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Approximation of the multiplicatives on random multi-normed space

Ravi P Agarwal[1], Reza Saadati[2*] and Ali Salamati[2]

*Correspondence: rsaadati@eml.cc
[2]Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, we consider random multi-normed spaces introduced by Dales and Polyakov (Multi-Normed Spaces, 2012). Next, by the fixed point method, we approximate the multiplicatives on these spaces.

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1 Introduction

The concept of random normed spaces and their properties are discussed in [2]. Also, the concept of multi-normed spaces was introduced by Dales and Polyakov. In this paper we combine the mentioned concepts and introduce random multi-normed spaces. Next, we get an approximation for homomorphisms in these spaces. For more results and applications, one can see [3–23].

Definition 1.1 Let $(E,\mu,*)$ be a random normed space. $*$ is a continuous t-norm. A multi-random norm on $\{E^k, k\in\mathbb{N}\}$ is sequence $\{N_k\}$ such that N_k is a random norm on E^k $(k\in\mathbb{N})$, $\mu_x^1(t)=\mu_x(t)$ for each $x\in E$ and $t\in\mathbb{R}$ and the following axioms are satisfied for each $k\in\mathbb{N}$ with $k\geq 2$:

(NF1) $\mu^k_{A_\sigma(x)}(t)=\mu^k_x(t)$, for each $\sigma\in\sigma_k$, $x\in E^k$, $t\in\mathbb{R}$,

(NF2) $\mu^k_{M_\alpha(x)}(t)\geq\mu^k_{\max_{i\in\mathbb{N}_k}|\alpha_i|x}(t)$, for each $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k$, $x\in E^k$, $t\in\mathbb{R}$,

(NF3) $\mu^{k+1}_{(x_1,\dots,x_k,0)}(t)=\mu^k_{(x_1,\dots,x_k)}(t)$, for each $x_1,\dots,x_k\in E$ and $t\in\mathbb{R}$,

(NF4) $\mu^{k+1}_{(x_1,\dots,x_k,x_k)}(t)=\mu^k_{(x_1,\dots,x_k)}(t)$, for each $x_1,\dots,x_k\in E$ and $t\in\mathbb{R}$.

In this case $\{(E^k,\mu^k,*),k\in\mathbb{N}\}$ is called a random multi-normed space. Moreover, if axiom (NF4) is replaced by the following axiom:

(DF4) $\mu^{k+1}_{(x_1,\dots,x_k,x_k)}(t)=\mu^k_{(x_1,\dots,2x_k)}(t)$, for each $x_1,\dots,x_k\in E$ and $t\in\mathbb{R}$,

then $\{\mu^k\}$ is called a dual random multi-normed and $\{(E^k,\mu^k,*),k\in\mathbb{N}\}$ is called a dual random multi-normed space.

2 Approximation of the multiplicatives

We apply fixed point theory [24] to get an approximation for multiplicatives. A metric d on non-empty set Υ with range $[0,\infty]$ is called a generalized metric.

Lemma 2.1 ([25, 26]) *Let $k \in \mathbb{N}$, and let E and F be linear spaces such that $(F^k, \mu^k, *)$ is a complete random multi-normed space. Let there exist $0 \le M < 1$, $\lambda > 0$, and a function $\psi : E^k \longrightarrow [0,\infty)$ such that*

$$\psi(\lambda x_1, \ldots, \lambda x_k) \le \lambda M \psi(x_1, \ldots, x_k) \quad (x_1, \ldots, x_k \in E). \tag{2.1}$$

We set $\Upsilon := \{\eta : E \longrightarrow F : \eta(0) = 0\}$, and define $d : \Upsilon \times \Upsilon$ on $[0,\infty]$ by

$$\begin{aligned} &d(\eta, \zeta) \\ &\quad = \inf\left\{c > 0 : \mu_{(\eta(x_1)-\zeta(x_1),\ldots,\eta(x_k)-\zeta(x_k))}(ct) \ge \frac{t}{t + \psi(x_1, \ldots, x_k)}, x_1, \ldots, x_k \in E\right\}. \end{aligned}$$

Then (Υ, d) is a complete generalized metric space, and the mapping $J : \Upsilon \longrightarrow \Upsilon$ defined by $(Jg)(x) := \frac{g(\lambda x)}{\lambda}$ $(x \in \Upsilon)$ is a strictly contractive mapping.

