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Area distortion under certain classes of quasiconformal mappings

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Abstract

In this paper we study the hyperbolic and Euclidean area distortion of measurable sets under some classes of K -quasiconformal mappings from the upper half-plane and the unit disk onto themselves, respectively.

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1 Introduction

Let $f : \Omega \rightarrow \mathbb{C}$ be an ACL (absolute continuous on lines) homeomorphism in a domain $\Omega \subset \mathbb{C}$ that preserves orientation. If f satisfies

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K \quad \text{a.e.}$$

for some $K \geq 1$, then f is a K -quasiconformal mapping, where

$$f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y),$$

and D_f is called the dilatation of f .

In 1956 Beurling and Ahlfors solved the boundary value problem for quasiconformal mappings [1]. If $M \geq 1$, they gave an explicit formula for the extension of an M -quasisymmetric function $h : \mathbb{R} \rightarrow \mathbb{R}$ to a quasiconformal mapping $f = u + iv$ from $\mathbb{H}$ onto itself, where $\mathbb{H}$ denotes the upper half-plane. The mapping f is called the Beurling-Ahlfors extension of h . In particular f satisfies (see [2])

$$\frac{1}{Cy^2} \leq \frac{J_f(z)}{v^2} \leq \frac{C}{y^2},$$

where J_f denotes the Jacobian of f and $C = C(K) > 0$ depends on $K = K(M)$, the maximal dilatation of f . Thus, for each measurable subset E of $\mathbb{H}$, it holds that

$$\frac{A_{\mathcal{H}}(E)}{C} \leq A_{\mathcal{H}}(f(E)) \leq CA_{\mathcal{H}}(E),$$

where $A_{\mathcal{H}}(\cdot)$ denotes the hyperbolic area in the half-plane $\mathbb{H}$.

In 1994 Astala [3] proved that if f is a K -quasiconformal mapping from the unit disk $\mathbb{D}$ onto itself, normalized by $f(0) = 0$, and if E is any measurable subset of the unit disk, then $A_e(f(E)) \leq a(K)A_e(E)^{1/K}$, where $A_e(\cdot)$ denotes the Euclidean area and $a(K) \rightarrow 1$ when $K \rightarrow 1^+$.

In 1998 Reséndis and Porter [4] obtained some results about area distortion under quasiconformal mappings on the unit disk $\mathbb{D}$ onto itself with respect to the hyperbolic measure. They also showed the existence of explodable sets; this kind of sets has bounded hyperbolic area, but under a specific quasiconformal mapping its image has infinite hyperbolic area.

In recent years harmonic quasiconformal mappings have been extensively studied, see [5–9] and the papers cited therein. The following two recent results are very close to the results presented in this paper.

In 2007 Knežević and Mateljević [10] proved the following Schwarz-Pick type distortion theorem.

Theorem 1 *Let f be a K -quasiconformal harmonic mapping from the unit disk $\mathbb{D}$ onto itself, then*

$$\frac{1}{1+k} \frac{1 - |f(z)|^2}{1 - |z|^2} \leq |f_z(z)| \leq \frac{1}{1-k} \frac{1 - |f(z)|^2}{1 - |z|^2},$$

where k is defined by $k := \frac{K-1}{K+1}$.

In 2012 Min Chen and Xingdi Chen [11] studied the class of (K, K') quasiconformal maps from $\mathbb{H}$ onto itself, and they obtained the following result about area distortion of harmonic mappings.

Theorem 2 *Let $f(z) = u(z) + iv(z)$ be a harmonic mapping from $\mathbb{H}$ onto itself and continuous on $\mathbb{H} \cup \mathbb{R}$ with $f(\infty) = \infty$. In particular, f has the form $f(z) = f(x + iy) = u(x, y) + icy$ for some $c > 0$ (see [12]). If f is K -quasiconformal and $E \subset \mathbb{H}$ is any measurable set, then*

- (i) $A_e(f(E)) \leq c^2 K A_e(E)$,
- (ii) $A_{\mathcal{H}}(f(E)) \leq K A_{\mathcal{H}}(E)$,
- (iii) $0 < u_x \leq \frac{c(K+1+\sqrt{(K+1)^2-4})}{2}$,
- (iv) $|u_y| \leq c \sqrt{(K+1) \frac{K+1+\sqrt{(K+1)^2-4}}{2}}$.

In this paper we use the following hyperbolic density definitions:

$$\frac{2|dz|}{1 - |z|^2} \quad \text{and} \quad \frac{|dw|}{\operatorname{Im} w}$$

for the unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{H}$, respectively, see [13] and [14]. We denote also by $A_{\mathcal{H}}$ the hyperbolic area in the unit disk $\mathbb{D}$.

2 Results and discussion

Our purpose in this article is to continue the study of the hyperbolic area distortion under K -quasiconformal mappings from the upper half-plane $\mathbb{H}$ onto itself or from the unit disk $\mathbb{D}$ onto itself. Due to the existence of explodable sets (see [4]), we study some particular

classes of quasiconformal mappings. First we apply the result of Knežević and Mateljević [10] to estimate the hyperbolic area distortion under harmonic quasiconformal mappings from $\mathbb{D}$ onto itself, and we remove the hypothesis of harmonicity in Theorem 2.

Additionally, we generalize even more the class studied by Chen and Chen in [11]. More precisely, the main result of this paper is the following.

Theorem 3 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself such that f maps a family of horocycles with a common tangent point onto a family of horocycles. Then, for each measurable set $E \subset \mathbb{H}$, the following inequalities hold:*

$$\frac{1}{K^9} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K^9 A_{\mathcal{H}}(E).$$

These bounds are asymptotically sharp when $K \rightarrow 1^+$.

Additionally, we obtain some results about radial and angular quasiconformal mappings. Motivated by the generalization mentioned above, we finally describe a set that contains the region of values of the partial derivatives of K -quasiconformal mappings.

2.1 Harmonic quasiconformal mappings

In this part we use Theorem 1 to estimate the hyperbolic area distortion under quasiconformal harmonic mappings from the unit disk onto itself and analyze the hyperbolic and Euclidean area distortion under quasiconformal mappings $f(z) = f(x + iy) = u(x, y) + icy$, with $c > 0$, from $\mathbb{H}$ onto itself, without the hypothesis of harmonicity of f (see [11]), and we sketch the proof of items (i) and (ii) of Theorem 2 as a corollary of this result and get better bounds than the ones obtained in items (iii) and (iv) in the same theorem. See [13] and [14] for results in hyperbolic geometry.

Theorem 4 *Let f be a K -quasiconformal harmonic mapping from the unit disk $\mathbb{D}$ onto itself. If $E \subset \mathbb{D}$ is a measurable set, then*

$$\frac{1}{K} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq \left(\frac{K+1}{2} \right)^2 A_{\mathcal{H}}(E).$$

These bounds are asymptotically sharp when $K \rightarrow 1^+$.

Proof Let $w = f(z) = u(z) + iv(z)$, $z \in \mathbb{D}$. By hypothesis and Theorem 1, the mapping f satisfies

$$\begin{aligned} \left(\frac{1}{1+k} \frac{1-|f(z)|^2}{1-|z|^2} \right)^2 &\leq |f_z(z)|^2 \\ &\leq \left(\frac{1}{1-k} \frac{1-|f(z)|^2}{1-|z|^2} \right)^2. \end{aligned}$$

Moreover, the Jacobian J_f of f satisfies

$$(1-k^2)|f_z|^2 \leq J_f = |f_z|^2 - |f_{\bar{z}}|^2 \leq |f_z|^2.$$

Thus, for each measurable set $E \subset \mathbb{D}$, the following equalities hold:

$$A_{\mathcal{H}}f(E) = \int_{f(E)} \frac{du dv}{(1 - |w|^2)^2} = \int_E \frac{J_f(z)}{(1 - |f(z)|^2)^2} dx dy,$$

$$\frac{1 - k^2}{(1 + k)^2} A_{\mathcal{H}}(E) \leq \int_E \frac{J_f(z) dx dy}{(1 - |f(z)|^2)^2} \leq \frac{1}{(1 - k)^2} A_{\mathcal{H}}(E)$$

and the result follows immediately. $\square$

Theorem 5 *Let f be a K -quasiconformal mapping from the upper half-plane $\mathbb{H}$ onto itself given by $f(z) = f(x + iy) = u(x, y) + icy$, with $c > 0$. If there exists $M > 0$ such that $|f_z(z)| - |\bar{f}_z(z)| \leq M$ a.e., then for any measurable subset E of $\mathbb{H}$ the following inequalities hold:*

- (i) $A_e(f(E)) \leq M^2 K A_e(E)$,
- (ii) $A_{\mathcal{H}}(f(E)) \leq \frac{M^2 K}{c^2} A_{\mathcal{H}}(E)$.

