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A generalization of a theorem of Bor

Hikmet Seyhan Özarslan* and Bağdagül Kartal

*Correspondence:
seyhan@erciyes.edu.tr
Department of Mathematics, Erciyes University, Kayseri, 38039, Turkey

Abstract

In this paper, a general theorem concerning absolute matrix summability is established by applying the concepts of almost increasing and δ-quasi-monotone sequences.

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1 Introduction

A positive sequence (y_n) is said to be almost increasing if there is a positive increasing sequence (u_n) and two positive constants K and M such that $Ku_n \leq y_n \leq Mu_n$ (see [1]). A sequence (c_n) is said to be δ-quasi-monotone, if $c_n \to 0$, $c_n > 0$ ultimately and $\Delta c_n \geq -\delta_n$, where $\Delta c_n = c_n - c_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $T = (t_{nv})$ be a normal matrix, *i.e.*, a lower triangular matrix of nonzero diagonal entries. At that time T describes the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $Ts = (T_n(s))$, where

$$T_n(s) = \sum_{v=0}^{n} t_{nv} s_v, \quad n = 0, 1, \dots \tag{1}$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |T, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|\bar{\Delta} T_n(s)\right|^k < \infty, \tag{2}$$

where

$$\bar{\Delta} T_n(s) = T_n(s) - T_{n-1}(s).$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |T, p_n|_k$ summability reduces to $|T, p_n|_k$ summability (see [4]). If we set $\varphi_n = n$ for all n, $\varphi - |T, p_n|_k$ summability is the same as $|T|_k$ summability (see [5]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $t_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability (see [6]).

2 Known result

In [7, 8], Bor has established the following theorem dealing with $|\bar{N},p_n|_k$ summability factors of infinite series.

Theorem 2.1 *Let (Y_n) be an almost increasing sequence such that $|\Delta Y_n| = O(Y_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Assume that there is a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nY_n\delta_n < \infty$, $\sum B_nY_n$ is convergent and $|\Delta\lambda_n| \le |B_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n}|\lambda_n| = O(1) \quad \text{as } m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} \frac{1}{n}|z_n|^k = O(Y_m) \quad \text{as } m \to \infty, \tag{4}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|z_n|^k = O(Y_m) \quad \text{as } m \to \infty, \tag{5}$$

where (z_n) is the nth $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \ge 1$.

3 Main result

The purpose of this paper is to generalize Theorem 2.1 to the $\varphi - |T,p_n|_k$ summability. Before giving main theorem, let us introduce some well-known notations. Let $T = (t_{nv})$ be a normal matrix. Lower semimatrices $\bar{T} = (\bar{t}_{nv})$ and $\hat{T} = (\hat{t}_{nv})$ are defined as follows:

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, v = 0, 1, \ldots \tag{6}$$

and

$$\hat{t}_{00} = \bar{t}_{00} = t_{00}, \qquad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}, \quad n = 1, 2, \ldots \tag{7}$$

Here, $\bar{T}$ and $\hat{T}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then we write

$$T_n(s) = \sum_{v=0}^{n} t_{nv}s_v = \sum_{v=0}^{n} \bar{t}_{nv}a_v \tag{8}$$

and

$$\bar{\Delta}T_n(s) = \sum_{v=0}^{n} \hat{t}_{nv}a_v. \tag{9}$$

By taking the definition of general absolute matrix summability, we established the following theorem.

Theorem 3.1 *Let $T = (t_{nv})$ be a positive normal matrix such that*

$$\bar{t}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{10}$$

$$t_{n-1,v} \geq t_{nv}, \quad \text{for } n \geq v + 1, \tag{11}$$

$$t_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{12}$$

and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If all conditions of Theorem 2.1 *with conditions* (4) *and* (5) *are replaced by*

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^{k-1} \frac{1}{n} |z_n|^k = O(Y_m) \quad \text{as } m \to \infty \tag{13}$$

and

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^{k} |z_n|^k = O(Y_m) \quad \text{as } m \to \infty, \tag{14}$$

then the series $\sum a_n \lambda_n$ is $\varphi - |T, p_n|_k$ summable, $k \geq 1$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2 ([7]) *Let (Y_n) be an almost increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. If (B_n) is δ-quasi-monotone with $\sum B_n Y_n$ is convergent and $|\Delta \lambda_n| \leq |B_n|$ for all n, then we have*

$$|\lambda_n| Y_n = O(1) \quad \text{as } n \to \infty. \tag{15}$$

Lemma 3.3 ([8]) *Let (Y_n) be an almost increasing sequence such that $n|\Delta Y_n| = O(Y_n)$. If (B_n) is δ-quasi monotone with $\sum n Y_n \delta_n < \infty$, and $\sum B_n Y_n$ is convergent, then*

$$n B_n Y_n = O(1) \quad \text{as } n \to \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} n Y_n |\Delta B_n| < \infty. \tag{17}$$

