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The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ_p $(1 < p < \infty)$

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Dedicated to BE Rhoades

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Abstract

The discrete generalized Cesàro matrix $A_t = (a_{nk})$ is the triangular matrix with nonzero entries $a_{nk} = t^{n-k}/(n+1)$, where $t \in [0,1]$. In this paper, boundedness, compactness, spectra, the fine spectra and subdivisions of the spectra of discrete generalized Cesàro operator on ℓ_p $(1 < p < \infty)$ have been determined.

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1 Introduction

The lower triangular matrix $A_t = (c_{nk})$ defined by $c_{nk} = t^{n-k}/(n+1)$, $0 < t \leq 1$ is called a discrete generalized Cesàro operator. The matrix reduces to the Cesàro matrix by setting $t = 1$. In 1982, Rhaly [1] showed that the discrete generalized Cesàro operator A_t on the ℓ_2 Hilbert space was a bounded compact linear operator and computed its spectrum. Also in [2], lower bounds for these classes were obtained under certain restrictions on ℓ_p $(1 < p < \infty)$ by Rhoades. In this article, we show that this operator is a compact linear operator, calculate its spectrum and get two subdivisions of this spectrum on the ℓ_p $(1 < p < \infty)$ sequence space.

2 Boundedness of discrete generalized Cesàro operator

In 1982, Rhaly [1] showed that the discrete generalized Cesàro operator A_t on the Hilbert space ℓ_2 is a bounded linear operator. We will show that A_t is a bounded linear operator on ℓ_p $(1 < p < \infty)$.

Theorem 1 ([3] (Hardy inequalities)) *If $p > 1$, $a_n \geq 0$, and $A_n = a_1 + a_2 + \cdots + a_n$, then unless all a_n's are 0,*

$$\sum \left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum a_n^p \tag{2.1}$$

inequality is provided. This constant is the best possible.

Theorem 2 *$A_t \in B(\ell_p)$ and $\|A_t\|_{B(\ell_p)} \leq \frac{p}{p-1}$ for $0 < t < 1$, where $1 < p < \infty$.*

Proof Using Theorem 1, since $0 < t < 1$, we have

$$\begin{aligned}
\|A_t x\|_p^p = \sum_{n=0}^{\infty} |y_n|^p &= \sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{k=0}^{n} t^{n-k} x_k \right|^p \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |t|^{n-k} |x_k| \right)^p \\
&\leq \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^p \\
&\leq \left(\frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |x_n|^p = \left(\frac{p}{p-1} \right)^p \|x\|_p^p.
\end{aligned}$$

Hence we get

$$A_t \in B(\ell_p) \quad \text{and} \quad \|A_t\| \leq \frac{p}{p-1}. \qquad \square$$

3 Compactness of discrete generalized Cesàro operator

Compact linear operators have a great deal of application in practice. For instance, they play a central role in the theory of integral equations and in various problems of mathematical physics.

Disentangling the historical development of the spectral theory of compact linear operators is particularly hard because many of the results were originally proved early in the twentieth century for integral equations acting on particular Banach spaces of functions. These operators behave very much like familiar finite dimensional matrices without necessarily having finite rank. For a compact linear operator, spectral theory can be treated fairly completely in the sense that Fredholm's famous theory of linear integral equations may be extended to linear functional equations $Tx - \lambda x = y$ with a complex parameter λ. This generalized theory is called the *Riesz-Schauder theory*.

Definition 1 ([4]) Let X and Y be normed spaces. An operator $T : X \to Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if, for every bounded subset M of X, the image $T(M)$ is relatively compact, that is, the closure $\overline{T(M)}$ is compact.

From the definition of compactness of a set, we readily obtain a useful criterion for the operator.

Theorem 3 ([4]) *Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.*

The following theorem makes it easy to show the compactness of a linear operator over a normed space.

Theorem 4 ([4]) *Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then*:

(a) *If T is bounded and* $\dim T(X) < \infty$, *the operator T is compact.*

(b) *If* $\dim X < \infty$, *the operator T is compact.*

The following is important as a tool for proving compactness of a given operator as the uniform operator limit of a sequence of compact linear operators.

Theorem 5 ([4]) *Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent, say, if $\|T_n - T\| \to 0$, then the limit operator T is compact.*

In 1982, Rhaly [1] showed that the discrete generalized Cesàro operator A_t on the Hilbert space ℓ_2 is a compact linear operator. We show that A_t is a compact linear operator on ℓ_p $(1 < p < \infty)$.

