)$ be a multivalued mapping. Suppose that A_0 is a pseudomonotone, coercive, potential, and inverse strongly monotone mapping. Let $\{h(u_n)\}$, M, $\tilde{F}$, $\tilde{S}_0$, $\tilde{\mu}_0$ and $\tilde{F}_0$, F_1, $\tilde{R}_0$, $\tilde{R}_1$ be the same as in Theorem 3.1.*

Then, for arbitrary $u_0 = u \in M$, the sequence $\{u_n\}$ defined by

$$(u_{n+1}-u_n,\eta-u_{n+1}) + \tau\bigl(F_1(\eta)-F_1(u_{n+1})\bigr) \geq \tau\bigl(f-h(u_n)w_n,\eta-u_{n+1}\bigr) \quad \forall\eta\in M, \tag{3.33}$$

with $0 < \tau < \tau_0 = \frac{2\gamma}{h}$, $h > 0$, converges strongly to $\tilde{u} = \Pi^{F_1}_{SOL(M,F_1,h(w)-f)}0$.

Proof Since any inverse strongly monotone mapping is $\frac{1}{\gamma}$-Lipschitzian mapping, i.e., bounded Lipschitz continuous with $\mu(\xi) = \frac{1}{\gamma_0}$ and $\Phi(\xi) = \xi$, then by simple modifications of the proof of Theorem 3.1, we can easily show that there exists a subsequence $\{u_{n_{k+1}}\}$ of $\{u_n\}$ such that $u_{n_{k+1}} \rightharpoonup \bar{u} \in SOL(M,F_1,h(w)-f)$ and $\lim_{k\to\infty}\|u_{n_{k+1}}\|_H = \|\bar{u}\|_H$.

Since every Hilbert space is uniformly convex, by virtue of the Kadec-Klee property of H, we have $\lim_{k\to\infty} u_{n_{k+1}} = \bar{u} \in SOL(M,F_1,h(w)-f)$.

Now, we prove that $u_n \rightharpoonup \bar{u}$ and $\lim_{n\to\infty}\|u_n\|_H = \|\bar{u}\|_H$.

From $\bar{u} \in SOL(M,F_1,h(w)-f)$, we have

$$\tau\bigl(F_1(\eta)-F_1(\bar{u})\bigr) \geq \tau\bigl(f-h(\bar{w}),\eta-\bar{u}\bigr), \quad \forall\eta\in M. \tag{3.34}$$

Setting $\eta = u_{n+1}$ in (3.34) and $\eta = \bar{u}$ in (3.33), we have

$$\tau\bigl(F_1(u_{n+1})-F_1(\bar{u})\bigr) \geq \tau\bigl(f-h(\bar{w}),u_{n+1}-\bar{u}\bigr) \tag{3.35}$$

and

$$\tau\bigl(F_1(\bar{u})-F_1(u_{n+1})\bigr) \geq (u_{n+1}-u_n,u_{n+1}-\bar{u}) + \tau\bigl(f-h(u_n)w_n,\bar{u}-u_{n+1}\bigr). \tag{3.36}$$

Adding (3.35) and (3.36), we have

$$\begin{aligned}(u_{n+1}-\bar{u},u_{n+1}-\bar{u}) &\leq (u_n-\bar{u},u_{n+1}-\bar{u})-\tau\big(h(u_n)w_n-h(\bar{w}),u_{n+1}-\bar{u}\big)\\ &= \big(u_n-\bar{u}-\tau\big(h(u_n)w_n-h(\bar{w})\big),u_{n+1}-\bar{u}\big),\end{aligned}$$

which implies that

$$\|u_{n+1}-\bar{u}\|_H \leq \big\|u_n-\bar{u}-\tau\big(h(u_n)w_n-h(\bar{w})\big)\big\|_H.$$

Then, by the inverse strong monotonicity of A_0, we obtain for all sufficiently large n

$$\begin{aligned}\|u_{n+1}-\bar{u}\|_H^2 &\leq \|u_n-\bar{u}\|_H^2-2\tau\big(h(u_n)w_n-h(\bar{w}),u_n-\bar{u}\big)+\tau^2\big\|h(u_n)w_n-h(\bar{w})\big\|_H^2\\ &= \|u_n-\bar{u}\|_H^2-2\tau h(w_n-\bar{w},u_n-\bar{u})+\tau^2h^2\|w_n-\bar{w}\|_H^2\\ &\leq \|u_n-\bar{u}\|_H^2-\tau h\left(2-\frac{\tau h}{\gamma}\right)\|w_n-\bar{w}\|_H^2.\end{aligned}$$

Since $2-\frac{\tau h}{\gamma}>0$, it follows that $\|u_{n+1}-\bar{u}\|_H \leq \|u_n-\bar{u}\|_H$ and so $\lim_{n\to\infty}\|u_n-\bar{u}\|_H=\sigma_{\bar{u}}$.