Proof Since f is a K -quasiconformal mapping, f satisfies $|f_z| + |\bar{f}_z| \leq K(|f_z| - |\bar{f}_z|)$ a.e. in $\mathbb{H}$, and by hypothesis we get $J_f = (|f_z| + |\bar{f}_z|)(|f_z| - |\bar{f}_z|) \leq KM^2$ a.e. in $\mathbb{H}$. Then, for any measurable set $E \subset \mathbb{H}$,

$$A_e(f(E)) = \int_{f(E)} du dv = \int_E J_f dx dy \leq KM^2 A_e(E)$$

and for the second result we have

$$A_{\mathcal{H}}(f(E)) = \int_{f(E)} \frac{du dv}{(\operatorname{Im} w)^2} = \int_E \frac{J_f dx dy}{(cy)^2} \leq \frac{M^2 K}{c^2} A_{\mathcal{H}}(E). \quad \square$$

If we additionally suppose that f is a harmonic mapping, then there exists a holomorphic function $g : \mathbb{H} \rightarrow \mathbb{C}$ such that $f(z) = \operatorname{Re} g(z) + icy$. Thus

$$|f_z| = \frac{1}{2} |g'(z) + c| \quad \text{and} \quad |\bar{f}_z| = \frac{1}{2} |g'(z) - c|.$$

Since

$$\frac{|\bar{f}_z|}{|f_z|} \leq k \quad \text{or equivalently} \quad |g'(z) - c|^2 \leq k^2 |g'(z) + c|^2,$$

we obtain that $g'(z)$ belongs to the circle D with center $(\frac{1+k^2}{1-k^2}c, 0)$ and radius $\frac{2ck}{1-k^2}$. Hence, for each $w \in D$, we get the estimations

$$|w + c| - |w - c| \leq 2c \quad \text{and} \quad |w + c| + |w - c| \leq 2Kc.$$

In particular we have

$$|f_z| - |\bar{f}_z| \leq c \quad \text{and} \quad |f_z| + |\bar{f}_z| \leq Kc.$$

So we obtain Theorem 2 as a corollary of Theorem 5.

We note that Theorem 5 gives some information about the partial derivatives, in specific we obtain that if f satisfies the hypothesis of the last theorem, then f_z and $\bar{f}_z$ are bounded.

The previous sketch gives us also some idea to study quasiconformal mappings of the form $f(x + iy) = u(x, y) + iv(y)$.

2.2 Quasiconformal mappings $f(x + iy) = u(x, y) + iv(y)$

We now generalize the class studied by Cheng and Chen (see [11]) in two directions. First, we will show that it is possible to avoid the harmonic hypothesis, and second, we will prove that the class of K -quasiconformal mappings given by $f(x + iy) = u(x, y) + iv(y)$ is a family that accepts asymptotically sharp bi-bounds for the area distortion.

Let $1 \leq K < \infty$ and $\Omega \subset \mathbb{C}$ be a domain. Suppose that $f : \Omega \rightarrow \mathbb{C}$ is a K -quasiconformal mapping given by

$$f(x + iy) = u(x, y) + iv(x, y). \quad (1)$$

Then f satisfies

$$\left| \frac{f_{\bar{z}}}{f_z} \right| \leq k, \quad \text{a.e.} \quad \text{or equivalently} \quad \left| \frac{f_{\bar{z}}}{f_z} \right|^2 \leq k^2, \quad \text{a.e.} \quad (2)$$

Inequality (2) is satisfied if and only if

$$(u_x - v_y)^2 + (v_x + u_y)^2 \leq k^2 ((u_x + v_y)^2 + (v_x - u_y)^2) \quad \text{a.e.}$$

or equivalently

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 - \frac{1+k^2}{1-k^2} 2u_x v_y + \frac{1+k^2}{1-k^2} 2u_y v_x \leq 0 \quad \text{a.e.}$$

Define

$$\alpha = \alpha(k) := \frac{1+k^2}{1-k^2} \geq 1. \quad (3)$$

Then f satisfies inequality (2) if and only if

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 - 2\alpha u_x v_y + 2\alpha u_y v_x \leq 0 \quad \text{a.e.} \quad (4)$$

From now on each expression that involves partial derivatives will be true almost everywhere (a.e.) and Ω will denote a domain of the complex plane $\mathbb{C}$.

In this part we focus on K -quasiconformal mappings f from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. In particular f can be extended homeomorphically to $\overline{\mathbb{H}}$, $u(x, y)$ is ACL and $v(y)$ is absolutely continuous. We know that f satisfies

$$u_x^2 + u_y^2 + v_x^2 - 2\alpha u_x v_y \leq 0 \quad \text{a.e.} \quad (5)$$

Despite the fact that v depends only on the variable y , we write v_y instead of v' to emphasize the dependence on y . The next result gives the principal characteristics of the mapping f , and most of them are consequences of its quasiconformal properties (the rest are easy to prove).

Proposition 1 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. Then*

- *The function $y \mapsto v(y)$ is a homeomorphism from $[0, \infty) \rightarrow [0, \infty)$ that is absolutely continuous and differentiable a.e.*
- *For almost every $y \in [0, \infty)$, the function $x \mapsto u(x, y)$ is a homeomorphism from $\mathbb{R}$ onto itself that is absolutely continuous and differentiable a.e.*
- *The K -quasiconformal inverse mapping $f^{-1} : \mathbb{H} \rightarrow \mathbb{H}$ has the same form as f , that is, $f^{-1}(x + iy) = w(x, y) + iv^{-1}(y)$.*
- *Let $y_0 \in (0, \infty)$ be fixed. The function $h : \mathbb{H} \rightarrow \mathbb{H}$ defined by $h(x + iy) = u(x, y + y_0) + i[v(y + y_0) - v(y_0)]$ is a K -quasiconformal mapping. In particular each function $x \mapsto u(x, y_0)$ is quasisymmetric.*
- *If $g : \mathbb{H} \rightarrow \mathbb{H}$ is a K' -quasiconformal mapping given by $g(x + iy) = l(x, y) + iw(y)$, then $f \circ g$ is a KK' -quasiconformal mapping of the same form.*
- *The mapping f leaves invariant the family of horocycles with tangential point at infinity.*

We study inequality (5) in more detail. To get this, we complete in (5) the square in v_y , so we obtain

$$(u_x - \alpha v_y)^2 + u_y^2 \leq v_y^2 (\alpha^2 - 1) \quad \text{a.e.}$$

This inequality defines a circle a.e., thus u_x and v_y satisfy in particular

$$\alpha v_y - v_y \sqrt{\alpha^2 - 1} \leq u_x \leq \alpha v_y + v_y \sqrt{\alpha^2 - 1} \quad \text{a.e.}$$

and

$$-v_y \sqrt{\alpha^2 - 1} \leq u_y \leq v_y \sqrt{\alpha^2 - 1} \quad \text{a.e.}$$

In fact, the circle is a subset of the square described by the previous inequalities. Observe that

$$K = \alpha + \sqrt{\alpha^2 - 1} \tag{6}$$

and

$$\frac{1}{K} = \alpha - \sqrt{\alpha^2 - 1},$$

where K is the maximal dilatation of f . Let $C = \sqrt{\alpha^2 - 1}$. Then $0 \leq C$. With this notation the last inequalities can be written as follows:

$$\frac{v_y}{K} \leq u_x \leq K v_y \quad \text{a.e.} \tag{7}$$

and

$$-C v_y \leq u_y \leq C v_y \quad \text{a.e.} \tag{8}$$

Given $0 \leq x$, we integrate (7) on the interval $[0, x]$

$$\int_0^x \frac{v_y(y)}{K} dt \leq \int_0^x u_x(t, y) dt \leq \int_0^x K v_y(y) dt.$$

If we choose any fixed $y \in (0, \infty)$ such that $u(x, y)$ is absolutely continuous with respect to x , then we obtain

$$\frac{x}{K} v_y(y) + u(0, y) \leq u(x, y) \leq K x v_y(y) + u(0, y) \quad (9)$$

for each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$. Using the left-hand side of the last inequality, we get

$$\limsup_{y \rightarrow 0^+} \left[\frac{v_y(y)x}{K} + u(0, y) \right] \leq \limsup_{y \rightarrow 0^+} u(x, y),$$

and since $u(x, y)$ is continuous, we obtain

$$\frac{x}{K} \limsup_{y \rightarrow 0^+} v_y(y) \leq u(x, 0) - u(0, 0) < \infty.$$

