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Optimal convex combination bounds of geometric and Neuman means for Toader-type mean

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Abstract

In this paper, we prove that the double inequalities

$$\alpha N_{QA}(a,b) + (1-\alpha)G(a,b) < TD\big[A(a,b),G(a,b)\big] < \beta N_{QA}(a,b) + (1-\beta)G(a,b),$$
$$\lambda N_{AQ}(a,b) + (1-\lambda)G(a,b) < TD\big[A(a,b),G(a,b)\big] < \mu N_{AQ}(a,b) + (1-\mu)G(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/8$, $\beta \geq 4/[\pi(\log(1+\sqrt{2})+\sqrt{2})] = 0.5546\cdots$, $\lambda \leq 3/10$ and $\mu \geq 8/[\pi(\pi+2)] = 0.4952\cdots$, where $TD(a,b)$, $G(a,b)$, $A(a,b)$ and $N_{QA}(a,b)$, $N_{AQ}(a,b)$ are the Toader, geometric, arithmetic and two Neuman means of a and b, respectively.

MSC: 26E60; 33E05

Keywords: Toader mean; geometric mean; Neuman mean

1 Introduction

For $x,y,z \geq 0$ with $xy + xz + yz \neq 0$ and $r \in (0,1)$, the symmetric integrals $R_F(x,y,z)$ and $R_G(x,y,z)$ [1] of the first and second kinds, and the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ of the first and second kinds are defined by

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \big[(t+x)(t+y)(t+z)\big]^{-1/2}\,dt,$$
$$R_G(x,y,z) = \frac{1}{4}\int_0^\infty \big[(t+x)(t+y)(t+z)\big]^{-1/2}\left(\frac{x}{t+x}+\frac{y}{t+y}+\frac{z}{t+z}\right)t\,dt,$$
$$\mathcal{K}(r) = \int_0^{\pi/2}\big[1-r^2\sin^2(t)\big]^{-1/2}\,dt, \qquad \mathcal{E}(r) = \int_0^{\pi/2}\big[1-r^2\sin^2(t)\big]^{1/2}\,dt,$$

respectively.

The well-known identities

$$\mathcal{K}(r) = R_F\big(0,1-r^2,1\big), \qquad \mathcal{E}(r) = 2R_G\big(0,1-r^2,1\big)$$

were established by Carlson in [1].

Let $a, b > 0$ with $a \neq b$. Then the Toader mean $TD(a,b)$ [2] and the Schwab-Borchardt mean $SB(a,b)$ [3–5] are respectively defined by

$$TD(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2(t) + b^2\sin^2(t)}\,dt = \begin{cases} 2a\mathcal{E}(\sqrt{1-(b/a)^2})/\pi, & a > b, \\ 2b\mathcal{E}(\sqrt{1-(a/b)^2})/\pi, & a < b, \end{cases} \tag{1.1}$$

and

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

Very recently, Neuman [6] introduced the Neuman mean $N(a,b)$ of the second kind as follows:

$$N(a,b) = \frac{1}{2}\left[a + \frac{b^2}{SB(a,b)}\right].$$

It is well known that the Toader mean $TD(a,b)$, the Schwab-Borchardt mean $SB(a,b)$ and the Neuman mean of the second kind $N(a,b)$ satisfy the identities (see [6, 7])

$$\begin{aligned} TD(a,b) &= \frac{4}{\pi}R_G(a^2, b^2, 0) \\ &= \frac{1}{\pi}\int_0^{\infty}\left[(t+a^2)(t+b^2)\right]^{-1/2}\left(\frac{a^2}{t+a^2} + \frac{b^2}{t+b^2}\right)t\,dt, \\ SB(a,b) &= 1/R_F(a^2, b^2, b^2) \\ &= 2/\int_0^{\infty}\left[(t+a^2)(t+b^2)(t+b^2)\right]^{-1/2}dt, \\ N(a,b) &= R_G(a^2, b^2, b^2) \\ &= \frac{1}{4}\int_0^{\infty}\left[(t+a^2)(t+b^2)(t+b^2)\right]^{-1/2}\left(\frac{a^2}{t+a^2} + \frac{b^2}{t+b^2} + \frac{b^2}{t+b^2}\right)t\,dt. \end{aligned}$$