Theorem 2.2 *Let E be a linear space and let $((F^n, \mu^n, *) : n \in \mathbb{N})$ be a complete random multi-normed space. Let $k \in \mathbb{N}$ and let there exist $0 \le M_0 < 1$ and a function $\varphi : E^{2k} \longrightarrow [0,\infty)$ satisfying*

$$\varphi(2x_1, 2y_1, \ldots, 2x_k, 2y_k) \le 2M_0 \varphi(x_1, y_1, \ldots, x_k, y_k) \tag{2.2}$$

for all $x_1, y_1, \ldots, x_k, y_k \in E$. Suppose that $f : E \longrightarrow F$ is a mapping with $f(0) = 0$ and

$$\begin{aligned} &\mu^k_{(f(\lambda x_1+\lambda y_1)-\lambda f(x_1)-\lambda f(y_1),\ldots,f(\lambda x_k+\lambda y_k)-\lambda f(x_k)-\lambda f(y_k))}(t) \\ &\quad \ge \frac{t}{t + \varphi(x_1, y_1, \ldots, x_k, y_k)}, \end{aligned} \tag{2.3}$$

$$\mu^k_{(f(x_1y_1)-f(x_1)f(y_1),\ldots,f(x_ky_k)-f(x_k)f(y_k))}(t) \ge \frac{t}{t + \varphi(x_1, y_1, \ldots, x_k, y_k)}, \tag{2.4}$$

for all $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $x_1, y_1, \ldots, x_k, y_k \in E$, $t > 0$.

Then

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.5}$$

exists for any $x_1, \ldots, x_k \in E$ and defines a random homomorphism $H : E \longrightarrow F$ such that

$$\mu_{(f(x_1)-H(x_1),\ldots,f(x_k)-H(x_k))}(t) \ge \frac{(1 - M_0)t}{(1 - M_0)t + M_0 \psi(x_1, \ldots, x_k)}, \tag{2.6}$$

$$\psi(x_1, \ldots, x_k) = \varphi\left(\frac{x_1}{2}, \frac{x_1}{2}, \ldots, \frac{x_k}{2}, \frac{x_k}{2}\right), \tag{2.7}$$

for all $x_1, \ldots, x_k \in E$ and $t > 0$.

Proof Let $x_1 = \frac{x_1}{2}, \ldots, x_k = \frac{x_k}{2}$, $y_1 = \frac{y_1}{2}, \ldots, y_k = \frac{y_k}{2}$ in (2.2). We get

$$\varphi(x_1, y_1, \ldots, x_k, y_k) \le 2M_0 \varphi\left(\frac{x_1}{2}, \frac{y_1}{2}, \ldots, \frac{x_k}{2}, \frac{y_k}{2}\right), \tag{2.8}$$

since f is odd, $f(0) = 0$. So $\mu_{f(0)}(\frac{t}{2}) = 1$. Letting $\lambda = 1$ and $y = x$, we get

$$\mu^k_{(f(2x_1)-2f(x_1),\ldots,f(2x_k)-2f(x_k))}(t) \geq \frac{t}{t + \varphi(x_1, x_1, \ldots, x_k, x_k)} \tag{2.9}$$

for all $x_1, y_1, \ldots, x_k, y_k \in E$. Consider the following set:

$$s =: \{g : E \longrightarrow F\}$$

and introduce the generalized metric on s:

$$\begin{aligned} &d(g,h) \\ &\quad = \inf\left\{\nu \in \mathbb{R}_+ : \mu^k_{(g(x_1)-h(x_1),\ldots,g(x_k)-h(x_k))}(\nu t) \geq \frac{t}{t + \varphi(x_1, \ldots, x_k)}, x_1, \ldots, x_k \in E, t > 0\right\}, \end{aligned}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show (s, d) is complete. Now, we consider the linear mapping $J : s \longrightarrow s$ such that

$$J\bigl(g(x)\bigr) := 2g\left(\frac{x}{2}\right)$$

for all $x \in E$. Let $g, h \in s$ be given such that $d(g,h) = \varepsilon$. Then we have

$$\mu^k_{(g(x_1)-h(x_1),\ldots,g(x_k)-h(x_k))}(\varepsilon t) \geq \frac{t}{t + \varphi(x_1, x_1, \ldots, x_k, x_k)},$$