By following the same arguments as in [1] and [32], we can readily claim that all weak limit points of the sequence $\{u_n\}$ coincide, and hence $u_n \rightharpoonup \bar{u}$ as $n\to\infty$.

By the weak lower semicontinuity of $\|\cdot\|_H$, this implies that

$$\liminf_{n\to\infty}\|u_n\|_H > \|\bar{u}\|_H. \tag{3.37}$$

Analogically to the proof of step 6 with obvious modifications, we have

$$\limsup_{n\to\infty}\|u_n\|_H \leq \|\bar{u}\|_H. \tag{3.38}$$

This, together with (3.37), implies that $\lim_{n\to\infty}\|u_n\|_H=\|\bar{u}\|_H$.

Applying again the virtue of the Kadec-Klee property of H, we obtain $\lim_{n\to\infty}u_n=\bar{u}$. This completes the proof of Theorem 3.2. □

Remark 3.1 Theorems 3.1 and 3.2 extend and improve the corresponding Theorems 1.1 and 1.2.

Example 3.1 (Axisymmetric shell problem) A quintessential example of a single-valued mapping satisfying all the assumptions contemplated in Theorems 3.1 and 3.2 which appears in determining the axisymmetric equilibrium position of a soft netlike rotation shell is as follows:

The shell surface (in a strainless state) is assumed to be a cylinder of length l and radius 1. Let s be a Lagrangian coordinate in the longitudinal direction such that $0<s<l$.

Let $V=[\overset{\circ}{W}{}_p^{(1)}(0,l)]^2$ and $V^*=[\overset{\circ}{W}{}_q^{(-1)}(0,l)]^2$, $q=\frac{p}{p-1}$, $p>1$. Set $u(s)=(u_1(s),u_2(s))$, $\eta(s)=(\eta_1(s),\eta_2(s))$, $M=\{u\in V: u_2(s)+1\geq 0\ \forall s\in(0,l)\}$, and $\lambda_1=[(1+\frac{du_1}{ds})^2+(\frac{du_2}{ds})^2]^{\frac{1}{2}}$, $\lambda_2=1+u_2$.

Consider the surface force is characterized by a known constant function $\mathbb{P}$. Let $T_i(\lambda_i)$, $i=1,2$, be two functions (tightening force) satisfying conditions (3)-(5) in Badriev and Banderov [33].

Consider the mappings $A, B, C, D : V \to V^*$ defined by

$$\langle Au, \eta \rangle = \int_0^l \frac{T_1(\lambda_1)}{\lambda_1} \left(\left(1 + \frac{du_1}{ds}, \frac{du_2}{ds} \right), \frac{d\eta}{ds} \right) ds;$$

$$\langle Bu, \eta \rangle = \int_0^l \left(\frac{1}{2} u_2^2 \frac{d\eta_1}{ds} + \left(1 + \frac{du_1}{ds} \right) u_2 \eta_2 \right) ds;$$

$$\langle Cu, \eta \rangle = \int_0^l \left(\left(1 + \frac{du_1}{ds} \right) \eta_2 + \frac{du_2}{ds} \frac{d\eta_1}{ds} \right) ds;$$

$$\langle Du, \eta \rangle = \int_0^l T_2(\lambda_2) \eta_2 \, ds.$$

If $A_0 = (A + D) + \mathbb{P}(B + C)$, then by Theorems 2 and 3 in [33] it follows that the mapping A_0 satisfies all the assumptions postulated in Theorems 3.1 and 3.2.

4 Conclusion

A generalized multivalued pseudomonotone mixed variational inequality is considered, and a modified two-layer iteration via a boundary point approach to find the minimum norm solution of such inequalities is introduced, and its strong convergence is proved in the framework of uniformly convex spaces. The results develop the corresponding recent results.

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The author declares that he has no competing interests.

Authors' contributions
I am the only author. I have read and approved the final manuscript.

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