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Absolute $\varphi - |C, \alpha, \beta; \delta|_k$ summability of infinite series

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Abstract

In this paper, we established a generalized theorem on a minimal set of sufficient conditions for absolute summability factors by applying a sequence of a wider class (quasi-power increasing sequence) and the absolute Cesàro $\varphi - |C, \alpha, \beta; \delta|_k$ summability for an infinite series. We further obtained well-known applications of the above theorem as corollaries, under suitable conditions.

MSC: 40F05; 40D15; 40G05

Keywords: absolute summability; infinite series; $\varphi - |C, \alpha, \beta; \delta|_k$ summability; quasi-f-power increasing sequence

1 Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and the nth sequence to sequence transformation (mean) of $\{s_n\}$ be given by u_n s.t.

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k. \tag{1}$$

Before discussing $\varphi - |C, \alpha, \beta; \delta|_k$ summability, let us introduce some well-known basic summabilities which are helpful in understanding the $\varphi - |C, \alpha, \beta; \delta|_k$ summability.

Definition 1 The series $\sum_{n=0}^{\infty} a_n$ is said to be absolute summable, if

$$\lim_{n\to\infty} u_n = s \tag{2}$$

and $\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty$.

Definition 2 ([1]) Let t_n represent the nth $(C, 1)$ means of the sequence (na_n), then the series $\sum_{n=0}^{\infty} a_n$ is said to be $|C, 1|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{3}$$

Definition 3 ([2]) The nth Cesáro means of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n) is denoted by $t_n^{\alpha,\beta}$, *i.e.*

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{4}$$

where

$$A_n^{\alpha+\beta}=\begin{cases}0, & n<0,\\ 1, & n=0,\\ O(n^{\alpha+\beta}), & n>0.\end{cases}$$

If the sequence $\{t_n^{\alpha,\beta}\}$ satisfies

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|t_n^{\alpha,\beta}\right|^k<\infty, \tag{5}$$

then the series $\sum_{n=0}^{\infty}a_n$ is said to be $\varphi-|C,\alpha,\beta|_k$ summable.

Definition 4 For the following condition:

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^{k-\delta k}}\left|t_n^{\alpha,\beta}\right|^k<\infty, \tag{6}$$

the series $\sum_{n=0}^{\infty}a_n$ is said to be $\varphi-|C,\alpha,\beta;\delta|_k$ summable, where $k\geq 1$, $\delta\geq 0$ and (φ_n) is a sequence of positive real numbers.

Bor gave a number of theorems on absolute summability. In 2002, Bor found the sufficient conditions for an infinite series to be $|C,\alpha|_k$ summable [3] and $|C,\alpha;\delta|_k$ summable [4]. In 2011, he generalized his previous results for $|C,\alpha,\beta|_k$ summability [5] and $|C,\alpha,\beta;\delta|_k$ summability [6], respectively. In 2014, Bor [7] generalized the $|C,\alpha|_k$ summability factor to the $|C,\alpha,\beta;\delta|_k$ summability of an infinite series and in [8], he discussed a general class of power increasing sequences and absolute Riesz summability factors of an infinite series. In [9], Bor applied $|C,\alpha,\gamma;\beta|_k$ summability to obtain the sufficient conditions for an infinite series to be absolute summable.

Bor [10] gave a new application of quasi-power increasing sequence by applying absolute Cesáro $\varphi-|C,\alpha|_k$ summability for an infinity series. Özarslan [11] generalized the result on $\varphi-|C,1|_k$ by a more general absolute $\varphi-|C,\alpha|_k$ summability. In 2016, Sonker and Munjal [12] determined a theorem on generalized absolute Cesáro summability with the sufficient conditions for an infinite series and in [13], they used the concept of triangle matrices for obtaining the minimal set of sufficient conditions of an infinite series to be bounded.

2 Known results

By using $|C,\alpha|_k$ summability, Bor [14] gave a minimal set of sufficient conditions for an infinite series to be absolute summable.