For this reason, $\limsup_{y \rightarrow 0^+} v_y(y)$ exists and consequently $\liminf_{y \rightarrow 0^+} v_y(y)$ exists too. We define $v_y^+(0) := \limsup_{y \rightarrow 0^+} v_y(y)$ and $v_y^-(0) := \liminf_{y \rightarrow 0^+} v_y(y)$. With this notation, we obtain from (9)

$$\frac{v_y^+(0)x}{K} + u(0, 0) \leq u(x, 0) \leq K v_y^-(0)x + u(0, 0). \quad (10)$$

On the other hand, we choose any fixed $x \in [0, \infty)$ such that $u(x, y)$ is absolutely continuous with respect to y , and we integrate (8) on the interval $[0, y]$. So

$$\int_0^y -C v_y(t) dt \leq \int_0^y u_y(x, t) dt \leq \int_0^y C v_y(t) dt$$

and, since $v(y)$ is absolutely continuous, we obtain

$$-C v(y) + u(x, 0) \leq u(x, y) \leq C v(y) + u(x, 0)$$

for $y \in [0, \infty)$ and almost every $x \in [0, \infty)$. By an argument of continuity of the mapping f and density, we have

$$-C v(y) + u(x, 0) \leq u(x, y) \leq C v(y) + u(x, 0) \quad (11)$$

for all $(x, y) \in [0, \infty) \times [0, \infty)$. Setting $x = 0$ in the previous inequality, we get

$$-C v(y) + u(0, 0) \leq u(0, y) \leq C v(y) + u(0, 0). \quad (12)$$

Thus, combining (9) and (12), we have

$$\frac{v_y(y)x}{K} - C v(y) + u(0, 0) \leq u(x, y) \leq K x v_y(y) + C v(y) + u(0, 0)$$

for each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$. In the same way we use (10) and (11) to obtain

$$\frac{x}{K} v_y^+(0) - Cv(y) + u(0, 0) \leq u(x, y) \leq Kxv_y^-(0) + Cv(y) + u(0, 0).$$

We combine the left- and right-hand sides of the previous inequalities to get

$$\frac{x}{K} v_y(y) - Cv(y) + u(0, 0) \leq u(x, y) \leq Kxv_y^-(0) + Cv(y) + u(0, 0) \quad (13)$$

and

$$\frac{x}{K} v_y^+(0) - Cv(y) + u(0, 0) \leq u(x, y) \leq Kxv_y(y) + Cv(y) + u(0, 0) \quad (14)$$

for each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$. Since the left- and right-hand sides of inequalities (13) and (14) represent linear equations in the variable x , we compare their slopes and the fact that $x \geq 0$ to conclude

$$\frac{v_y(y)}{K} \leq Kv_y^-(0) \quad \text{and} \quad \frac{v_y^+(0)}{K} \leq Kv_y(y)$$

for each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$. Hence

$$\frac{v_y^+(0)}{K^2} \leq v_y(y) \leq K^2 v_y^-(0)$$

for each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$. We recall that $v(y)$ is absolutely continuous, and we integrate the above inequalities on the interval $[0, y]$

$$\frac{1}{K^2} \int_0^y v_y^+(0) dt \leq \int_0^y v_y(t) dt \leq K^2 \int_0^y v_y^-(0) dt$$

to get

$$\frac{1}{K^2} v_y^+(0) y \leq v(y) \leq K^2 v_y^-(0) y. \quad (15)$$

In particular $0 < v_y^-(0) \leq v_y^+(0) < \infty$ and v has right Dini's derivatives at 0. If $x < 0$, we obtain the same relation as (15) and have the next result.

Theorem 6 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. Then $v_y^*(0) = \limsup_{y \rightarrow 0^+} v_y(y)$ and $v_y^-(0) = \liminf_{y \rightarrow 0^+} v_y(y)$ are finite, and the partial derivatives of f satisfy the following inequalities:*

1. $\frac{1}{K^2} v_y^+(0) \leq v_y(y) \leq K^2 v_y^-(0)$ for almost every $y \in (0, \infty)$.
2. $\frac{1}{K^3} v_y^+(0) \leq u_x(x, y) \leq K^3 v_y^-(0)$ for almost every $x + iy \in \mathbb{H}$.
3. $|u_y(x, y)| \leq \frac{K(K^2-1)}{2} v_y^-(0)$ for almost every $x + iy \in \mathbb{H}$.

In particular by Proposition 6 the partial derivatives of f belong to some truncate solid cone.

We can combine the previous result with items (iii) and (iv) of Theorem 2 to obtain the following result.

Corollary 1 *Let f be a harmonic K -quasiconformal mapping from $\mathbb{H}$ onto itself with $f(\infty) = \infty$. In particular there exists $c > 0$ such that $f(x + iy) = u(x, y) + icy$. Then*

$$\frac{c}{K^3} \leq u_x \leq A(K)c$$

and

$$|u_y| \leq B(K)c$$

with

$$A(K) = \begin{cases} K^3 & \text{if } K \in [1, a], \\ \frac{K+1+\sqrt{(K+1)^2-4}}{2} & \text{if } K \in [a, \infty), \end{cases}$$

$$B(K) = \begin{cases} \frac{K(K^2-1)}{2} & \text{if } K \in [1, b], \\ \sqrt{(K+1) \frac{K+1+\sqrt{(K+1)^2-4}}{2}} & \text{if } K \in [b, \infty), \end{cases}$$

where $a = 1.12373\dots$ and $b = 1.95371\dots$ are the solutions of the equations $K^3 = \frac{K+1+\sqrt{(K+1)^2-4}}{2}$ and $\frac{K(K^2-1)}{2} = \sqrt{(K+1) \frac{K+1+\sqrt{(K+1)^2-4}}{2}}$, respectively.

Proof The left-hand side of the first inequality is immediate from item 1 of Theorem 6. We now consider the right-hand side estimations of u_x in item 2 of Theorem 6, with $v_y^-(0) = v_y^+(0) = c$ and item (iii) of Theorem 2. Then

$$u_x \leq \min \left\{ cK^3, \frac{c(K+1+\sqrt{(K+1)^2-4})}{2} \right\}.$$

We consider the equation

$$K^3 = \frac{K+1+\sqrt{(K+1)^2-4}}{2}$$

and obtain the value of a by solving $K^6 - K^4 - K^3 + 1 = 0$ that has only two real solutions $K = 1$ and $K = 1.12373\dots = a$.

We now consider the estimations of $|u_y|$ in item 3 of Theorem 6 and item (iv) of Theorem 2. Thus

$$|u_y| \leq \min \left\{ \frac{cK(K^2-1)}{2}, c\sqrt{(K+1) \frac{K+1+\sqrt{(K+1)^2-4}}{2}} \right\}.$$

The value of b is given by the real solution of

$$\frac{K(K^2-1)}{2} = \sqrt{(K+1) \frac{K+1+\sqrt{(K+1)^2-4}}{2}}$$

that can be reduced to the equation

$$\left(\frac{K+1}{2} \right) (K^4 - 2K^3 + K^2 - 2) = \sqrt{(K+1)^2 - 4},$$

and the real solution of this equation is $K = 1.95371 \dots = b$. This value is obtained through the equation

$$\left(\frac{(K+1)^2}{4}\right)(K^4 - 2K^3 + K^2 - 2)^2 = (K+1)^2 - 4$$

discarding the value $K = 1.55739 \dots$ that obviously is not a solution of the original equation. $\square$

The last corollary shows that these estimations are asymptotically sharp when $K \rightarrow 1^+$ improving the result obtained in [11].

Theorem 7 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. Then*

1. *There exists $M > 0$ such that $|f_z| - |f_{\bar{z}}| \leq M$ and $|f_z| + |f_{\bar{z}}| \leq KM$ a.e.*
2. *The mapping f is Lipschitz in $\mathbb{H}$.*
3. *The mapping f is hyperbolically Lipschitz in $\mathbb{H}$.*

Proof

1. Since

$$\begin{aligned} |f_z| - |f_{\bar{z}}| &= \frac{1}{2} \sqrt{(u_x + v_y)^2 + u_y^2} - \frac{1}{2} \sqrt{(u_x - v_y)^2 + u_y^2} \\ &\leq \frac{1}{2} \sqrt{2(u_x^2 + v_y^2 + u_y^2)} \quad \text{a.e.} \end{aligned}$$

We estimate the last expression using Theorem 6 to obtain

$$\begin{aligned} &\frac{1}{2} \sqrt{2(u_x^2 + v_y^2 + u_y^2)} \\ &\leq \frac{\sqrt{2}}{2} \sqrt{(K^3 v_y^-(0))^2 + (K^2 v_y^-(0))^2 + \left(\frac{K^2 - 1}{2} K v_y^-(0)\right)^2} \\ &= \frac{K^2 v_y^-(0)}{\sqrt{2}} \sqrt{K^2 + 1 + \left(\frac{K^2 - 1}{2K}\right)^2} \\ &= \frac{K^2 v_y^-(0)}{\sqrt{2}} \sqrt{\frac{5K^4 + 2K^2 + 1}{4K^2}} \\ &= \frac{K v_y^-(0)}{2\sqrt{2}} \sqrt{5K^4 + 2K^2 + 1} \quad \text{a.e.} \end{aligned}$$

Thus we choose $M = \frac{K v_y^-(0)}{2\sqrt{2}} \sqrt{5K^4 + 2K^2 + 1}$.