for all $x_1, \ldots, x_k \in E$ and all $t > 0$ and hence we have

$$\begin{aligned} \mu^k_{(Jg(x_1)-Jh(x_1),\ldots,Jg(x_k)-Jh(x_k))}(M_0\varepsilon t) &= \mu^k_{2g(\frac{x_1}{2})-2h(\frac{x_1}{2}),\ldots,2g(\frac{x_k}{2})-2h(\frac{x_k}{2}))}(M_0\varepsilon t) \\ &= \mu^k_{g(\frac{x_1}{2})-h(\frac{x_1}{2}),\ldots,g(\frac{x_k}{2})-h(\frac{x_k}{2}))}\left(\frac{M_0}{2}\varepsilon t\right) \\ &\geq \frac{\frac{M_0}{2}t}{\frac{M_0}{2} + \varphi(\frac{x_1}{2}, \frac{x_1}{2}, \ldots, \frac{x_k}{2}, \frac{x_k}{2})} \\ &\geq \frac{\frac{M_0}{2}t}{\frac{M_0}{2} + \frac{M_0}{2}\varphi(x_1, x_1, \ldots, x_k, x_k)} \\ &= \frac{t}{t + \varphi(x_1, x_1, \ldots, x_k, x_k)} \end{aligned}$$

for all $x_1, \ldots, x_k \in E$ and $t > 0$. Then $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq M_0\varepsilon$. This means that

$$d(Jg, Jh) \leq M_0\varepsilon$$

for all $g, h \in s$. It follows that

$$\mu_{(f(x_1)-2f(\frac{x_1}{2}),\ldots,f(x_k)-2f(\frac{x_k}{2}))}\left(\frac{M_0}{2}t\right) \geq \frac{t}{t + \varphi(x_1, x_1, \ldots, x_k, x_k)}$$

for all $x_1, \ldots, x_k \in E$ and $t > 0$. So $d(f, Jf) \leq \frac{M_0}{2}$.

Now, there exists a mapping $H : E \longrightarrow F$ satisfying the following:

(1) H is a fixed point of J, *i.e.*,

$$H\left(\frac{x}{2}\right) = \frac{1}{2}H(x) \tag{2.10}$$

for all $x \in E$. Since $f : E \longrightarrow E$ is odd, $H : E \longrightarrow F$ is an odd mapping. The mapping H is a unique fixed point of J in the set

$$M = \{g \in s : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.10) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu^k_{(f(x_1)-H(x_1),\dots,f(x_k)-H(x_k))}(\nu t) \geq \frac{t}{t + \varphi(x_1,\dots,x_k)}$$

for all $x_1,\dots,x_k \in E$,

(2) $d(J^n f,H) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2_n}\right) = H(x)$$

for all $x \in E$,

(3) $d(f,H) \leq \frac{1}{1-M_0} d(f,Jf)$, which implies

$$d(f,H) \leq \frac{M_0}{2-2M_0}.$$

Put $\lambda = 1$ in (2.3). Then

$$\begin{aligned}&\mu^k_{(2^n(f(\frac{x_1}{2^n}+\frac{y_1}{2^n})-f(\frac{x_1}{2^n})-f(\frac{y_1}{2^n})),\dots,2^n(f(\frac{x_k}{2^n}+\frac{y_k}{2^n})-f(\frac{x_k}{2^n})-f(\frac{y_k}{2^n})))}(t)\\ &\quad\geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{M_0^n}{2^n}\varphi(x_1,y_1,\dots,x_k,y_k)}\end{aligned}$$

for all $x_1,\dots,x_k,y_1,\dots,y_k \in E$, $t > 0$ and $n \geq 1$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{M_0^n}{2^n}\varphi(x_1,y_1,\dots,x_k,y_k)} = 1$$

for all $x_1,\dots,x_k,y_1,\dots,y_k \in E$, $t > 0$. It follows that

$$\mu^k_{(H(x_1+y_1)-H(x_1)-H(y_1),\dots,H(x_k+y_k)-H(x_k)-H(y_k))}(t) = 1$$

for all $x_1,\dots,x_k,y_1,\dots,y_k \in E$, $t > 0$. So mapping $H : E \longrightarrow F$ is Cauchy additive.