Theorem 6 *A_t is a compact linear operator over ℓ_p $(1 < p < \infty)$ for* $0 < t < 1$.

Proof Let

$$A_t^r(x) = \left(x_0, \frac{1}{2}(tx_0 + x_1), \frac{1}{3}\left(t^2 x_0 + tx_1 + x_2\right), \dots, \frac{1}{r+1}\sum_{k=0}^{r} t^{r-k} x_k, 0, 0, \dots\right).$$

For $\forall r \in \mathbb{N}$, we obtain that $\dim(A^r) = r + 1 < \infty$. Hence, from Theorem 4, for all $r \in \mathbb{N}$, the operator A^r is compact on ℓ_p. With triangular inequality and Hölder's inequality, for all $x \in \ell_p$, we have

$$\begin{aligned}
\left\|\left(A_t^r - A_t\right)(x)\right\|_p^p &= \sum_{n=r+1}^{\infty}\left|\frac{1}{n+1}\sum_{k=0}^{n} t^{n-k} x_k\right|^p \le \sum_{n=r+1}^{\infty}\left\{\frac{1}{n+1}\sum_{k=0}^{n} t^{n-k}|x_k|\right\}^p \\
&\le \sum_{n=r+1}^{\infty}\frac{1}{(n+1)^p}\left\{\left[\sum_{k=0}^{n} t^{(n-k)q}\right]^{\frac{1}{q}}\left[\sum_{k=0}^{n}|x_k|^p\right]^{\frac{1}{p}}\right\}^p \\
&\le \|x\|_p^p \sum_{n=r+1}^{\infty}\frac{1}{(n+1)^p}\left[\sum_{k=0}^{n} t^{(n-k)q}\right]^{\frac{p}{q}} \\
&= \|x\|_p^p \sum_{n=r+1}^{\infty}\frac{1}{(n+1)^p}\left[1 + t^q + \cdots + \left(t^q\right)^n\right]^{\frac{p}{q}} \\
&= \|x\|_p^p \sum_{n=r+1}^{\infty}\frac{1}{(n+1)^p}\left[\frac{1-(t^q)^{n+1}}{1-t^q}\right]^{\frac{p}{q}}.
\end{aligned} \tag{3.1}$$

Then we get

$$\left\|A_t^r - A_t\right\|_p^p \le \sum_{n=r+1}^{\infty}\frac{1}{(n+1)^p}\left[\frac{1-(t^q)^{n+1}}{1-t^q}\right]^{\frac{p}{q}}. \tag{3.2}$$

Hence, we obtain

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)^p}{(n+2)^p}\left[\frac{1-(t^q)^{n+2}}{1-(t^q)^{n+1}}\right]^{\frac{p}{q}} \to 1,$$

where

$$c_n = \frac{1}{(n+1)^p}\left[\frac{1-(t^q)^{n+1}}{1-t^q}\right]^{\frac{p}{q}}.$$

After that, we get

$$\begin{aligned}
&n\left(\frac{c_{n+1}}{c_n}-1\right) \\
&\quad = n\left\{\frac{(n+1)^p}{(n+2)^p}\left[\frac{1-(t^q)^{n+2}}{1-(t^q)^{n+1}}\right]^{\frac{p}{q}}-1\right\}, \quad t^q =: \beta \\
&\quad = n\left\{\left(1-\frac{1}{n+2}\right)^p\left[1-\frac{\beta^{n+1}-\beta^{n+2}}{1-\beta^{n+1}}\right]^{\frac{p}{q}}-1\right\} \\
&\quad = n\left\{\left[1-\frac{p}{n+2}+o\left(\frac{1}{n+2}\right)\right]\left[1+\frac{p}{q}\frac{\beta^{n+1}-\beta^{n+2}}{1-\beta^{n+1}}+o\left(\frac{\beta^{n+1}-\beta^{n+2}}{1-\beta^{n+1}}\right)\right]-1\right\},
\end{aligned}$$

that is,

$$n\left(\frac{c_{n+1}}{c_n}-1\right) \to -p < -1.$$

Thus, from the Raabe test, $\sum_{n=0}^{\infty} c_n$ converges, and therefore $\sum_{k=n}^{\infty} c_k \to 0$ (for $n \to \infty$). From (3.1), we have $\|A_t^r - A_t\| \to 0$ (for $r \to \infty$). Thus, A_t is the compact linear operator over ℓ_p $(1 < p < \infty)$ for $0 < t < 1$ from Theorem 5. □