Theorem 2.1 *Let X_n be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$. Suppose also that there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying the following*:

$$\sum n\xi_n X_n = O(1), \tag{7}$$

$$\Delta A_n \le \xi_n, \tag{8}$$

$$|\Delta\lambda_n| \le |A_n|, \tag{9}$$

$$\sum A_n X_n \quad \text{is convergent for all } n. \tag{10}$$

If the conditions

$$|\lambda_n| X_n = O(1) \quad \text{as } n \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} = O(X_m) \quad \text{as } m \to \infty, \tag{12}$$

are satisfied, then the series $\sum a_n\lambda_n$ is $|C,\alpha|_k$ summable, $0 < \alpha \le 1$ *and* $k \ge 1$.

3 Main results

A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X,f) \ge 1$ such that $Kf_n X_n \ge f_m X_m$ for all $n \ge m \ge 1$, where $f = [f_n(\eta,\zeta)] = \{n^{\eta}(\log n)^{\zeta}, \zeta \ge 0, 0 < \eta < 1\}$ [15]. If we set $\zeta = 0$, then we get a quasi-η-power increasing sequence [16].

With the help of generalized Cesáro $\varphi - |C,\alpha,\beta;\delta|_k$ summability, we modernized the results of Bor [14] and established the following theorem.

Theorem 3.1 *Let X_n be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$. Suppose also that there exists a ξ-quasi-monotone sequence of numbers (A_n) such that*

$$\sum n\xi_n X_n = O(1), \tag{13}$$

$$\Delta A_n \le \xi_n, \tag{14}$$

$$|\Delta\lambda_n| \le |A_n|, \quad \text{and} \tag{15}$$

$$\sum A_n X_n \quad \text{is convergent for all } n. \tag{16}$$

Then the series $\sum a_n\lambda_n$ is $\varphi - |C,\alpha,\beta;\delta|_k$ summable for $k \ge 1$, $0 < \alpha \le 1$, $\beta > -1$, $\alpha + \beta > 0$ *and* $\delta \ge 0$, *if the following conditions are satisfied*:

$$|\lambda_n| X_n = O(1) \quad \text{as } n \to \infty, \tag{17}$$

$$\sum_{n=\nu}^{m} \frac{\varphi_n^{k-1}}{n^{(\alpha+\beta-\delta+1)k}} = O\left(\frac{\varphi_\nu^{k-1}}{\nu^{(\alpha+\beta-\delta+1)k-1}}\right), \tag{18}$$

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1}(w_n^{\alpha,\beta})^k}{n^{k-\delta k}} = O(X_m) \quad \text{as } m \to \infty, \tag{19}$$

where $w_n^{\alpha,\beta}$ is given by [17]

$$w_n^{\alpha,\beta} = \begin{cases} \max_{1\le v\le n} |t_v^{\alpha,\beta}|, & \beta > -1, 0 < \alpha < 1, \\ |t_n^{\alpha,\beta}|, & \beta > -1, \alpha = 1. \end{cases} \tag{20}$$

4 Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 4.1 ([18]) *If $0 < \alpha \le 1$, $\beta > -1$ and $1 \le v \le n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p\right| \le \max_{1\le m\le v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p\right|. \tag{21}$$

Lemma 4.2 ([19]) *Let (X_n) be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$. If (A_n) is a ξ-quasi-monotone sequence with $\Delta A_n \le \xi_n$ and $\sum n\xi_n X_n < \infty$, then*

$$\sum_{n=1}^{\infty} nX_n|A_n| < \infty, \tag{22}$$

$$nA_nX_n = O(1) \quad \text{as } n \to \infty. \tag{23}$$

5 Proof of the theorem

Let $t_n^{\alpha,\beta}$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then the series will be $\varphi - |C,\alpha,\beta;\delta|_k$ summable (by Definition 4), if

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_n^{\alpha,\beta}\right|^k < \infty. \tag{24}$$