Let $y_1 = x_1,\dots,y_k = x_k$ in (2.3). Then we have

$$\mu^k_{2^n(f(\frac{\beta x_1}{2^n})-f(\frac{\beta x_1}{2^n}),\dots,f(\frac{\beta x_k}{2^n})-f(\frac{\beta x_k}{2^n}))}(2^n t) \geq \frac{t}{t + \varphi(\frac{x_1}{2^n},\frac{x_1}{2^n},\dots,\frac{x_k}{2^n},\frac{x_k}{2^n})}$$

for all $\lambda, \beta \in \mathbb{T}$, $\lambda = \frac{\beta}{2}$, $x_1, \ldots, x_k \in E$, $t > 0$ and $n \geq 1$. So we have

$$\mu^k_{2^n(f(\frac{\beta x_1}{2^n})-f(\frac{\beta x_1}{2^n}),\ldots,f(\frac{\beta x_k}{2^n})-f(\frac{\beta x_k}{2^n}))}(t) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{M_0^n}{2^n}\varphi(x_1, x_1, \ldots, x_k, x_k)}$$

for all $\beta \in \mathbb{T}$, $x_1, \ldots, x_k \in E$, $t > 0$ and $n \geq 1$. We have

$$\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{M_0^n}{2^n}\varphi(x_1, x_1, \ldots, x_k, x_k)} = 1$$

for all $x_1, \ldots, x_k \in E$, $t > 0$, and

$$\mu^k_{(H(\beta x_1)-\beta H(x_1),\ldots,H(\beta x_k)-\beta H(x_k))}(t) = 1$$

for all $\beta \in \mathbb{T}$, $x_1, \ldots, x_k \in E$, $t > 0$. Thus, the additive mapping $H : E \longrightarrow F$ is $\mathbb{R}$-linear. From (2.4), we have

$$\begin{aligned}
&\mu^k_{(4^n f(\frac{x_1}{2^n}\frac{y_1}{2^n})-2^n f(\frac{x_1}{2^n})2^n f(\frac{y_1}{2^n}),\ldots,4^n f(\frac{x_k}{2^n}\frac{y_k}{2^n})-2^n f(\frac{x_k}{2^n})2^n f(\frac{y_k}{2^n}))}\bigl(4^n t\bigr) \\
&\quad\geq \frac{t}{t + \varphi(\frac{x_1}{2^n}, \ldots, \frac{x_k}{2^n}, \frac{y_1}{2^n}, \ldots, \frac{y_k}{2^n})}
\end{aligned}$$

for all $x_1, \ldots, x_k \in E$, $t > 0$ and $n \geq 1$.

Then we have

$$\begin{aligned}
&\mu^k_{(4^n f(\frac{x_1}{2^n}\frac{y_1}{2^n})-2^n f(\frac{x_1}{2^n})2^n f(\frac{y_1}{2^n}),\ldots,4^n f(\frac{x_k}{2^n}\frac{y_k}{2^n})-2^n f(\frac{x_k}{2^n})2^n f(\frac{y_k}{2^n}))}\bigl(4^n t\bigr) \\
&\quad\geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{M_0^n}{t^n}\varphi(x_1, \ldots, x_k, y_1, \ldots, y_k)}
\end{aligned}$$

for all $x_1, \ldots, x_k \in E$, $t > 0$ and $n \geq 1$.

Since

$$\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{M_0^n}{t^n}\varphi(x_1, \ldots, x_k, y_1, \ldots, y_k)} = 1$$

for all $x_1, \ldots, x_k \in E$, $t > 0$, we have

$$\mu^k_{(H(x_1y_1)-H(x_1)H(y_1),\ldots,H(x_ky_k)-H(x_k)H(y_k))}(t) = 1$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$, $t > 0$. Thus, the mapping $H : E \longrightarrow F$ is multiplicative. Therefore, there exists a unique random homomorphism $H : E \longrightarrow F$ satisfying (2.6), and this completes the proof. □

3 Approximation in dual random multi-normed space

The following lemma is an immediate result of the definition of random multi-normed space.