2. Let $z_1, z_2 \in \mathbb{H}$ and l be the Euclidean segment that joints z_1 with z_2 . Then

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq \int_{f(l)} |df| \\ &= \int_l (|f_z| + |f_{\bar{z}}|) |dz| \\ &\leq MK |z_1 - z_2|. \end{aligned}$$

3. Let $z_1, z_2 \in \mathbb{H}$ and l be the hyperbolic segment that joints z_1 with z_2 . Then

$$\begin{aligned} d_{\mathcal{H}}(f(z_1), f(z_2)) &\leq \int_{f(l)} \frac{|dw|}{\operatorname{Im} w} = \int_l \frac{|df|}{v} \\ &\leq \int_l (|f_z| + |f_{\bar{z}}|) \frac{|dz|}{v} \leq MK \int_l \frac{|dz|}{v} \leq LMK d_{\mathcal{H}}(z_1, z_2), \end{aligned}$$

where $L = \frac{K^2}{v_y^+(0)}$ and $d_{\mathcal{H}}$ denotes the hyperbolic metric. $\square$

Now we can extend even more the result obtained by Min Chen and Xindy Chen [11].

Theorem 8 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. Then, for each measurable set $E \subset \mathbb{H}$,*

1. $\frac{(v_y^+(0))^2}{K^5} A_e(E) \leq A_e(f(E)) \leq K^5 (v_y^-(0))^2 A_e(E)$.
2. $\frac{1}{K^9} \left(\frac{v_y^+(0)}{v_y^-(0)} \right)^2 A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K^9 \left(\frac{v_y^-(0)}{v_y^+(0)} \right)^2 A_{\mathcal{H}}(E)$.

Since $v_y^-(0) \leq v_y^+(0)$, we have

$$\frac{1}{K^9} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K^9 A_{\mathcal{H}}(E),$$

and these inequalities are asymptotically sharp when $K \rightarrow 1^+$.

Proof Let $E \subset \mathbb{H}$ be a measurable set. The Jacobian of f is $J_f = u_x v_y$. By (7) and item 1 of Theorem 6,

$$\frac{(v_y^+(0))^2}{K^5} \leq J_f \leq K^5 (v_y^-(0))^2 \quad \text{a.e.}$$

The Euclidean area of $f(E)$ is

$$\int_E J_f \, dx \, dy = A_e(f(E))$$

and in consequence

$$\frac{(v_y^+(0))^2}{K^5} A_e(E) \leq A_e(f(E)) \leq K^5 (v_y^-(0))^2 A_e(E).$$

On the other hand, for the hyperbolic area only, we see that from (15) it follows

$$\frac{1}{K^9} \left(\frac{v_y^+(0)}{v_y^-(0)} \right)^2 A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K^9 \left(\frac{v_y^-(0)}{v_y^+(0)} \right)^2 A_{\mathcal{H}}(E). \quad \square$$

Now, we can prove Theorem 3.

Proof of Theorem 3 If $f : \mathbb{H} \rightarrow \mathbb{H}$ is a K -quasiconformal mapping that leaves invariant the family of horocycles with common tangential point at ∞ , then $f(x + iy) = u(x, y) + iv(y)$ and is exactly Theorem 8.

We now suppose that $f : \mathbb{H} \rightarrow \mathbb{H}$ is a K -quasiconformal mapping such that f maps horocycles with tangential point at $x_0 \in \mathbb{R}$ onto horocycles with tangential point at $x_1 \in \mathbb{R}$.

For $i = 0, 1$, let H_i be a horocyclic with tangential point at x_i and $H_\infty^i \subset \mathbb{H}$ be a horocyclic with tangential point at infinity. Then there exist Möbius transformations T_i such that $T_i(H_i) = H_\infty^i$. We define $\hat{f} : H_\infty^0 \rightarrow H_\infty^1$ by $\hat{f}(z) = (T_1 \circ f \circ T_0^{-1})(z)$, then $\hat{f}$ is K -quasiconformal and can be written in the form $\hat{f}(x + iy) = u(x, y) + iv(y)$. By Theorem 8 we have that if $E \subset \mathbb{H}$ is a measurable set, then $\hat{f}$ satisfies

$$\frac{1}{K^9} A_{\mathcal{H}}(T_0(E)) \leq A_{\mathcal{H}}(\hat{f}(T_0(E))) \leq K^9 A_{\mathcal{H}}(T_0(E)).$$

Since the hyperbolic area in $\mathbb{H}$ is invariant under the Möbius transformations T_i , the result follows immediately. The other case is similar. $\square$

Corollary 2 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. Then, for each measurable set $E \subset \mathbb{H}$,*

(i) *If v is differentiable at 0, then*

$$\frac{(v'_y(0))^2}{K^9} A_e(E) \leq A_e(f(E)) \leq K^9 (v'_y(0))^2 A_e(E).$$

(ii) *If v is continuously differentiable in a neighborhood of 0,*

$$\frac{(v'_y(0))^2}{K^5} A_e(E) \leq A_e(f(E)) \leq K^5 (v'_y(0))^2 A_e(E).$$

These inequalities are asymptotically sharp when $K \rightarrow 1^+$.

Proof If v is differentiable at 0, then from (15) we get $\frac{v_y^-(0)}{K^2} \leq v'_y(0) \leq K^2 v_y^+(0)$. If v is continuously differentiable in a neighborhood of 0, then $v_y^-(0) = v'_y(0) = v_y^+(0)$. We prove the corollary applying item 1 of Theorem 8. $\square$

The following examples of quasiconformal mappings are not harmonic, thus we are generalizing the results obtained in [11].

Example 1

1. Let $f : \mathbb{H} \rightarrow \mathbb{H}$ given by

$$f(x + iy) = x + \frac{i}{2} (2y \arctan y + 2\pi y - \ln(1 + y^2)).$$

Then f is a $\frac{3\pi}{2}$ -quasiconformal mapping with

$$v_y^+(0) = v_y^-(0) = v'_y(0) = \pi.$$

Thus, for each measurable set $E \subset \mathbb{H}$, we have

$$\frac{32}{243\pi^3} A_e(E) \leq A_e(f(E)) \leq \frac{243\pi^7}{32} A_e(E)$$

and

$$\frac{512}{19,683\pi^9} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq \frac{19,683\pi^9}{512} A_{\mathcal{H}}(E).$$

2. Let $f: \mathbb{H} \rightarrow \mathbb{H}$ given by $f(x + iy) = 2x + \sin(x + y) + iy$. Then f is a $\frac{11+\sqrt{85}}{6}$ -quasiconformal mapping with

$$\nu_y^+(0) = \nu_y^-(0) = \nu'(0) = 1.$$

It is enough to observe that the dilatation of f is given by

$$\begin{aligned} D_f(x, y) &= \frac{\sqrt{\cos^2(x + y) + (3 + \cos(x + y))^2} + \sqrt{\cos^2(x + y) + (1 + \cos(x + y))^2}}{\sqrt{\cos^2(x + y) + (3 + \cos(x + y))^2} - \sqrt{\cos^2(x + y) + (1 + \cos(x + y))^2}} \end{aligned}$$

and the function $D_f(x, a - x)$ depends only on a , more precisely

$$D_f(x, a - x) = \frac{\sqrt{\cos^2 a + (3 + \cos a)^2} + \sqrt{\cos^2 a + (1 + \cos a)^2}}{\sqrt{\cos^2 a + (3 + \cos a)^2} - \sqrt{\cos^2 a + (1 + \cos a)^2}}.$$

The maximum of $D_f(x, a - x)$ is attained at the critical points $a = 2l\pi$, $l = 0, \pm 1, \pm 2, \dots$ and the maximal dilatation of f is

$$\frac{\sqrt{17} + \sqrt{5}}{\sqrt{17} - \sqrt{5}} = \frac{11 + \sqrt{85}}{6}.$$

Thus, for each measurable set $E \subset \mathbb{H}$, we have

$$\left(\frac{6}{11 + \sqrt{85}} \right)^5 A_e(E) \leq A_e(f(E)) \leq \left(\frac{11 + \sqrt{85}}{6} \right)^5 A_e(E)$$

and

$$\left(\frac{6}{11 + \sqrt{85}} \right)^9 A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq \left(\frac{11 + \sqrt{85}}{6} \right)^9 A_{\mathcal{H}}(E).$$

2.3 Angular and radial quasiconformal mappings

In this part we obtain the results of area distortion for radial and angular mappings. In the case of angular mappings, we use the hyperbolic model of the unit disk $\mathbb{D}$.