Let $p \in \mathbb{R}$ and $a, b > 0$. Then the pth power mean $M_p(a,b)$ is defined by

$$M_p(a,b) = \left[(a^p + b^p)/2\right]^{1/p}(p \neq 0), \qquad M_0(a,b) = \sqrt{ab}. \tag{1.2}$$

We clearly see that $M_p(a,b)$ is symmetric and homogeneous of degree one with respect to a and b, strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) < Q(a,b) = M_2(a,b)$$

hold for $a,b>0$ with $a\neq b$, where $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$ and $Q(a,b)=\sqrt{(a^2+b^2)/2}$ are the geometric, arithmetic and quadratic means of a and b, respectively.

In [6], Neuman presented the explicit formula for $N_{QA}(a,b)\equiv N[Q(a,b),A(a,b)]$ and $N_{AQ}(a,b)\equiv N[A(a,b),Q(a,b)]$ as follows:

$$N_{QA}(a,b)=\frac{1}{2}A(a,b)\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right], \tag{1.3}$$

$$N_{AQ}(a,b)=\frac{1}{2}A(a,b)\left[1+\left(1+v^2\right)\frac{\tan^{-1}(v)}{v}\right] \tag{1.4}$$

and proved that the inequalities

$$A(a,b)<N_{QA}(a,b)<N_{AQ}(a,b)<Q(a,b) \tag{1.5}$$

hold for $a,b>0$ with $a\neq b$, where $v=(a-b)/(a+b)$.

Recently, the Toader mean has been the subject of intensive research. In particular, many remarkable inequalities for Toader mean and other related means can be found in the literature [8–41].

In [42], Vuorinen conjectured that

$$TD(a,b)>M_{3/2}(a,b)$$

for all $a,b>0$ with $a\neq b$. This conjecture was proved by Qiu and Shen [43], and Barnard et al. [44], respectively, and Alzer and Qiu [45] presented the best possible upper power mean bound for the Toader mean as follows:

$$TD(a,b)<M_{\log 2/\log(\pi/2)}(a,b)$$

for all $a,b>0$ with $a\neq b$.

Li, Qian and Chu [46] proved that the inequality

$$\alpha N_{AQ}(a,b)+(1-\alpha)A(a,b)<TD(a,b)<\beta N_{AQ}(a,b)+(1-\beta)A(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 3/4$ and $\beta\geq 4(4-\pi)/[\pi(\pi-2)]=0.9573\cdots$.

Note that

$$G(a,b)<TD\big[A(a,b),G(a,b)\big]<A(a,b) \tag{1.6}$$

for all $a,b>0$ with $a\neq b$.

From inequalities (1.5) and (1.6) we clearly see that

$$G(a,b)<TD\big[A(a,b),G(a,b)\big]<N_{QA}(a,b)<N_{AQ}(a,b)$$

for all $a,b>0$ with $a\neq b$.

The main purpose of this paper is to find the greatest values α, λ and the least values β, μ such that the double inequalities

$$\alpha N_{QA}(a,b)+(1-\alpha)G(a,b)<TD\big[A(a,b),G(a,b)\big]<\beta N_{QA}(a,b)+(1-\beta)G(a,b),$$
$$\lambda N_{AQ}(a,b)+(1-\lambda)G(a,b)<TD\big[A(a,b),G(a,b)\big]<\mu N_{AQ}(a,b)+(1-\mu)G(a,b)$$

hold for all $a,b>0$ with $a\neq b$. As applications, we get two new bounds for the complete elliptic integral of the second kind in terms of elementary functions.