4 Spectrum of discrete generalized Cesàro operator

Definition 2 Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$. A number $\lambda \in \mathbb{C}$ that provides the following conditions is called the regular value of T, and the set of all regular values of T will be denoted by $\rho(T)$ and it is called the resolvent set of T:

(R1) $R_\lambda(T) := T_\lambda^{-1} := (T - \lambda I)^{-1}$ resolvent operator exists,
(R2) $R_\lambda(T)$ is bounded, and
(R3) $R_\lambda(T)$ is defined on a set which is dense in X.

Moreover, $\sigma(T) = \mathbb{C} - \rho(T)$ is called the spectrum of T.

Furthermore, the spectrum $\sigma(T)$ naturally splits into three disjoint sets, some of which may be empty. The discrete splitting of the spectrum can be defined as the point spectrum, the continuous spectrum and the residual spectrum as follows.

Definition 3 ([4])

(a) The point spectrum or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T.

(b) The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and satisfies (R3) but not (R2), that is, $R_\lambda(T)$ is unbounded.
(c) The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of $R_\lambda(T)$ is not dense in X.

Spectral theory is an important part of functional analysis. It plays a crucial role in many branches of mathematics such as function theory, complex analysis, differential and integral equations, control theory and also in numerous applications as they are intimately related to the stability of the underlying physical systems. For more information on spectrum, see [4].

The following theorem tells us that the point spectrum of a compact linear operator is not complicated. In fact, we also know that each spectral value $\lambda \neq 0$ of a compact linear operator is an eigenvalue from the next theorem. The spectrum of a compact linear operator largely resembles the spectrum of an operator on a finite dimensional space.

Theorem 7 ([4]) *A compact linear operator $T : X \to X$ on a normed space X has the following properties*:
(a) *The set of the eigenvalues of T is countable* (*perhaps finite or even empty*).
(b) $\lambda = 0$ *is the only possible point of accumulation of that set.*
(c) *Every spectral value $\lambda \neq 0$ is an eigenvalue.*
(d) *If X is infinite dimensional*, *then* $0 \in \sigma(T)$.

4.1 Spectrum of discrete generalized Cesàro operator on ℓ_p ($1 < p < \infty$)

Spectrum of compact Rhaly operator was specified in [5] and [6]. The spectrum of discrete generalized Cesàro operator A_t on the Hilbert space ℓ_2 was examined by Rhaly [1] in 1982. We determine the spectrum of A_t on ℓ_p ($1 < p < \infty$). Let $S := \{\frac{1}{n} : n = 1, 2, \ldots\}$.

In this section, we will compute the spectrum of the generalized discrete generalized Cesàro matrix, the compact linear operator A_t, where $0 < t < 1$.

Theorem 8 $\sigma_p(A_t, \ell_p) = S$ *for* $0 < t < 1$, *where* $1 < p < \infty$.

Proof Let

$$A_t x = \lambda x \quad \text{for } 1 < p < \infty,$$

where $x \neq \theta$. In this case, equations

$$\begin{aligned}
&x_0 = \lambda x_0,\\
&\frac{1}{2}(tx_0 + x_1) = \lambda x_1,\\
&\frac{1}{3}\left(t^2x_0 + tx_1 + x_2\right) = \lambda x_2,\\
&\frac{1}{4}\left(t^3x_0 + t^2x_1 + tx_2 + x_3\right) = \lambda x_3,\\
&\vdots\\
&\frac{1}{n+1}\left(\sum_{k=0}^{n} t^{n-k}x_k\right) = \lambda x_n,\\
&\vdots
\end{aligned} \tag{4.1}$$

are provided.