4 Proof of Theorem 3.1

Let (I_n) indicate the T-transform of the series $\sum a_n \lambda_n$. Then we obtain

$$\bar{\Delta} I_n = \sum_{v=0}^{n} \hat{t}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{t}_{nv} \lambda_v}{v} v a_v \tag{18}$$

by means of (8) and (9).

Using Abel's formula for (18), we obtain

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{\nu=1}^{n-1} \Delta_\nu\left(\frac{\hat{t}_{n\nu}\lambda_\nu}{\nu}\right)\sum_{r=1}^{\nu} ra_r + \frac{\hat{t}_{nn}\lambda_n}{n}\sum_{r=1}^{n} ra_r \\ &= \sum_{\nu=1}^{n-1} \frac{\nu+1}{\nu}\Delta_\nu(\hat{t}_{n\nu})\lambda_\nu z_\nu + \sum_{\nu=1}^{n-1}\frac{\nu+1}{\nu}\hat{t}_{n,\nu+1}\Delta\lambda_\nu z_\nu \\ &\quad + \sum_{\nu=1}^{n-1}\hat{t}_{n,\nu+1}\lambda_{\nu+1}\frac{z_\nu}{\nu} + \frac{n+1}{n}t_{nn}\lambda_n z_n \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.\end{aligned}$$

For the proof of Theorem 3.1, it suffices to prove that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1}|I_{n,r}|^k < \infty$$

for $r = 1, 2, 3, 4$.

By Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{\nu=1}^{n-1}\left|\Delta_\nu(\hat{t}_{n\nu})\right||\lambda_\nu||z_\nu|\right)^k \\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{\nu=1}^{n-1}\left|\Delta_\nu(\hat{t}_{n\nu})\right||\lambda_\nu|^k|z_\nu|^k\right)\left(\sum_{\nu=1}^{n-1}\left|\Delta_\nu(\hat{t}_{n\nu})\right|\right)^{k-1}.\end{aligned}$$

By (6) and (7), we have

$$\begin{aligned}\Delta_\nu(\hat{t}_{n\nu}) &= \hat{t}_{n\nu} - \hat{t}_{n,\nu+1} \\ &= \bar{t}_{n\nu} - \bar{t}_{n-1,\nu} - \bar{t}_{n,\nu+1} + \bar{t}_{n-1,\nu+1} \\ &= t_{n\nu} - t_{n-1,\nu}.\end{aligned} \tag{19}$$

Thus using (6), (10) and (11)

$$\sum_{\nu=1}^{n-1}\left|\Delta_\nu(\hat{t}_{n\nu})\right| = \sum_{\nu=1}^{n-1}(t_{n-1,\nu} - t_{n\nu}) \le t_{nn}.$$

Hence, we get

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,1}|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}t_{nn}^{k-1}\left(\sum_{\nu=1}^{n-1}\left|\Delta_\nu(\hat{t}_{n\nu})\right||\lambda_\nu|^k|z_\nu|^k\right)$$

by using (12)

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\bigl|\Delta_v(\hat{t}_{nv})\bigr||\lambda_v|^k|z_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^k|z_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\bigl|\Delta_v(\hat{t}_{nv})\bigr|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_v|^k|z_v|^k\sum_{n=v+1}^{m+1}\bigl|\Delta_v(\hat{t}_{nv})\bigr|.\end{aligned}$$

Now, using (11) and (19), we obtain

$$\sum_{n=v+1}^{m+1}\bigl|\Delta_v(\hat{t}_{nv})\bigr| = \sum_{n=v+1}^{m+1}(t_{n-1,v}-t_{nv}) \leq t_{vv}.$$

Thus, by using Abel's formula, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,1}|^k &= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_v|^{k-1}|\lambda_v||z_v|^k t_{vv}\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}|\lambda_v||z_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^{k}|z_r|^k + O(1)|\lambda_m|\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k}|z_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|Y_v + O(1)|\lambda_m|Y_m\\ &= O(1)\sum_{v=1}^{m-1}|B_v|Y_v + O(1)|\lambda_m|Y_m\\ &= O(1) \quad \text{as } m\to\infty,\end{aligned}$$

in view of (14) and (15).