2 Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

For $r\in(0,1)$, we clearly see that

$$\mathcal{K}\big(0^+\big)=\mathcal{E}\big(0^+\big)=\pi/2,\qquad \mathcal{K}\big(1^-\big)=+\infty,\qquad \mathcal{E}\big(1^-\big)=1,$$

and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the formulas (see[21], Appendix E, pp.474-475)

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\qquad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$
$$\frac{d[\mathcal{E}(r)-\mathcal{K}(r)]}{dr}=-\frac{r\mathcal{E}(r)}{1-r^2}.$$

Lemma 2.1 (see [21], Theorem 1.25) *For $-\infty<a<b<+\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad\textit{and}\quad\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (see [21], Theorem 3.21(1), Exercise 3.43(11) and Exercise 3.43(29))

(1) *The function $r\mapsto[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;*
(2) *The function $r\mapsto[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,+\infty)$;*
(3) *The function $r\mapsto[(2-r^2)\mathcal{K}(r)-2\mathcal{E}(r)]/r^4$ is strictly increasing from $(0,1)$ onto $(\pi/16,+\infty)$.*

Lemma 2.3 *The function $r\mapsto\varphi_1(r)=\{\frac{2}{\pi}\sqrt{1-r^2}[2\mathcal{E}(r)-\mathcal{K}(r)]+2r^2-1\}/r^2$ is strictly increasing from $(0,1)$ onto $(3/4,1)$.*

Proof Simple computations lead to

$$\varphi_1\big(0^+\big)=\frac{3}{4},\qquad \varphi_1\big(1^-\big)=1, \tag{2.1}$$

$$\varphi_1'(r)=\frac{2}{\pi r^3}\gamma_1(r), \tag{2.2}$$

where

$$\gamma_1(r) = \frac{\mathcal{K}(r) - 3\mathcal{E}(r)}{\sqrt{1-r^2}} + \pi, \tag{2.3}$$

$$\gamma_1(0^+) = 0, \tag{2.4}$$

$$\gamma_1'(r) = \frac{r^3}{(1-r^2)^{3/2}} \frac{(2-r^2)\mathcal{K}(r) - 2\mathcal{E}(r)}{r^4}. \tag{2.5}$$

From (2.5) and Lemma 2.2(3) we get

$$\gamma_1'(r) > \frac{\pi r^3}{16(1-r^2)^{3/2}} > 0. \tag{2.6}$$

Therefore, Lemma 2.3 follows easily from (2.1), (2.2), (2.4) and (2.6). □

Lemma 2.4 *The function* $r \mapsto \varphi_2(r) = (2r^2 + \sqrt{1-r^4} - 1)/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(1,2)$.

Proof It is easy to verify that

$$\varphi_2(0^+) = 2, \qquad \varphi_2(1^-) = 1, \tag{2.7}$$

$$\varphi_2'(r) = \frac{2(\sqrt{1-r^4} - 1)}{r^3\sqrt{1-r^4}} < 0 \tag{2.8}$$

for $r \in (0,1)$.

Therefore, Lemma 2.4 follows easily from (2.7) and (2.8). □

Lemma 2.5 *The function* $r \mapsto \varphi_3(r) = [2r^2\mathcal{K}(r) - 5\mathcal{E}(r)]/\sqrt{1-r^2}$ *is strictly increasing from* $(0,1)$ *onto* $(-5\pi/2, +\infty)$.

Proof It is not difficult to verify that

$$\varphi_3(0^+) = -\frac{5}{2}\pi, \qquad \varphi_3(1^-) = +\infty, \tag{2.9}$$

$$\varphi_3'(r) = \frac{r}{(1-r^2)^{3/2}} \left[(5-3r^2) \frac{\mathcal{K}(r) - \mathcal{E}(r)}{r^2} - \mathcal{E}(r) \right]. \tag{2.10}$$

From (2.10) and Lemma 2.2(2) together with the monotonicity of $\mathcal{E}(r)$ on $(0,1)$ we clearly see that

$$\varphi_3'(r) > \frac{r}{(1-r^2)^{3/2}} \left[(5-3r^2) \times \frac{\pi}{4} - \frac{\pi}{2} \right] = \frac{3\pi}{4} \frac{r}{\sqrt{1-r^2}} > 0 \tag{2.11}$$

for $r \in (0,1)$.