Lemma 3.1 *Let $\{(E^k,\mu^k,*),k\in\mathbb{N}\}$ be a dual random multi-normed space, $k,n\in\mathbb{N}$, $x_1,x_2,\dots,x_k,x_{k+1},\dots,x_{k+n}\in\mathbb{E}$ and $\lambda_1,\dots,\lambda_k$ be real numbers of absolute value 1. Then we have:*

(i) $\mu^k_{(\lambda_1x_1,\dots,\lambda_kx_k)}(t)=\mu^k_{(x_1,\dots,x_k)}(t)$,
(ii) $\mu^k_{(x_1,\dots,x_k)}(t)\geq\mu^{k+1}_{(x_1,\dots,x_k,x_{k+1})}(t)$,
(iii) $\mu^{k+n}_{(x_1,\dots,x_k,x_{k+1},\dots,x_{k+n})}(t)\geq T_M(\mu^k_{(x_1,\dots,x_k)}(\alpha t),\mu^n_{(x_{k+1},\dots,x_{k+n})}(\beta t))$, *where $\alpha,\beta\geq 0$ and* $\alpha+\beta=1$,
(iv) $\min_{i\in\mathbb{N}_k}\mu_{x_i}(t)\geq\mu^k_{(x_1,\dots,x_k)}(t)\geq\min_{i\in\mathbb{N}_k}\mu_{x_i}(\alpha_i t)$,

where $\alpha_1,\dots,\alpha_k\geq 0$ and $\sum_{i=1}^k\alpha_i=1$. In particular, we have

$$\mu^k_{(x_1,\dots,x_k)}(t)\geq\min_{i\in\mathbb{N}_k}\mu_{kx_i}(t).$$

Theorem 3.2 *Let E be a linear space, and $\{(E^k,\mu^k,*),k\in\mathbb{N}\}$ be a random multi space. Let $\alpha\in(0,1)$ and $f:E\longrightarrow F$ is a mapping satisfying $f(0)=0$ and*

$$\mu^k_{(f(\frac{x_1+y_1}{2})-\frac{f(x_1)}{2}-\frac{f(y_1)}{2},\dots,f(\frac{x_k+y_k}{2})-\frac{f(x_k)}{2}-\frac{f(y_k)}{2})}\left(\frac{t}{s}\right)\geq 1-\frac{\alpha}{t},\tag{3.1}$$

where $x_1,\dots,x_k,y_1,\dots,y_k\in E$ and $t,s\in\mathbb{N}$ with the greatest common divisor $(t,s)=1$.

Then there exists a unique additive mapping $T:E\longrightarrow F$ such that

$$\mu^k_{(f(x_1)-T(x_1),\dots,f(x_k)-T(x_k))}\left(\frac{2t}{s}\right)\geq 1-\frac{\alpha}{t}\tag{3.2}$$

for all $x_1,\dots,x_k\in E$ and $t,s\in\mathbb{N}$ with $(t,s)=1$.

Proof Replacing $x_1,\dots,x_k$ and $y_1,\dots,y_k$ by $2x_1,\dots,2x_k$ and $0,\dots,0$ in (3.1), respectively, yields

$$\mu^k_{(2f(x_1)-f(2x_1),\dots,2f(x_k)-f(2x_k))}\left(\frac{2t}{s}\right)\geq 1-\frac{\alpha}{t}.\tag{3.3}$$

Replacing $x_1,\dots,x_k$, t, s by $2x_1,\dots,2x_k$, $2t$, $2s$, respectively, in (3.3) and repeating this process for n-time ($n\in\mathbb{N}$), it follows that

$$\mu^k_{(\frac{f(2^{n-1}x_1)}{2^{n-1}}-\frac{f(2^nx_1)}{2^n},\dots,\frac{f(2^{n-1}x_k)}{2^{n-1}}-\frac{f(2^nx_k)}{2^n})}\left(\frac{t}{2^{n-1}s}\right)\geq 1-\frac{\alpha}{2^{n-1}t}\tag{3.4}$$

for $n,m\in\mathbb{N}$ with $n>m$. Using (3.4) and (RN2) we get

$$\mu^k_{(\frac{f(2^mx_1)}{2^m}-\frac{f(2^nx_1)}{2^n},\dots,\frac{f(2^mx_k)}{2^m}-\frac{f(2^nx_k)}{2^n})}\left(\sum_{i=m}^{n-1}2^{-i}\frac{t}{s}\right)\geq 1-\frac{\alpha}{2^mt}.$$