(i) From equation $(1-\lambda)x_0 = 0$, if $x_0 \neq 0$ then $\lambda = 1$. From (4.1), we have

$$\begin{aligned}
&\Rightarrow \quad \frac{1}{2}(tx_0 + x_1) = x_1 &&\Rightarrow \quad \frac{1}{2}tx_0 = \frac{1}{2}x_1 &&\Rightarrow \quad x_1 = tx_0\\
&\Rightarrow \quad \frac{1}{3}(t^2x_0 + tx_1 + x_2) = x_2 &&\Rightarrow \quad \frac{2}{3}t^2x_0 = \frac{2}{3}x_2 &&\Rightarrow \quad x_2 = t^2x_0\\
&\Rightarrow && &&\Rightarrow \quad x_n = t^nx_0, \quad x_0 \neq 0, 0 < t < 1\\
& &&\quad\vdots
\end{aligned}$$

Since

$$\left|\frac{x_{n+1}}{x_n}\right|^p = t^p \to t^p < 1,$$

we get $\sum_k |x_k|^p = |x_0| \sum_k |t^p|^k < \infty$. Hence, we have $(x_n) = (t^nx_0) \in \ell_p$. Therefore, the eigenvector corresponding to $\lambda = 1$, $x = (x_0, tx_0, t^2x_0, t^3x_0, \dots) \in \ell_p$, that is, we have $\lambda = 1 \in \sigma_p(A_t, \ell_p)$.

(ii) Let $x_0 = 0$. Therefore, we obtain

$$\frac{1}{2}x_1 = \lambda x_1 \quad \Rightarrow \quad \left(\lambda - \frac{1}{2}\right)x_1 = 0$$

from the second equation in (4.1). If $x_1 \neq 0$, then $\lambda = \frac{1}{2}$. Hence, we obtain

$$\begin{aligned}
&\frac{1}{3}(tx_1 + x_2) = \frac{1}{2}x_2 &&\Rightarrow \quad \frac{1}{3}tx_1 = \frac{1}{6}x_2 &&\Rightarrow \quad x_2 = 2tx_1,\\
&\frac{1}{4}(t^2x_1 + tx_2 + x_3) = \frac{1}{2}x_3 &&\Rightarrow \quad \frac{4}{3}t^2x_1 = \frac{1}{4}x_3 &&\Rightarrow \quad x_3 = 3t^2x_1,\\
&\frac{1}{5}(t^3x_1 + t^2x_2 + tx_3 + x_4) = \frac{1}{2}x_4 &&\Rightarrow \quad \frac{6}{5}t^3x_1 = \frac{3}{10}x_4 &&\Rightarrow \quad x_4 = 4t^3x_1,\\
& &&\quad\vdots &&\Rightarrow \quad x_n = nt^{n-1}x_1
\end{aligned}$$

from the other equations in (4.1). Then, since

$$\left|\frac{x_{n+1}}{x_n}\right|^p = \left(\frac{n+1}{n}\right)^p t^p \to t^p < 1,$$

we have $\sum_n |x_n|^p < \infty$, that is, $x = (x_n) \in \ell_p$. Thus, the eigenvector corresponding to $\lambda = \frac{1}{2}$ is $x = (0, x_1, 2tx_1, 3t^2x_1, \ldots) \in \ell_p$, *i.e.*, $\lambda = 1/2 \in \sigma_p(A_t, \ell_p)$.

(iii) If x_m is the first nonzero component of the sequence $x = (x_n)$, then from mth equation in (4.1), *i.e.*,

$$\frac{1}{m+1}\left(\sum_{k=0}^{m} t^{m-k}x_k\right) = \lambda x_m,$$

we get

$$\frac{1}{m+1}x_m = \lambda x_m \quad \Rightarrow \quad \left(\lambda - \frac{1}{m+1}\right)x_m = 0, \quad x_m \neq 0 \quad \Rightarrow \quad \lambda = \frac{1}{m+1}.$$

In this case, we have

$$x_{m+n} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}t^n x_m \quad \text{for all } n \geq 1$$

from other equations in (4.1). Since $t \in (0,1)$,

$$\left|\frac{x_{m+n+1}}{x_{m+n}}\right|^p = \left(\frac{m+n+1}{n}\right)^p t^p \to t^p < 1 \quad (\text{by } n \to \infty),$$

the eigenvector corresponding to $\lambda = 1/(m+1)$ is

$$x = \left(0, 0, \ldots, x_m, (m+1)tx_m, \frac{(m+1)(m+2)}{2}t^2x_m, \right.$$
$$\left. \ldots, \frac{(m+1)(m+2)\cdots(m+n)}{n!}t^n x_m, \ldots\right) \in \ell_p,$$

i.e., $\lambda = 1/(m+1) \in \sigma_p(A_t, \ell_p)$. Hence, $\sigma_p(A_t, \ell_p) = S = \{\frac{1}{m} : m = 1, 2, \ldots\}$. □

We will use the following lemma to find the adjoint on the ℓ_p $(1 < p < \infty)$ sequence space of a linear transform.