Proposition 2 *Let $f: \Omega \rightarrow \mathbb{C}$ be an ACL mapping. If $f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$, then for a.e. in Ω*

$$4|f_z|^2 = \left(u_r + \frac{v_\theta}{r} \right)^2 + \left(v_r - \frac{u_\theta}{r} \right)^2, \quad (16)$$

$$4|f_{\bar{z}}|^2 = \left(u_r - \frac{v_\theta}{r} \right)^2 + \left(v_r + \frac{u_\theta}{r} \right)^2 \quad (17)$$

and the Jacobian of f is

$$J_f = \frac{1}{r}(u_r v_\theta - u_\theta v_r). \quad (18)$$

A mapping $f : \mathbb{H} \rightarrow \mathbb{H}$ is said to be radial at $x \in \mathbb{R}$ if f leaves invariant all Euclidean rays in $\mathbb{H}$ that meet at x .

Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be a radial mapping at x . Since hyperbolic area is invariant under horizontal translations, we can assume that the point $x \in \mathbb{R}$, where the Euclidean rays meet, is $x = 0$. If f is a radial mapping, then f can be written in polar coordinates (r, θ) as $f(z) = f(re^{i\theta}) = \rho(r, \theta)e^{i\theta}$, with $\rho(r, \theta) : (0, \infty] \times (0, \pi) \rightarrow (0, \infty)$ if $z = re^{i\theta}$.

Lemma 1 *Let f be an ACL mapping from $\mathbb{H}$ onto itself. Suppose that f is a radial mapping at 0. Then its Jacobian is $J_f = \frac{\rho \rho_r}{r}$ a.e. If f preserves orientation, then $\rho_r > 0$ a.e.*

Proof Since $f(z) = f(re^{i\theta}) = \rho(r, \theta)e^{i\theta}$, then $u(r, \theta) = \rho(r, \theta) \cos \theta$ and $v(r, \theta) = \rho(r, \theta) \sin \theta$, and the proof is immediate from (18). $\square$

Proposition 3 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself. Suppose that f is a radial mapping at 0. Then the function ρ satisfies the following:*

1. For $1 \leq r < \infty$,

$$r^{\frac{1}{K}} \leq \frac{\rho(r, \theta)}{\rho(1, \theta)} \leq r^K;$$

2. For $0 < r < 1$,

$$r^K \leq \frac{\rho(r, \theta)}{\rho(1, \theta)} \leq r^{\frac{1}{K}}.$$

Proof We first prove that the function $(0, \infty) \ni r \mapsto \ln \rho(r, \theta)$ is absolutely continuous for almost every $\theta \in (0, \pi)$. It is enough to prove that for every $M > 1$, the function $[\frac{1}{M}, M] \ni r \mapsto \ln \rho(r, \theta)$ is absolutely continuous for almost every $\theta \in (0, \pi)$. Let $\Omega = \{z = x + i\theta \in \mathbb{C} : (x, \theta) \in (-\infty, \infty) \times (0, \pi)\}$. Then the mapping $\log \circ f \circ \exp : \Omega \rightarrow \Omega$ is K -quasiconformal. Thus the function $(-\infty, \infty) \ni x \mapsto \ln \rho(e^x, \theta)$ is absolutely continuous for almost every $\theta \in (0, \pi)$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for every finite collection of disjoint intervals $(a_j, b_j) \subset \mathbb{R}$, $j = 1, 2, \dots, n$, with $\sum_{j=1}^n (b_j - a_j) < \delta$, then

$$\sum_{j=1}^n (\ln \rho(e^{b_j}, \theta) - \ln \rho(e^{a_j}, \theta)) < \varepsilon.$$

Since $\ln r$ is absolutely continuous on $[\frac{1}{M}, M]$, there exists $\delta' > 0$ such that for every finite collection of disjoint intervals $(c_l, d_l) \subset [\frac{1}{M}, M]$, $l = 1, 2, \dots, m$, with $\sum_{l=1}^m (d_l - c_l) < \delta'$, then

$$\sum_{l=1}^m (\ln d_l - \ln c_l) < \delta,$$

and by the previous inequality

$$\sum_{l=1}^m (\ln \rho(d_l, \theta) - \ln \rho(c_l, \theta)) < \varepsilon.$$

If $f(z) = \rho(r, \theta)e^{i\theta}$, from (16) and (17) we get

$$|f_z|^2 = \frac{1}{4} \left(\rho_r^2 + 2 \frac{\rho_r \rho}{r} + \frac{\rho^2}{r^2} + \frac{\rho_\theta^2}{r^2} \right) \quad \text{a.e.} \quad (19)$$

and

$$|f_{\bar{z}}|^2 = \frac{1}{4} \left(\rho_r^2 - 2 \frac{\rho_r \rho}{r} + \frac{\rho^2}{r^2} + \frac{\rho_\theta^2}{r^2} \right) \quad \text{a.e.} \quad (20)$$

By (2), (19) and (20) we have

$$\frac{1}{4} \left[\left(\rho_r - \frac{\rho}{r} \right)^2 + \frac{\rho_\theta^2}{r^2} \right] \leq k^2 \frac{1}{4} \left[\left(\rho_r + \frac{\rho}{r} \right)^2 + \frac{\rho_\theta^2}{r^2} \right] \quad \text{a.e.}$$

or equivalently

$$\frac{2(r^2 \rho_r^2 + \rho^2 + \rho_\theta^2)}{4r \rho_r \rho} \leq \frac{k^2 + 1}{1 - k^2} = \alpha \quad \text{a.e.}$$

Then

$$\frac{1}{2} \left[\frac{r \rho_r}{\rho} + \frac{\rho}{r \rho_r} \right] \leq \alpha \quad \text{a.e.}$$

and thus

$$\left(\frac{r \rho_r}{\rho} \right)^2 - 2\alpha \left(\frac{r \rho_r}{\rho} \right) + 1 \leq 0 \quad \text{a.e.}$$

Solving this inequality, we obtain

$$\frac{1}{Kr} \leq \frac{\rho_r}{\rho} \leq \frac{K}{r} \quad \text{a.e.} \quad (21)$$

or equivalently

$$\frac{1}{Kr} \leq \frac{\partial}{\partial r} \ln \rho \leq \frac{K}{r} \quad \text{a.e.}$$

We choose any fixed $\theta \in (0, \pi)$ such that $\ln \rho(r, \theta)$ is absolutely continuous on r , and we integrate the previous inequality on the interval $[1, R]$ to get

$$\int_1^R \frac{1}{Kr} dr \leq \int_1^R \frac{\partial}{\partial r} \ln \rho dr \leq \int_1^R \frac{K}{r} dr.$$

Thus

$$\frac{1}{K} \ln r \Big|_1^R \leq \ln \rho(r, \theta) \Big|_1^R \leq K \ln r \Big|_1^R \quad \text{for almost every } \theta \in (0, \pi) \text{ and } R \in [1, \infty).$$

By an argument of continuity of f and density, we finally obtain

$$R^{\frac{1}{K}} \leq \frac{\rho(R, \theta)}{\rho(1, \theta)} \leq R^K \quad \text{for all } (R, \theta) \in [1, \infty) \times (0, \pi).$$

In a similar way, if we suppose that $0 < R < 1$, then

$$R^K \leq \frac{\rho(R, \theta)}{\rho(1, \theta)} \leq R^{\frac{1}{K}} \quad \text{for all } (R, \theta) \in (0, 1) \times (0, \pi). \quad \square$$

Theorem 9 *Let f be a K -quasiconformal mapping from $\mathbb{H}$ onto itself that leaves invariant each ray in $\mathbb{H}$ that meets a real base point. If $E \subset \mathbb{H}$ is a measurable set, then*

$$\frac{1}{K} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K A_{\mathcal{H}}(E).$$

These inequalities are asymptotically sharp when $K \rightarrow 1^+$.