Then

$$\mu^k_{(\frac{f(2^mx_1)}{2^m}-\frac{f(2^nx_1)}{2^n},\dots,\frac{f(2^mx_k)}{2^m}-\frac{f(2^nx_k)}{2^n})}\left(\frac{t}{s}\right)\geq 1-\frac{\alpha}{2^mt}\tag{3.5}$$

for $x \in E$. Then, replacing $x_1, \ldots, x_k$ by $x, 2x, \ldots, 2^{k-1}x$ in (3.5), we have

$$\begin{aligned}
&\mu^k_{(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \ldots, \frac{f(2^{m+k-1}x)}{2^{m+k-1}} - \frac{f(2^{n+k-1}x)}{2^{n+k-1}})}\left(\frac{t}{s}\right) \\
&\quad \geq 1 - \frac{\alpha}{2^m t} \\
&\quad \geq 1 - \frac{\alpha}{2^m}. \qquad (3.6)
\end{aligned}$$

Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\alpha}{2^{n_0}} < \varepsilon$. Now we substitute m, n with n, $n+p$ $(p \in \mathbb{N})$, respectively, in (3.6), for each $n \geq n_0$, and we get

$$\begin{aligned}
\mu^k_{(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p} x)}{2^{n+p}}, \ldots, \frac{f(2^{n+k-1}x)}{2^{n+k-1}} - \frac{f(2^{n+p+k-1}x)}{2^{n+p+k-1}})}\left(\frac{t}{s}\right) &\geq 1 - \frac{\alpha}{2^n t} \\
&> 1 - \varepsilon.
\end{aligned}$$

By Lemma 3.1, we have

$$\mu_{\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p} x)}{2^{n+p}}}\left(\frac{t}{s}\right) > 1 - \varepsilon \qquad (3.7)$$

for all $n > n_0$ and $p \in \mathbb{N}$. The density of rational numbers in $\mathbb{R}$ is useful in checking correctness of (3.6) with positive real number r instead of $\frac{t}{s}$. Then we have

$$\mu_{\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p} x)}{2^{n+p}}}(r) > 1 - \varepsilon$$

for each $x \in E$, $r \in \mathbb{R}^+$, $n \geq n_0$ and $p \in \mathbb{N}$. Then $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence, so it is convergent in the random multi-Banach space $\{(E^k, \mu^k, *), k \in \mathbb{N}\}$. Setting $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ and applying again Lemma 3.1, for each $r > 0$, we have

$$\mu^k_{(\frac{f(2^n x_1)}{2^n} - T(x_1), \ldots, \frac{f(2^n x_k)}{2^n} - T(x_k))}(r) \geq \min_{i \in \mathbb{N}_k} \mu_{\frac{f(2^n x_i)}{2^n} - T(x_i)}\left(\frac{r}{k}\right),$$

and

$$\lim_{n\to\infty} \frac{f(2^n x_k)}{2^n} = T(x_k).$$

We put $m = 0$ in (3.5), and we get

$$\mu^k_{(f(x_1) - \frac{f(2^n x_1)}{2^n}, \ldots, f(x_k) - \frac{f(2^n x_k)}{2^n})}\left(\frac{t}{s}\right) \geq 1 - \frac{\alpha}{t}. \qquad (3.8)$$

Then

$$\begin{aligned}
&\mu^k_{f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)}\left(\frac{2t}{s}\right) \\
&\quad \geq T_M\left(\mu^k_{(f(x_1) - \frac{f(2^n x_1)}{2^n}, \ldots, f(x_k) - \frac{f(2^n x_k)}{2^n})}\left(\frac{t}{s}\right),\right.
\end{aligned}$$

$$\begin{aligned}
&\mu^k_{(\frac{f(2^n x_1)}{2^n}-T(x_1),\ldots,\frac{f(2^n x_k)}{2^n}-T(x_k))}\left(\frac{t}{s}\right)\Big) \\
&\geq 1-\frac{\alpha}{t}
\end{aligned} \tag{3.9}$$

by (3.8) and when $n \to \infty$, which implies that (3.2).