Lemma 1 ([7], p. 215) *If $A \in B(\ell_p)$ $(1 < p < \infty)$, then A can be represented by an infinite matrix and A^*, which is an element of $B(\ell_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$, can be represented by the transpose of A matrix.*

The adjoint matrix of A_t on ℓ_p $(1 < p < \infty)$ is as follows:

Lemma 2 *The adjoint operator over ℓ_p $(p > 1)$ of the matrix A_t can be given as its transposition. That is, the matrix $(A_t)^* = (a^*_{nk})$ is given by*

$$a^*_{nk} = \begin{cases} \frac{t^{k-n}}{k+1}, & 0 \leq n \leq k, \\ 0, & n > k. \end{cases} \tag{4.2}$$

Theorem 9 $\sigma_p(A_t^*, \ell_p^* \cong \ell_q) = S$ *for* $0 < t < 1$, *where* $1 < p < \infty$.

Proof Let $x \neq \theta$ and $A_t^* x = \lambda x$. Then, for all $n \geq 1$, the equations

$$\begin{aligned}
&x_0 + \frac{t}{2}x_1 + \frac{t^2}{3}x_2 + \frac{t^3}{4}x_3 + \cdots = \lambda x_0,\\
&\frac{1}{2}x_1 + \frac{t}{3}x_2 + \frac{t^2}{4}x_3 + \cdots = \lambda x_1,\\
&\frac{1}{3}x_2 + \frac{t}{4}x_3 + \cdots = \lambda x_2,\\
&\frac{1}{4}x_3 + \cdots = \lambda x_3,\\
&\vdots
\end{aligned}$$

are realized from Lemma 2. Therefore $0 \notin \sigma_p(A_t^*, \ell_q)$ because if $\lambda = 0$ then $x_n = 0$ for all $n = 0, 1, 2, \ldots$. Hence, we get

$$x_n = \frac{1}{t^n} \frac{(\lambda - \frac{1}{n})(\lambda - \frac{1}{n-1}) \cdots (\lambda - 1)}{\lambda^n} x_0, \quad x_0 \neq 0$$

because $x \neq \theta$. That is, we have

$$x_n = \frac{1}{t^n} \prod_{k=1}^{n} \left(1 - \frac{1}{k\lambda}\right) x_0 \quad \text{for all } n \geq 1,$$

where $x_0 \neq 0$. If $\lambda = \frac{1}{m}$ for an integer m, then we have $\sum_n |x_n|^q < \infty$ because $x_n = 0$ for every $n \geq m$, so that, $x = (x_n) \in \ell_q$ is obtained. Hence, we get $\lambda = \frac{1}{m} \in \sigma_p(A_t^*, \ell_p^* \cong \ell_q)$ for all integers m. Let $\lambda \neq \frac{1}{m}$ for all integers m. Since

$$\left|\frac{x_{n+1}}{x_n}\right|^q = \frac{1}{t^q}\left|1 - \frac{1}{\lambda(n+1)}\right|^q \to \frac{1}{t^q} > 1 \quad (n \to \infty)$$

$\sum_n |x_n|^q$ series is divergent. So, there is no other eigenvalue, *i.e.*, we have

$$\sigma_p\left(A_t^*, \ell_q\right) = S. \qquad \square$$

Theorem 10 $\sigma(A_t, \ell_p) = S \cup \{0\}$ *for* $0 < t < 1$, *where* $1 < p < \infty$.

Proof Since $\dim \ell_p = \infty$, we have $0 \in \sigma(A_t, \ell_p)$ from Theorem 7. Also, since A_t is a compact linear operator by Theorem 6, each nonzero spectral value of A_t is an eigenvalue from Theorem 7. Therefore, $\sigma(A_t, \ell_p) = S \cup \{0\}$ is obtained from Theorem 8. $\square$

4.2 The fine spectrum of discrete generalized Cesàro operator on ℓ_p ($1 < p < \infty$)

Let X be a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and $T \in B(X)$. Then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$,