Proof We can suppose that the base point is 0 since hyperbolic area is invariant under horizontal translations. Let $E \subset \mathbb{H}$ be a measurable set and $\widehat{E}$ denote the set E in polar coordinates. If $f : \mathbb{H} \rightarrow \mathbb{H}$ is given by $f(z) = \rho(r, \theta)e^{i\theta}$, then

$$A_{\mathcal{H}}(f(E)) = \iint_{f(E)} \frac{du dv}{(\operatorname{Im} w)^2} = \iint_E \frac{J_f dx dy}{(\operatorname{Im} f(z))^2} = \iint_{\widehat{E}} \frac{\rho_r dr d\theta}{\rho \sin^2 \theta}.$$

By (21) we have

$$\frac{1}{Kr} \leq \frac{\rho_r}{\rho} \leq \frac{K}{r} \quad \text{a.e.}$$

then

$$\frac{1}{K} \iint_{\widehat{E}} \frac{r dr d\theta}{r^2 \sin^2 \theta} \leq \iint_{\widehat{E}} \frac{\rho_r dr d\theta}{\rho \sin^2 \theta} \leq K \iint_{\widehat{E}} \frac{r dr d\theta}{r^2 \sin^2 \theta}$$

or equivalently, in rectangular coordinates,

$$\frac{1}{K} \iint_E \frac{dx dy}{y^2} \leq A_{\mathcal{H}}(f(E)) \leq K \iint_E \frac{dx dy}{y^2},$$

that is,

$$\frac{1}{K} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K A_{\mathcal{H}}(E). \quad \square$$

Example 2 Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be the $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ -quasiconformal mapping which is radial at 0 given by

$$f(re^{i\theta}) = r \left(\frac{\theta^2}{2} + \theta + 1 \right) e^{i\theta}.$$

Then, for all measurable set $E \subset \mathbb{H}$, the mapping f satisfies

$$\frac{\sqrt{2}-1}{\sqrt{2}+1} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq \frac{\sqrt{2}+1}{\sqrt{2}-1} A_{\mathcal{H}}(E).$$

In the following case we consider the hyperbolic model in the unit disk $\mathbb{D}$.

A mapping $f: \mathbb{D} \rightarrow \mathbb{D}$ is said to be angular at $0 \in \mathbb{D}$ if f leaves invariant each circle in $\mathbb{D}$ with center at 0.

An angular mapping f at 0 can be written as $f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}$, where $\varphi: [0,1) \times [0,2\pi] \rightarrow \mathbb{R}$.

Lemma 2 *Let f be an ACL mapping from $\mathbb{D}$ onto itself. Suppose that f is angular at 0. Then its Jacobian is $J_f = \varphi_\theta$. If f preserves orientation, then $\varphi_\theta > 0$ a.e.*

Proof Since $f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}$, then $u(r,\theta) = r \cos \varphi(r,\theta)$ and $v(r,\theta) = r \sin \varphi(r,\theta)$, and the proof is immediate from (18). $\square$

Proposition 4 *Let f be a K -quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0. Then*

$$\frac{1}{K} \leq \varphi_\theta \leq K \quad \text{a.e. in } [0,1) \times [0,2\pi]. \quad (22)$$

Proof If $f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}$, from (16) and (17) we get

$$\begin{aligned} 4|f_z|^2 &= (1 + \varphi_\theta)^2 + r^2 \varphi_r^2 \quad \text{a.e.} \\ 4|\bar{f}_z|^2 &= (1 - \varphi_\theta)^2 + r^2 \varphi_r^2 \quad \text{a.e.} \end{aligned} \quad (23)$$

Since

$$\frac{(1 - \varphi_\theta)^2}{(1 + \varphi_\theta)^2} \leq \frac{|\bar{f}_z|^2}{|f_z|^2} \leq k^2 \quad \text{a.e.,}$$

we get the result. $\square$

Corollary 3 *Let f be a K -quasiconformal mapping from $\mathbb{D}$ onto itself. Suppose that f is angular at 0. Then, for each $0 < \theta_1 < \theta_2 < 2\pi$ and $r \in (0,1)$, the following holds:*

$$\frac{\theta_2 - \theta_1}{K} \leq \varphi(r, \theta_2) - \varphi(r, \theta_1) \leq K(\theta_2 - \theta_1).$$

Proof It is immediate integrating (22) and applying the continuity of $\varphi(r,\theta)$ and density. Thus, it is enough to prove that $\theta \mapsto \varphi(r,\theta)$, $\theta \in (0,2\pi)$, is absolutely continuous for almost every $r \in (0,1)$.

Let $\Omega_1 = \{z = x + i\theta \in \mathbb{C} : -\infty < x < 0 \text{ and } 0 < \theta < 2\pi\}$ and $\Omega_2 = \{z = x + i\theta \in \mathbb{C} : -\infty < x < 0 \text{ and } \varphi(r,0) < \theta < 2\pi + \varphi(r,0)\}$. Then the mapping $\log \circ f \circ \exp : \Omega_1 \rightarrow \Omega_2$ is K -quasiconformal. Thus the function $(0,2\pi) \ni \theta \mapsto \varphi(e^x, \theta)$ is absolutely continuous for almost every $x \in (-\infty, 0)$, or equivalently $\theta \mapsto \varphi(r, \theta)$, $\theta \in (0,2\pi)$, is absolutely continuous for almost every $r \in (0,1)$. $\square$

Theorem 10 *Let f be a K -quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0. If $E \subset \mathbb{H}$ is a measurable set, then*

$$\frac{1}{K} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq K A_{\mathcal{H}}(E).$$

These inequalities are asymptotically sharp when $K \rightarrow 1^+$.

Proof Let $E \subset \mathbb{D}$ be a measurable set and $\widehat{E}$ denote the set E in polar coordinates. If $f : \mathbb{D} \rightarrow \mathbb{D}$ is given as before by $f(z) = re^{i\varphi(r,\theta)}$, then

$$A_{\mathcal{H}}(f(E)) = \iint_{f(E)} \frac{4 du dv}{(1 - |w|^2)^2} = \iint_E \frac{4 J_f dx dy}{(1 - |f(z)|^2)^2} = \iint_{\widehat{E}} \frac{4 \varphi_\theta r dr d\theta}{(1 - r^2)^2}.$$

By (22) we have

$$\frac{1}{K} \iint_{\widehat{E}} \frac{4r dr d\theta}{(1 - r^2)^2} \leq \iint_{\widehat{E}} \frac{4\varphi_\theta r dr d\theta}{(1 - r^2)^2} \leq K \iint_{\widehat{E}} \frac{4r dr d\theta}{(1 - r^2)^2}$$

or equivalently, in rectangular coordinates,

$$\frac{1}{K} \iint_E \frac{4 dx dy}{(1 - |z|^2)^2} \leq A_{\mathcal{H}}(f(E)) \leq K \iint_E \frac{4 dx dy}{(1 - |z|^2)^2},$$

and this concludes the proof. $\square$

Example 3 Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a mapping given by $f(re^{i\theta}) = re^{i\varphi(r,\theta)}$, where $\varphi(r, \theta) = \theta e^{(\theta-2\pi)r}$. Then by (23)

$$\begin{aligned} \frac{|f_z| + |\bar{f}_z|}{|f_z| - |\bar{f}_z|} &\leq \frac{4|f_z|^2}{|f_z|^2 - |\bar{f}_z|^2} \\ &= \frac{e^{2r(\theta-2\pi)} r^2 \theta^2 (\theta - 2\pi)^2 + (1 + e^{r(\theta-2\pi)} (1 + r\theta))^2}{(1 + r\theta) e^{r(\theta-2\pi)}} \\ &\leq e^{r(\theta-2\pi)} r^2 \theta^2 (\theta - 2\pi)^2 + \frac{1}{e^{r(\theta-2\pi)}} + 2 + (1 + r\theta) e^{r(\theta-2\pi)} \\ &\leq \pi^4 + e^{2\pi r} + 2 + (1 + 2\pi) \\ &\leq \pi^4 + 2\pi + 3 + e^{2\pi}. \end{aligned}$$

Then f is a $\pi^4 + 2\pi + 3 + e^{2\pi}$ -quasiconformal mapping. Thus, for each measurable set E of the unit disk $\mathbb{D}$, the following inequalities hold:

$$\frac{1}{\pi^4 + 2\pi + 3 + e^{2\pi}} A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq (\pi^4 + 2\pi + 3 + e^{2\pi}) A_{\mathcal{H}}(E).$$

Numerical evidence says that the maximal dilatation of f can be $e^{2\pi}$.

The following example shows that the result of Theorem 10 can not be generalized to radial mappings at 0.