Now, we show that T is additive. Let $x, y \in E$ and replace $x_1, \ldots, x_k$ by $2^n x$, $y_1, \ldots, y_k$ by $2^n y$, and t by $2^n t$ in (3.1). We get

$$\mu^k_{(f(2^n \frac{x+y}{2})-\frac{f(2^n x)}{2}-\frac{f(2^n y)}{2},\ldots,f(2^n \frac{x+y}{2})-\frac{f(2^n x)}{2}-\frac{f(2^n y)}{2})}\left(\frac{2^n t}{s}\right) \geq 1-\frac{\alpha}{2^n t}.$$

Using (NF4), we conclude that

$$\mu_{\frac{f 2^n(\frac{x+y}{2})}{2^n}-\frac{1}{2}\frac{f(2^n x)}{2^n}-\frac{1}{2}\frac{f(2^n y)}{2^n}}\left(\frac{t}{s}\right) \geq 1-\frac{\alpha}{2^n t}. \tag{3.10}$$

On the other hand, we obtain that

$$\begin{aligned}
\mu_{T(\frac{x+y}{2})-\frac{1}{2}T(x)-\frac{1}{2}T(y)}\left(\frac{4t}{s}\right) \geq{}& T_M\Big(\mu_{T(\frac{x+y}{2})-\frac{f(2^n(\frac{x+y}{2}))}{2^n}}\left(\frac{t}{s}\right), \\
&\mu_{\frac{T(x)}{2}-\frac{1}{2}\frac{f(2^n x)}{2^n}}\left(\frac{t}{s}\right), \\
&\mu_{\frac{T(y)}{2}-\frac{1}{2}\frac{f(2^n y)}{2^n}}\left(\frac{t}{s}\right), \\
&\mu_{\frac{f(2^n(\frac{x+y}{2}))}{2^n}-\frac{1}{2}\frac{f(2^n)}{2^n x}-\frac{1}{2}\frac{f(2^n y)}{2^n}}\left(\frac{t}{s}\right)\Big) \\
\geq{}& 1-\frac{\alpha}{2^n}
\end{aligned} \tag{3.11}$$

for each $x, y \in E$, $t, s \in \mathbb{N}$ with $(t, s) = 1$. Utilizing again the density of $\mathbb{Q}$ in $\mathbb{R}$, we find that (3.11) remains true if $\frac{4t}{s}$ is substituted with a positive real number r.

Consequently,

$$\mu_{T(\frac{x+y}{2})-\frac{1}{2}T(x)-\frac{1}{2}T(y)}(r) \geq 1-\frac{\alpha}{2^n}$$

for each $x, y \in E$ and $r \in \mathbb{R}$. Letting $n \to \infty$ reveals that T complies with Jensen, and using the fact that $T(0) = 0$, we conclude that T is additive [27, Theorem 6].

It remains to show the uniqueness of T. Suppose that T' is another additive mapping satisfying (3.2). Then, for each $t, s \in \mathbb{N}$, sufficiently large n in $\mathbb{N}$ and $x \in E$,

$$\begin{aligned}
\mu_{T'(x)-T(x)}\left(\frac{t}{s}\right) &= \mu_{\frac{T'(2^n x)}{2^n}-\frac{T(2^n x)}{2^n}}\left(\frac{t}{s}\right) \\
&\geq T_M\left(\mu_{T'(2^n x)-f(2^n x)}\left(\frac{2^{n-1}t}{s}\right), \mu_{T(2^n x)-f(2^n x)}\left(\frac{2^{n-1}t}{s}\right)\right) \\
&\geq 1-\frac{\alpha}{2^{n-2}t} \\
&\geq 1-\frac{\alpha}{2^{n-2}}.
\end{aligned}$$

This inequality holds for each $r \in \mathbb{R}^{+}$ instead of $\frac{t}{s}$, too. Therefore, for each $r \in \mathbb{R}^{+}$, $n \in \mathbb{N}$, $\mu_{T'(x)-T(x)}(r) \geq 1 - \frac{\alpha}{2^{n-2}}$, letting $n \to \infty$, it follows that $T = T'$. □

4 Conclusion

In this paper, we consider multi-Banach spaces, approximate by multiplicatives, and provide some controlled mappings, which are stable by control functions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Author details
[1]Department of Mathematics, Texas A & M University - Kingsville, Kingsville, TX 78363, USA. [2]Department of Mathematics, Iran University of Science and Technology, Tehran, Iran.

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