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

Table 1 Goldberg's decomposition of the spectrum

		(1) $R(\lambda;T)$ exists and is bounded	(2) $R(\lambda;T)$ exists and is unbounded	(3) $R(\lambda;T)$ does not exists
(I)	$R(\lambda I-T)=X$	$\lambda \in \rho(T)$	-	$\lambda \in \sigma_p(T)$
(II)	$\overline{R(\lambda I-T)}=X$	$\lambda \in \rho(T)$	$\lambda \in \sigma_c(T)$	$\lambda \in \sigma_p(T)$
(III)	$\overline{R(\lambda I-T)}\neq X$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_p(T)$

(III) $\overline{R(T)} \neq X$,

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labeled by I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. For example, let an operator be in state III_2. Then $\overline{R(T)} \neq X$ and T^{-1} exist and T^{-1} is unbounded. From the closed graph theorem, I_2 is empty (see [8]).

Applying the Goldberg classification to the operator $T_\lambda := \lambda I - T$, we have

(I) $T_\lambda = \lambda I - T$ is surjective,

(II) $\overline{R(T_\lambda)} = X$, but $R(T_\lambda) \neq X$,

(III) $\overline{R(T_\lambda)} \neq X$,

and three possibilities for T_λ^{-1}:

(1) $T_\lambda = \lambda I - T$ is injective and T_λ^{-1} is bounded,

(2) $T_\lambda = \lambda I - T$ is injective and T_λ^{-1} is unbounded, and

(3) $T_\lambda = \lambda I - T$ is not injective.

If λ is a complex number such that $T_\lambda = \lambda I - T \in \mathrm{I}_1$ or $T_\lambda = \lambda I - T \in \mathrm{II}_1$, then $\lambda \in \rho(T,X)$. All scalar values of λ not in $\rho(T,X)$ comprise the spectrum of T. The further classification of $\sigma(T,X)$ gives rise to the fine spectrum of T. That is, $\sigma(T,X)$ can be divided into the subsets $\mathrm{I}_2\sigma(T,X)$, $\mathrm{I}_3\sigma(T,X)$, $\mathrm{II}_2\sigma(T,X)$, $\mathrm{II}_3\sigma(T,X)$, $\mathrm{III}_1\sigma(T,X)$, $\mathrm{III}_2\sigma(T,X)$, $\mathrm{III}_3\sigma(T,X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in \mathrm{III}_2\sigma(T,X)$.

We can summarize the above in Table 1.

This classification of the spectrum is called the Goldberg classification. Let us give the theorems that will help the Goldberg classification.

Theorem 11 ([8], p. 58) *If T^* has a bounded inverse, then $R(T^*)$ is closed.*

Theorem 12 ([8], p. 59) *T has a dense range if and only if T^* is* 1-1.

Theorem 13 ([8], p. 60) *$R(T^*) = X^*$ if and only if T has a bounded inverse.*

Theorem 14 ([8], p. 60) *$\overline{R(T)} = X$ and T has a bounded inverse if and only if $R(T^*) = X^*$ and T^* has a bounded inverse.*

The fine spectra of bounded linear operators defined by some particular limitation matrices over some sequence spaces were first discussed in [5, 9–11] and [12].

Then the spectra and fine spectra of some operators have been studied by various authors [13–22] and are still being studied.

We will examine the fine spectrum of a discrete generalized Cesàro operator on ℓ_p $(1 < p < \infty)$, which is compact in this section.

Theorem 15 $0 \in \mathrm{II}_2\sigma(A_t, \ell_p)$ *for* $0 < t < 1$, *where* $1 < p < \infty$.

Proof Since $\sigma_p(A_t, \ell_p) = S$, we have $0 \notin \sigma_p(A_t, \ell_p)$. Thus, $(A_t)^{-1}$ exists. Hence $A_t \in (1) \cup (2)$. The operator A_t^* is 1-1 because $0 \notin \sigma_p(A_t^*, \ell_q)$. Hence, we have $\overline{R(A_t)} = \ell_p$ from Theorem 12. If $A_t x = y$, we obtain

$$y_n = \frac{1}{n+1}\sum_{k=0}^{n} t^{n-k}x_k.$$

Therefore, we get

$$x_0 = y_0 \quad \text{and} \quad x_n = (n+1)y_n - tny_{n-1}$$

from

$$\begin{aligned}(n+1)y_n &= t^n x_0 + t^{n-1}x_1 + \cdots + tx_{n-1} + x_n,\\ tny_{n-1} &= t\bigl(t^{n-1}x_0 + t^{n-2}x_1 + \cdots + x_{n-1}\bigr).\end{aligned}$$