Again, using Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\Delta\lambda_v||z_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||B_v||z_v|^k\right)\\ &\quad\times\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||B_v|\right)^{k-1}.\end{aligned}$$

By means of (6), (7) and (11), we have

$$\begin{aligned}
\hat{t}_{n,v+1} &= \bar{t}_{n,v+1} - \bar{t}_{n-1,v+1} \\
&= \sum_{i=v+1}^{n} t_{ni} - \sum_{i=v+1}^{n-1} t_{n-1,i} \\
&= t_{nn} + \sum_{i=v+1}^{n-1} (t_{ni} - t_{n-1,i}) \\
&\leq t_{nn}.
\end{aligned}$$

In this way, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} t_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{t}_{n,v+1}||B_v||z_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{t}_{n,v+1}||B_v||z_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} |B_v||z_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\hat{t}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |B_v||z_v|^k \sum_{n=v+1}^{m+1} |\hat{t}_{n,v+1}|.
\end{aligned}$$

By (6), (7), (10) and (11), we obtain

$$|\hat{t}_{n,v+1}| = \sum_{i=0}^{v} (t_{n-1,i} - t_{ni}).$$

Thus, using (6) and (10), we have

$$\sum_{n=v+1}^{m+1} |\hat{t}_{n,v+1}| = \sum_{n=v+1}^{m+1} \sum_{i=0}^{v} (t_{n-1,i} - t_{ni}) \leq 1,$$

then we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,2}|^k &= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^{k-1} v|B_v| \frac{1}{v} |z_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta\big(v|B_v|\big) \sum_{r=1}^{v} \varphi_r^{k-1} \left(\frac{p_r}{P_r} \right)^{k-1} \frac{1}{r} |z_r|^k \\
&\quad + O(1) m|B_m| \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^{k-1} \frac{1}{v} |z_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta B_v| Y_v + O(1) \sum_{v=1}^{m-1} |B_v| Y_v + O(1) m|B_m| Y_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

in view of (13), (16) and (17).

Also, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,3}|^k &\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\lambda_{v+1}|\frac{|z_v|}{v}\right)^k \\
&\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\lambda_{v+1}|\frac{|z_v|^k}{v}\right)\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\
&\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}t_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\lambda_{v+1}|\frac{|z_v|^k}{v}\right)\left(\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\lambda_{v+1}|\frac{|z_v|^k}{v}\right) \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{t}_{n,v+1}||\lambda_{v+1}|\frac{|z_v|^k}{v}\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|z_v|^k}{v}\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{t}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_{v+1}|\frac{|z_v|^k}{v}\sum_{n=v+1}^{m+1}|\hat{t}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}|\lambda_{v+1}|\frac{|z_v|^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^{k-1}\frac{1}{r}|z_r|^k \\
&\quad + O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}\frac{1}{v}|z_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|B_{v+1}|Y_{v+1} + O(1)|\lambda_{m+1}|Y_{m+1} \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

in view of (3), (12), (13) and (15).

Finally, as in $I_{n,1}$, we have

$$\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}t_{nn}^k|\lambda_n|^k|z_n|^k \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k|\lambda_n|^{k-1}|\lambda_n||z_n|^k \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k|\lambda_n||z_n|^k = O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

in view of (12), (14) and (15). Finally, the proof of Theorem 3.1 is completed.

5 Corollary

If we take $\varphi_n = \frac{P_n}{p_n}$ and $t_{n\nu} = \frac{p_\nu}{P_n}$ in Theorem 3.1, then we get Theorem 2.1. In this case, conditions (13) and (14) reduce to conditions (4) and (5), respectively. Also, the condition '$(\frac{\varphi_n p_n}{P_n})$ is a non-increasing sequence' and the conditions (10)-(12) are clearly satisfied.

6 Conclusions

In this study, we have generalized a well-known theorem dealing with an absolute summability method to a $\varphi - |T, p_n|_k$ summability method of an infinite series by using almost increasing sequences and δ-quasi-monotone sequences.

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Competing interests
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Authors' contributions
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