Example 4 Let $K \geq 1$. Let $f, g : \mathbb{D} \rightarrow \mathbb{D}$ be the K -quasiconformal mappings given by $f(re^{i\theta}) = r^{\frac{1}{K}} e^{i\theta}$ and $g(re^{i\theta}) = r^K e^{i\theta}$. For each $r \in (0, 1)$, define $E_r = \{z \in \mathbb{D} : |z| < r\}$. Then

$$A_{\mathcal{H}}(E_r) = \frac{4\pi r^2}{1 - r^2}, \quad A_{\mathcal{H}}(f(E_r)) = \frac{4\pi r^{\frac{2}{K}}}{1 - r^{\frac{2}{K}}}, \quad A_{\mathcal{H}}(g(E_r)) = \frac{4\pi r^{2K}}{1 - r^{2K}}.$$

Then if $K > 1$, there is not $C > 0$ such that

$$A_{\mathcal{H}}(f(E_r)) \leq C A_{\mathcal{H}}(E_r) \quad \text{or} \quad \frac{A_{\mathcal{H}}(E_r)}{C} \leq A_{\mathcal{H}}(g(E_r)) \quad \text{for all } r \in (0, 1).$$

2.4 The Beurling-Ahlfors extension

Using the Beurling-Ahlfors (BA) extension, we give explicit examples of quasiconformal mappings of the form $f(x, y) = u(x, y) + iv(x, y)$ and their associated bi-bounds for the hyperbolic area distortion.

More precisely, let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism and define its Beurling-Ahlfors extension $f : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ by $f(x + iy) = u(x, y) + iv(x, y)$, where

$$u(x, y) = \frac{1}{2y} \int_{-y}^y h(x+t) dt, \quad v(x, y) = \frac{1}{2y} \int_0^y (h(x+t) - h(x-t)) dt.$$

Let $M \geq 1$. An increasing homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ is M -quasisymmetric if

$$\frac{1}{M} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq M$$

for all $x \in \mathbb{R}$ and $t > 0$. It is well known that its BA-extension is a $K = K(M) \geq 1$ quasiconformal mapping, even more; Ahlfors proved in [2] that this extension is a quasi-isometry, that is, there exists $0 < C = C(K) < \infty$ such that

$$\frac{1}{Cy^2} \leq \frac{J(z)}{v^2} \leq \frac{C}{y^2} \quad \text{a.e.}$$

Example 5 Let $h(x) = x$. Then h is a 1-quasisymmetric function and its BA-extension is the 2-quasiconformal and harmonic mapping defined by $f(x + iy) = x + \frac{iy}{2}$. Moreover, for each measurable set $E \subset \mathbb{H}$,

$$A_e(f(E)) = \frac{1}{2}A_e(E) \quad \text{and} \quad A_{\mathcal{H}}(f(E)) = 2A_{\mathcal{H}}(E).$$

Example 6 Let $g(x) = x^3$. Then g is a $7 + 4\sqrt{3}$ -quasisymmetric function, and its BA-extension is the 20.7872... quasiconformal mapping $f(x + iy) = x^3 + xy^2 + \frac{i}{4}(6x^2y + y^3)$. In particular $f(x, y)$ does not have bounded derivatives. Moreover,

$$\frac{y^2 J(x + iy)}{v(x, y)^2} = \frac{12(6x^4 - 3x^2y^2 + y^4)}{(6x^2 + y^2)^2}.$$

Setting $y = cx$ in the right-hand side, we get for $x \neq 0$

$$r(c) = \frac{12(6 - 3c^2 + c^4)}{(6 + c^2)^2}$$

and it is easy to see that

$$\frac{3}{4} \leq r(c) < 12 \quad \text{for each } c \in \mathbb{R}.$$

Thus, for each measurable set $E \subset \mathbb{H}$, we have

$$\frac{3}{4}A_{\mathcal{H}}(E) \leq A_{\mathcal{H}}(f(E)) \leq 12A_{\mathcal{H}}(E).$$

Let

$$E = \left\{ (x, y) \in \mathbb{H} : x \in [1, \infty), 0 < y \leq \frac{1}{x^2} \right\}.$$

We have that $A_e(E) = 1$ and, since $J_f(x + iy) = \frac{3}{4}(6x^4 - 3x^2y^2 + y^4)$, it holds $A_e(f(E)) = \infty$. Thus f explodes Euclidean area.

Example 7 Let $g(x + iy) = 2x + \sin(x + y) + iy$. Then g is a $\frac{11+\sqrt{85}}{6}$ -quasiconformal mapping from $\mathbb{H}$ onto itself. Thus the function $h(x) = 2x + \sin x$ is 2.2...-quasisymmetric and its BA-extension is 3.2...-quasiconformal, given by

$$f(x + iy) = 2x + \frac{\sin x \sin y}{y} + \frac{i}{y} \left(y^2 + \cos x - \frac{\cos(x + y)}{2} - \frac{\cos(x - y)}{2} \right).$$

In a forthcoming paper we study more deeply quasi-isometric properties of the BA extension.

2.5 A set that contains the region of values of the partial derivatives of K -quasiconformal mappings

In this part we study some particular forms of the mapping f in (1). First, suppose that f is a K -quasiconformal mapping given by $f(x + iy) = u(x) + iv(y)$. Then, by (4), its partial derivatives satisfy the inequality

$$u_x^2 + v_y^2 - 2\alpha u_x v_y \leq 0 \quad \text{a.e.} \quad (24)$$

Since $\alpha \geq 1$, the discriminant of $u_x^2 + v_y^2 - 2\alpha u_x v_y$ is non-negative and (24) defines the interior of an angular region with the identification $u_x \sim x$ - axis and $v_y \sim y$ - axis. Thus we have proved.

Theorem 11 Let $1 \leq K < \infty$. If $f : \Omega \rightarrow \mathbb{C}$ is a K -quasiconformal mapping given by $f(x + iy) = u(x) + iv(y)$, then its partial derivatives belong to one of the angular regions defined by (24).

Proof The proof follows from the fact that the Jacobian of f is always positive. $\square$

If f is a K -quasiconformal mapping given by $f(x + iy) = u(x, y) + iv(y)$, then (4) reduces to

$$u_x^2 + u_y^2 + v_y^2 - 2\alpha u_x v_y \leq 0 \quad \text{a.e.} \quad (25)$$

Inequality (25) suggests studying the quadratic form $Q(x, y, w) = x^2 + y^2 + w^2 - 2\alpha xw$, whose associated symmetric matrix is

$$N = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ -\alpha & 0 & 1 \end{pmatrix}.$$

Proposition 5 *There exists an invertible matrix P such that $P^{-1}NP = D$, where*

$$D = \begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\alpha \end{pmatrix}.$$

Proof The eigenvalues of N are $\lambda_1 = 1 - \alpha$, $\lambda_2 = 1$ and $\lambda_3 = 1 + \alpha$ with eigenvectors $(1, 0, 1)$, $(0, 1, 0)$ and $(1, 0, -1)$, respectively. After normalization we obtain the basis $\mathcal{B} := \{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (0, 1, 0), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})\}$. Set

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = P^{-1}.$$

A simple calculus ends the proof. $\square$

Corollary 4 *The quadratic form $\widehat{Q}(\widehat{x}, \widehat{y}, \widehat{w}) = (1 - \alpha)\widehat{x}^2 + \widehat{y}^2 + (1 + \alpha)\widehat{w}^2$ represents the quadratic form $Q(x, y, w)$ with basis $\mathcal{B}$, where*

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{w} \end{pmatrix}.$$

In particular $Q(x, y, w) \leq 0$ if and only if $\widehat{Q}(\widehat{x}, \widehat{y}, \widehat{w}) \leq 0$.

The solution of $\widehat{Q}(\widehat{x}, \widehat{y}, \widehat{w}) = 0$ is a double cone if $\alpha > 1$. In fact, $\widehat{Q}(\widehat{x}, \widehat{y}, \widehat{w}) = 0$ if and only if

$$(\alpha - 1)\widehat{x}^2 = \widehat{y}^2 + (1 + \alpha)\widehat{w}^2$$

or equivalently

$$\widehat{x}^2 = \frac{\widehat{y}^2}{(\alpha - 1)} + \frac{(1 + \alpha)}{(\alpha - 1)}\widehat{w}^2$$

and this is the equation of a double elliptic cone in $\mathbb{R}^3$ in the basis $\mathcal{B}$.

Proposition 6 *Let $1 < K < \infty$. If $f : \Omega \rightarrow \mathbb{C}$ is a K -quasiconformal mapping given by $f(x + iy) = u(x, y) + iv(y)$, then its partial derivatives u_x , u_y and v_y belong to one branch of the elliptic cone (25).*

Proof As we saw f is K -quasiconformal if and only if u_x , u_y and v_y satisfy $Q(u_x, u_y, v_y) \leq 0$ that describes the solid cone (25). As f preserves orientation, then $J_f = u_x v_y > 0$ a.e. Since $v_y > 0$ a.e., then necessarily $u_x > 0$ a.e., and the result follows. $\square$

We do not study the case $f(x + iy) = u(x, y) + iv(x)$ because this kind of mapping is not a homeomorphism from $\mathbb{H}$ onto itself and in consequence is not quasiconformal.