Therefore, Lemma 2.5 follows from (2.9) and (2.11). □

Lemma 2.6 *The function* $r \mapsto \varphi_4(r) = \{\frac{2}{\pi}\sqrt{1-r^2}[2\mathcal{E}(r) - (1+r^2)\mathcal{K}(r)] + 3r^2 - 1\}/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(3/4, 2)$.

Proof Let $\phi_1(r) = \frac{2}{\pi}\sqrt{1-r^2}[2\mathcal{E}(r) - (1+r^2)\mathcal{K}(r)] + 3r^2 - 1$, $\phi_2(r) = r^2$. Then simple computations give

$$\phi_1(0^+) = \phi_2(0) = 0, \qquad \varphi_4(r) = \phi_1(r)/\phi_2(r), \tag{2.12}$$

$$\varphi_4(1^-) = 2, \tag{2.13}$$

$$\frac{\phi_1'(r)}{\phi_2'(r)} = 3 + \frac{1}{\pi\sqrt{1-r^2}}\left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right] + \frac{1}{\pi}\varphi_3(r). \tag{2.14}$$

It follows from Lemma 2.2(1), Lemma 2.5 and the function $r \mapsto \sqrt{1-r^2}$ strictly decreasing that $\phi_1'(r)/\phi_2'(r)$ is strictly increasing on $(0,1)$ and

$$\varphi_4(0^+) = \lim_{r\to 0^+}\frac{\phi_1'(r)}{\phi_2'(r)} = \frac{3}{4}. \tag{2.15}$$

Therefore, Lemma 2.6 follows from Lemma 2.1, (2.12), (2.13) and (2.15) together with the monotonicity of $\phi_1'(r)/\phi_2'(r)$. □

Lemma 2.7 *The function* $\varphi_5(r) = [3r^2 + \sqrt{1-r^2} - 1]/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(2, 5/2)$.

Proof We clearly see that

$$\varphi_5(0^+) = \frac{5}{2}, \qquad \varphi_5(1^-) = 2, \tag{2.16}$$

$$\varphi_5'(r) = -\frac{(1-\sqrt{1-r^2})^2}{r^3\sqrt{1-r^2}} < 0 \tag{2.17}$$

for $r \in (0,1)$.

Therefore, Lemma 2.7 follows easily from (2.16) and (2.17). □

3 Main results

Theorem 3.1 *The double inequality*

$$\alpha N_{QA}(a,b) + (1-\alpha)G(a,b) < TD\big[A(a,b), G(a,b)\big] < \beta N_{QA}(a,b) + (1-\beta)G(a,b) \tag{3.1}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leq 3/8$ *and* $\beta \geq 4/[\pi(\log(1+\sqrt{2}) + \sqrt{2})] = 0.5546\cdots$.

Proof Since $G(a,b)$, $TD(a,b)$ and $N_{QA}(a,b)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $a > b > 0$ and let $r = (a-b)/(a+b) \in (0,1)$. Then (1.1)-(1.3) lead to

$$TD\big[A(a,b), G(a,b)\big] = \frac{2}{\pi}A(a,b)\mathcal{E}(r), \tag{3.2}$$

$$G(a,b) = A(a,b)\sqrt{1-r^2}, \qquad N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]. \tag{3.3}$$

It follows from (3.2)-(3.3) that

$$\begin{aligned}&\frac{T[A(a,b),G(a,b)]-G(a,b)}{N_{QA}(a,b)-G(a,b)}\\&=\frac{\frac{2}{\pi}\varepsilon(r)-\sqrt{1-r^2}}{\frac{1}{2}[\sqrt{1+r^2}+\frac{\sinh^{-1}(r)}{r}]-\sqrt{1-r^2}}\\&=\frac{\frac{4}{\pi}r\varepsilon(r)-2r\sqrt{1-r^2}}{\sinh^{-1}(r)+(r\sqrt{1+r^2}-2r\sqrt{1-r^2})}.\end{aligned}\tag{3.4}$$