Then we give the matrix $A_t^{-1} = (c_{nk})$ with

$$c_{nk} = \begin{cases} n+1, & k = n,\\ -tn, & k = n-1,\\ 0, & \text{otherwise.}\end{cases}$$

If we take $y = (y_n) = (\frac{(-1)^n}{n+1}) \in \ell_p$ $(1 < p < \infty)$, then we have

$$(x_n) = \left((n+1)\frac{(-1)^n}{n+1} - (-1)^{n-1}\frac{nt}{n}\right) = \bigl((-1)^n(1+t)\bigr) \notin \ell_p.$$

Hence A_t is not onto, that is, $R(A_t) \neq \ell_p$. Therefore, $A_t \in \mathrm{II}$. As a consequence, $A_t \in \mathrm{II}_1$ or $A_t \in \mathrm{II}_2$. We have $A_t \notin \mathrm{II}_1$ because $0 \in \sigma(A_t, \ell_p)$. Then we have $A_t \in \mathrm{II}_2$, *i.e.*, $0 \in \mathrm{II}_2\sigma(A_t, \ell_p)$. □

Theorem 16 $\mathrm{III}_3\sigma(A_t, \ell_p) = S$ *for* $0 < t < 1$, *where* $1 < p < \infty$.

Proof If $\lambda = \frac{1}{m}$, then $T_\lambda = (\lambda I - A_t)$ has no inverse because $\sigma_p(A_t, \ell_p) = S = \{\frac{1}{m} : m = 1, 2, \dots\}$, that is, we have $T_\lambda \in (3)$. Since $\lambda = \frac{1}{m} \in \sigma_p(A_t^*, \ell_p)$, operator $T_\lambda^* = \lambda I - A_t^*$ is not 1-1 for $\lambda = \frac{1}{m}$. $T_\lambda = \lambda I - A_t$ does not have a dense image by Theorem 12. Hence, $\overline{R(T_\lambda)} \neq \ell_p$, *i.e.*, $T_\lambda \in \mathrm{III}$. Accordingly, $T_{\frac{1}{m}} = \frac{1}{m}I - A_t \in \mathrm{III}_3$, and hence, we have $\lambda = \frac{1}{m} \in \mathrm{III}_3\sigma(A_t, \ell_p)$. □

5 Subdivision of the spectrum of discrete generalized Cesàro operator on ℓ_p $(1 < p < \infty)$

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X a Weyl sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$ as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{K} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{5.1}$$

the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not surjective}\} \tag{5.2}$$

is called the defect spectrum of T.

The two subspectra (5.1) and (5.2) form a (not necessarily disjoint) subdivision

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_\delta(T) \tag{5.3}$$

of the spectrum. There is another subspectrum

$$\sigma_{co}(T) = \left\{\lambda \in \mathbb{K} : \overline{R(\lambda I - T)} \neq X\right\} \tag{5.4}$$

which is often called compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{co}(T) \tag{5.5}$$

of the spectrum. Clearly, $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma_{co}(T) \subseteq \sigma_\delta(T)$. Moreover, comparing these subspectra, we note that

$$\sigma_r(T) = \sigma_{co}(T) \setminus \sigma_p(T) \tag{5.6}$$

and

$$\sigma_c(T) = \sigma(T) \setminus \left[\sigma_p(T) \cup \sigma_{co}(T)\right]. \tag{5.7}$$

It can sometimes be useful to establish a relationship between the spectra of a bounded linear operator and its adjoint.

Proposition 1 ([23], Proposition 1.3) *The spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations*:

(a) $\sigma_{ap}(T^*) = \sigma_\delta(T)$;
(b) $\sigma_\delta(T^*) = \sigma_{ap}(T)$;
(c) $\sigma_p(T^*) = \sigma_{co}(T)$;
(d) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*) = \sigma_p(T) \cup \sigma_{ap}(T^*)$.

By the definitions given above, we can write Table 2.

This separation of the spectrum of some operator has been studied by various authors in [18, 24–26, 28] and is still being studied.

Theorem 17 *For $0 < t < 1$ and $1 < p < \infty$, we have*

(a) $\sigma_{ap}(A_t, \ell_p) = S \cup \{0\}$;
(b) $\sigma_\delta(A_t, \ell_p) = S \cup \{0\}$;
(c) $\sigma_{co}(A_t, \ell_p) = S$.