We consider now the general case, that is, a quasiconformal mapping $f : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$ given by $f(x + iy) = u(x, y) + iv(x, y)$, then by (4) we have that

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 - 2\alpha u_x v_y + 2\alpha v_x u_y \leq 0 \quad \text{a.e.}$$

In this case we study the quadratic form $Q(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz$ with the associated symmetric matrix

$$N = \begin{pmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ -\alpha & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 7 *There exists an invertible matrix P such that $P^{-1}NP = D$, where*

$$D = \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 \\ 0 & 0 & 0 & 1+\alpha \end{pmatrix}.$$

Proof The characteristic polynomial of the matrix N is $(1-\lambda)^4 - \alpha^2(1-\lambda)^2 - \alpha^2(1-\lambda)^2 + \alpha^4$ with eigenvalues $\lambda_1 = 1 + \alpha$, $\lambda_2 = 1 - \alpha$, and both with multiplicity two. The eigenvectors of λ_1 are $(1, 0, 0, -1)$ and $(0, 1, 1, 0)$ and for λ_2 are $(1, 0, 0, 1)$ and $(0, 1, -1, 0)$. After normalization we obtain the matrix

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

with inverse

$$P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Thus

$$P^{-1}NP = D.$$

□

Corollary 5 *The quadratic form*

$$\widehat{Q}(\widehat{x}, \widehat{y}, \widehat{z}, \widehat{w}) = (1-\alpha)\widehat{x}^2 + (1-\alpha)\widehat{y}^2 + (1+\alpha)\widehat{z}^2 + (1+\alpha)\widehat{w}^2$$

represents the quadratic form

$$Q(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz$$

in the basis

$$C = \left\{ \left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}} \right), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}} \right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right\},$$

where

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ \hat{w} \end{pmatrix}.$$

In particular $Q(x, y, z, w) \leq 0$ if and only if $\widehat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) \leq 0$.

Proof We have the relations

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{x} + \hat{z}) \\ \frac{1}{\sqrt{2}}(\hat{y} + \hat{w}) \\ \frac{1}{\sqrt{2}}(\hat{w} - \hat{y}) \\ \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}) \end{pmatrix}.$$

Thus

$$\begin{aligned} Q(x, y, z, w) &= x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz \\ &= \frac{1}{2}(\hat{x} + \hat{z})^2 + \frac{1}{2}(\hat{y} + \hat{w})^2 + \frac{1}{2}(\hat{w} - \hat{y})^2 + \frac{1}{2}(\hat{x} - \hat{z})^2 \\ &\quad - 2\alpha \left[\frac{1}{\sqrt{2}}(\hat{x} + \hat{z}) \right] \left[\frac{1}{\sqrt{2}}(\hat{x} - \hat{z}) \right] + 2\alpha \left[\frac{1}{\sqrt{2}}(\hat{w} + \hat{y}) \right] \left[\frac{1}{\sqrt{2}}(\hat{w} - \hat{y}) \right] \\ &= \frac{1}{2}[\hat{x}^2 + 2\hat{x}\hat{z} + \hat{z}^2] + \frac{1}{2}[\hat{y}^2 + 2\hat{y}\hat{w} + \hat{w}^2] \\ &\quad + \frac{1}{2}[\hat{x}^2 - 2\hat{x}\hat{z} + \hat{z}^2] + \frac{1}{2}[\hat{w}^2 - 2\hat{w}\hat{y} + \hat{y}^2] \\ &\quad - \alpha[\hat{x}^2 - \hat{z}^2] + \alpha[\hat{w}^2 - \hat{y}^2] \\ &= (1 - \alpha)\hat{x}^2 + (1 - \alpha)\hat{y}^2 + (1 + \alpha)\hat{z}^2 + (1 + \alpha)\hat{w}^2 \\ &= \widehat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w}); \end{aligned}$$

and this means that $Q(x, y, z, w) \leq 0$ if and only if $\widehat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) \leq 0$. $\square$

Proposition 8 Let $1 \leq K < \infty$. Let $f : \Omega \rightarrow \mathbb{C}$ be a K -quasiconformal mapping given by $f(x + iy) = u(x, y) + iv(x, y)$. Then its partial derivatives u_x , u_y and v_x , v_y belong to the solid bounded by $Q(x, y, z, w) = \widehat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = 0$.

3 Conclusions

The classes of mappings introduced in this paper have precise geometrical meaning, in particular, the class of quasiconformal mappings $f(z) = u(x, y) + iv(y)$; see, for example, Proposition 1.

As it is showed in Theorems 3, 6, 8, 9, 10 and Corollaries 1 and 2, we have obtained left and right asymptotic bounds for the hyperbolic or Euclidean area distortion. In some previous results only right bounds were known. Moreover, the examples showed that the different classes of mappings defined in the paper are not empty.

We do not know whether the branch of the elliptic cone (25), mentioned in Proposition 6, coincides or not with the region of variation of the partial derivatives of quasiconformal mappings $f(z) = u(x, y) + iv(y)$.

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The material of this paper belongs to the theory of plane quasiconformal mappings. The omnipresence of double inequalities associated to fundamental facts is well known in this theory, for example:

- (a) modulus of normalized quasiconformal mappings from the unit disk onto itself;
- (b) quasi-isometry of the Beurling-Ahlfors extension;
- (c) bi-Lipschitz inequalities for quasiconformal harmonic mappings; etc.

In this paper we study the hyperbolic or Euclidean area distortion under certain classes of quasiconformal mappings, defined in the upper half-plane or the unit disk. We found left and right bounds for the mentioned distortions. In particular, we are generalizing the results obtained for harmonic quasiconformal mappings from the upper half-plane onto itself (see Chen and Chen (*K, K'*)-quasiconformal harmonic mappings of the upper half plane onto itself Ann. Aca. Scien. Fenn. pp.265-276 (2012)). We also give a straightforward application of a recent result to this topic (see Knežević and Mateljević *On the quasi-isometries of harmonic quasiconformal mappings* Journal of Mathematical Analysis and Applications, pp.404-413 (2007)). Of course the generality of results is very desirable in mathematics; however, some particular cases have their own interest.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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References

1. Beurling, A, Ahlfors, LV: The boundary correspondence under quasiconformal mappings. *Acta Math.* **96**, 125-142 (1956)
2. Ahlfors, LV: Quasiconformal reflections. *Acta Math.* **109**, 291-301 (1963)
3. Astala, K: Area distortion of quasiconformal mappings. *Acta Math.* **173**, 37-60 (1994)
4. Porter, RM, Reséndis, LF: Quasiconformally explodable sets. *Complex Var. Theory Appl.* **36**, 379-392 (1998)
5. Emerenko, A, Hamilton, DH: On the area distortion by quasiconformal mappings. *Proc. Am. Math. Soc.* **123**, 2793-2797 (1995)
6. Chen, X, Qian, T: Estimate of hyperbolically partial derivatives of p -harmonic quasiconformal mappings and its applications. *Complex Var. Elliptic Equ.* **60**, 875-892 (2015)
7. Dongmian, F, Xinzhong, H: Harmonic K -quasiconformal mappings from the unit disk onto half planes. *Bull. Malays. Math. Soc.* **39**(1), 339-347 (2016)
8. Kalaj, D, Mateljević, M: Quasiconformal harmonic mappings and generalizations. In: *Proceedings of the ICM2010 Satellite Conference International Workshop on Harmonic and Quasiconformal Mappings (HQM2010)*, vol. 18, pp. 239-260 (2010)
9. Partyka, D, Sakan, K: On a asymptotically sharp variant of Heinz's inequality. *Ann. Acad. Sci. Fenn., Math.* **30**, 167-182 (2005)
10. Knežević, M, Mateljević, M: On the quasi-isometries of harmonic quasiconformal mappings. *J. Math. Anal. Appl.* **334**(1), 404-413 (2007)

11. Chen, M, Chen, X: (K, K') -quasiconformal harmonic mappings of the upper half plane onto itself. *Ann. Acad. Sci. Fenn., Math.* **37**, 265-276 (2012)
12. Axler, S Bourdon, P, Ramey, W: *Harmonic Function Theory*, pp. 1-259. Springer, New York (2001)
13. Anderson, J: *Hyperbolic Geometry*, pp. 1-230. Springer, London (2003)
14. Beardon, A: *The Geometry of Discrete Groups*, pp. 1-338. Springer, New York (1983)

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