Let $f_1(r)=\frac{4}{\pi}r\varepsilon(r)-2r\sqrt{1-r^2}$, $f_2(r)=\sinh^{-1}(r)+(r\sqrt{1+r^2}-2r\sqrt{1-r^2})$ and

$$f(r)=\frac{\frac{4}{\pi}r\varepsilon(r)-2r\sqrt{1-r^2}}{\sinh^{-1}(r)+(r\sqrt{1+r^2}-2r\sqrt{1-r^2})}.\tag{3.5}$$

Then simple computations lead to

$$f_1(0^+)=f_2(0)=0,\tag{3.6}$$

$$\frac{f_1'(r)}{f_2'(r)}=\frac{\frac{2}{\pi}\sqrt{1-r^2}[2\varepsilon(r)-\kappa(r)]+2r^2-1}{2r^2+\sqrt{1-r^4}-1}=\frac{\varphi_1(r)}{\varphi_2(r)},\tag{3.7}$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are defined as in Lemmas 2.3 and 2.4.

It follows from Lemmas 2.3-2.4 and (3.7) that $f_1'(r)/f_2'(r)$ is strictly increasing on $(0,1)$. Then (3.5), (3.6) and Lemma 2.1 lead to the conclusion that $f(r)$ is strictly increasing.

Moreover,

$$\lim_{r\to 0^+}\frac{\frac{4}{\pi}r\varepsilon(r)-2r\sqrt{1-r^2}}{\sinh^{-1}(r)+(r\sqrt{1+r^2}-2r\sqrt{1-r^2})}=\frac{3}{8},\tag{3.8}$$

$$\lim_{r\to 1^-}\frac{\frac{4}{\pi}r\varepsilon(r)-2r\sqrt{1-r^2}}{\sinh^{-1}(r)+(r\sqrt{1+r^2}-2r\sqrt{1-r^2})}=\frac{4}{\pi[\pi(\log(1+\sqrt{2})+\sqrt{2})]}.\tag{3.9}$$

Therefore, Theorem 3.1 follows easily from (3.4), (3.8) and (3.9) together with the monotonicity of $f(r)$. □

Theorem 3.2 *The double inequality*

$$\begin{aligned}\lambda N_{AQ}(a,b)+(1-\lambda)G(a,b)&<TD\big[A(a,b),G(a,b)\big]\\&<\mu N_{AQ}(a,b)+(1-\mu)G(a,b)\end{aligned}\tag{3.10}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\lambda\leq 3/10$ *and* $\mu\geq 8/[\pi(\pi+2)]=0.4952\cdots$.

Proof Without loss of generality, we assume that $a>b>0$ and let $r=(a-b)/(a+b)\in(0,1)$. Then from (1.4) we get

$$N_{AQ}(a,b)=\frac{1}{2}A(a,b)\left[1+\left(1+r^2\right)\frac{\tan^{-1}(r)}{r}\right].\tag{3.11}$$

It follows from (3.2), (3.11) and $G(a,b) = A(a,b)\sqrt{1-r^2}$ that

$$\begin{aligned}\frac{TD[A(a,b),G(a,b)] - G(a,b)}{N_{AQ}(a,b) - G(a,b)} & \frac{\frac{2}{\pi}\mathcal{E}(r) - \sqrt{1-r^2}}{\frac{1}{2}[1 + (1+r^2)\frac{\tan^{-1}(r)}{r}] - \sqrt{1-r^2}} \\ &= \frac{[\frac{4}{\pi}r\mathcal{E}(r) - 2r\sqrt{1-r^2}]/(1+r^2)}{\tan^{-1}(r) + (r - 2r\sqrt{1-r^2})/(1+r^2)}. \end{aligned} \tag{3.12}$$