Proof (a) We have $\mathrm{III}_1\sigma(A_t,\ell_p) = \emptyset$ from Table 2 because $\sigma(A_t,\ell_p) = S \cup \{0\}$ by Theorem 10, $\mathrm{III}_3\sigma(A_t,\ell_p) = S$ by Theorem 16 and $\mathrm{II}_2\sigma(A_t,\ell_p) = \{0\}$ by Theorem 15. Hence, we get

$$\sigma_{ap}(A_t,\ell_p) = \sigma(A_t,\ell_p) \setminus \mathrm{III}_1\sigma(A_t,\ell_p) = S \cup \{0\}$$

by Table 2.

(b) Since $\sigma(A_t,\ell_p) = S \cup \{0\}$, $\mathrm{III}_3\sigma(A_t,\ell_p) = S$ and $\mathrm{II}_2\sigma(A_t,\ell_p) = \{0\}$ from respectively Theorems 10, 16 and 15, we have $\mathrm{I}_3\sigma(A_t,\ell_p) = \emptyset$ by Table 2. Therefore, we obtain

$$\sigma_\delta(A_t,\ell_p) = \sigma(A_t,\ell_p) \setminus \mathrm{I}_3\sigma(A_t,\ell_p) = S \cup \{0\}$$

by Table 2.

(c) Since $\sigma(A_t,\ell_p) = S \cup \{0\}$, $\mathrm{III}_3\sigma(A_t,\ell_p) = S$ and $\mathrm{II}_2\sigma(A_t,\ell_p) = \{0\}$ from Theorems 10, 16 and 15 respectively, we obtain $\mathrm{III}_1\sigma(A_t,\ell_p) = \emptyset$ from Table 2. As a result,

$$\sigma_{co}(A_t,\ell_p) = \mathrm{III}_1\sigma(A_t,\ell_p) \cup \mathrm{III}_2\sigma(A_t,\ell_p) \cup \mathrm{III}_3\sigma(A_t,\ell_p) = S$$

by Table 2. □

Lemma 3 *For* $0 < t < 1$ *and* $1 < p < \infty$, *we have*

(a) $\sigma_{ap}(A_t^*,\ell_q) = S \cup \{0\}$;

(b) $\sigma_\delta(A_t^*,\ell_q) = S \cup \{0\}$.

Proof Since $\sigma_{ap}(A_t^*,\ell_q) = \sigma_\delta(A_t,\ell_p)$ and $\sigma_\delta(A_t^*,\ell_q) = \sigma_{ap}(A_t,\ell_p)$ from Proposition 1, the proof is clear. □

6 Conclusions

The spectra of summability methods, the Goldberg classification of the spectrum and the non-discrete spectral separation of this summability methods were discussed by various authors earlier. Still, a lot of mathematicians work on this subject. The spectrum of the discrete generalized Cesàro operator on a Hilbert space ℓ_2 was calculated by Rhaly [1] in 1982. In this article, we have obtained the spectra and various spectral separations of this operator over ℓ_p $(1 < p < \infty)$ sequence spaces. In [27], Yildirim *et al.* gave the spectra and spectral division of this operator over the c_0 and c sequence spaces. Also, a Mercerian

Table 2 Separations of the spectrum [24]

		(1) $R(\lambda;T)$ exists and is bounded	(2) $R(\lambda;T)$ exists and is unbounded	(3) $R(\lambda;T)$ does not exists
(I)	$R(\lambda I - T) = X$	$\lambda \in \rho(T)$	-	$\lambda \in \sigma_p(T)$ $\lambda \in \sigma_{ap}(T)$
(II)	$\overline{R(\lambda I - T)} \neq X$ $\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T)$	$\lambda \in \sigma_c(T)$ $\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$	$\lambda \in \sigma_p(T)$ $\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$
(III)	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T)$ $\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$	$\lambda \in \sigma_r(T)$ $\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$	$\lambda \in \sigma_p(T)$ $\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$

theorem was given in [27]. The spectra and spectral separation of this operator over the other sequence spaces are left as clear problems.

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Competing interests
The authors declare that they have no competing interest.

Authors' contributions
The authors have already had many joint publications. This work was carried out in collaboration between all authors.

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