Let $g_1(r) = [\frac{4}{\pi}r\mathcal{E}(r) - 2r\sqrt{1-r^2}]/(1+r^2)$, $g_2(r) = \tan^{-1}(r) + (r - 2r\sqrt{1-r^2})/(1+r^2)$ and

$$g(r) = \frac{[\frac{4}{\pi}r\mathcal{E}(r) - 2r\sqrt{1-r^2}]/(1+r^2)}{\tan^{-1}(r) + (r - 2r\sqrt{1-r^2})/(1+r^2)}. \tag{3.13}$$

Then simple computations lead to

$$g_1(0^+) = g_2(0) = 0, \tag{3.14}$$

$$\frac{g_1'(r)}{g_2'(r)} = \frac{\frac{2}{\pi}\sqrt{1-r^2}[2\varepsilon(r) - (1+r^2)\kappa(r)] + 3r^2 - 1}{3r^2 + \sqrt{1-r^2} - 1} = \frac{\varphi_4(r)}{\varphi_5(r)}, \tag{3.15}$$

where $\varphi_4(r)$ and $\varphi_5(r)$ are defined as in Lemmas 2.6 and 2.7.

It follows from Lemmas 2.6-2.7 and (3.15) that $g_1'(r)/g_2'(r)$ is strictly increasing on $(0,1)$. Then (3.13), (3.14) and Lemma 2.1 lead to the conclusion that $g(r)$ is strictly increasing.

Moreover,

$$\lim_{r\to 0^+} \frac{[\frac{4}{\pi}r\varepsilon(r) - 2r\sqrt{1-r^2}]/(1+r^2)}{\tan^{-1}(r) + (r - 2r\sqrt{1-r^2})/(1+r^2)} = \frac{3}{10}, \tag{3.16}$$

$$\lim_{r\to 1^-} \frac{[\frac{4}{\pi}r\varepsilon(r) - 2r\sqrt{1-r^2}]/(1+r^2)}{\tan^{-1}(r) + (r - 2r\sqrt{1-r^2})/(1+r^2)} = \frac{8}{\pi(\pi+2)}. \tag{3.17}$$

Therefore, Theorem 3.2 follows from (3.12), (3.16) and (3.17) together with the monotonicity of $g(r)$. □

From Theorems 3.1-3.2 we get the following Corollary 3.3 immediately.

Corollary 3.3 *Let* $\alpha = 3/8$, $\beta = 4/[\pi(\log(1+\sqrt{2}) + \sqrt{2})] = 0.5546\cdots$, $\lambda = 3/10$ *and* $\mu = 8/[\pi(\pi+2)] = 0.4952\cdots$. *Then the double inequalities*

$$\begin{aligned} &\frac{1}{4}\pi\alpha\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right] + \frac{1}{2}\pi(1-\alpha)\sqrt{1-r^2} \\ &\quad < \mathcal{E}(r) < \frac{1}{4}\pi\beta\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right] + \frac{1}{2}\pi(1-\beta)\sqrt{1-r^2}, \\ &\frac{1}{4}\pi\lambda\left[1 + (1+r^2)\frac{\tan^{-1}(r)}{r}\right] + \frac{1}{2}\pi(1-\lambda)\sqrt{1-r^2} \\ &\quad < \mathcal{E}(r) < \frac{1}{4}\pi\mu\left[1 + (1+r^2)\frac{\tan^{-1}(r)}{r}\right] + \frac{1}{2}\pi(1-\mu)\sqrt{1-r^2} \end{aligned}$$

hold for all $r \in (0,1)$.

4 Results and discussion

In this paper, we provide the sharp bounds for the Toader-type mean in terms of the convex combination of geometric and Neuman means. As applications, we find new bounds for the complete elliptic integral of the second kind.

5 Conclusion

In the article, we present the optimal convex combination bounds of the geometric and Neuman means for the Toader-type mean, and give several new upper and lower bounds for the complete elliptic integral of the second kind. The given results are the improvements of some previously known results.

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Competing interests
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Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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