Applying Abel's transformation and Lemma 4.1, we have

$$\begin{aligned} T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v \\ &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \end{aligned} \tag{25}$$

$$\begin{aligned} \left|T_n^{\alpha,\beta}\right| &\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p\right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v\right| \\ &\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha+\beta} w_v^{\alpha,\beta} |\Delta\lambda_v| + |\lambda_n| w_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned} \tag{26}$$

We use Minkowski's inequality,

$$\left|T_n^{\alpha,\beta}\right|^k = \left|T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}\right|^k \le 2^k\left(\left|T_{n,1}^{\alpha,\beta}\right|^k + \left|T_{n,2}^{\alpha,\beta}\right|^k\right). \tag{27}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,r}^{\alpha,\beta}\right|^k < \infty, \quad \text{for } r = 1, 2. \tag{28}$$

By using Hölder's inequality, Abel's transformation and the conditions of Lemma 4.2 [19], we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,1}^{\alpha,\beta}\right|^k &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \frac{1}{(A_n^{\alpha+\beta})^k} \left(\sum_{\nu=1}^{n-1} A_\nu^{\alpha+\beta} w_\nu^{\alpha,\beta} |\Delta\lambda_\nu|\right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{(1+\alpha+\beta-\delta)k}} \sum_{\nu=1}^{n-1} \nu^{(\alpha+\beta)k} \left(w_\nu^{\alpha,\beta}\right)^k |A_\nu| \left(\sum_{\nu=1}^{n-1} |A_\nu|\right)^{k-1} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{(\alpha+\beta)k} \left(w_\nu^{\alpha,\beta}\right)^k |A_\nu| \sum_{n=\nu+1}^{m+1} \frac{\varphi_n^{k-1}}{n^{(1+\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{\nu=1}^{m} \nu^{(\alpha+\beta)k} \left(w_\nu^{\alpha,\beta}\right)^k |A_\nu| \frac{\varphi_\nu^{k-1}}{\nu^{(1+\alpha+\beta-\delta)k-1}} \\
&= O(1) \sum_{\nu=1}^{m} \nu |A_\nu| \left(w_\nu^{\alpha,\beta}\right)^k \frac{\varphi_\nu^{k-1}}{\nu^{k-\delta k}} \\
&= O(1) \sum_{\nu=1}^{m-1} \Delta\left(\nu|A_\nu|\right) \sum_{r=1}^{\nu} \left(w_r^{\alpha,\beta}\right)^k \frac{\varphi_r^{k-1}}{r^{k-\delta k}} \\
&\quad + O(1) m|A_m| \sum_{\nu=1}^{m} \left(w_\nu^{\alpha,\beta}\right)^k \frac{\varphi_\nu^{k-1}}{\nu^{k-\delta k}} \\
&= O(1) \sum_{\nu=1}^{m-1} \left|(\nu+1)\Delta|A_\nu| - |A_\nu|\right| X_\nu + O(1) m|A_m| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \nu|\Delta A_\nu| X_\nu + O(1) \sum_{\nu=1}^{m-1} |A_\nu| X_\nu + O(1) m|A_m| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \nu \xi_\nu X_\nu + O(1) \sum_{\nu=1}^{m-1} |A_\nu| X_\nu + O(1) m|A_m| X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\sum_{n=2}^{m} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left|T_{n,2}^{\alpha,\beta}\right|^k &= O(1) \sum_{n=1}^{m} |\lambda_n| \left(w_n^{\alpha,\beta}\right)^k \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{\nu=1}^{n} \left(w_\nu^{\alpha,\beta}\right)^k \frac{\varphi_\nu^{k-1}}{\nu^{k-\delta k}} + O(1)|\lambda_m| \sum_{n=1}^{m} \left(w_n^{\alpha,\beta}\right)^k \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} |A_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \quad \text{as } m \to \infty.
\end{aligned} \tag{30}$$

Collecting (24)-(30), we have

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} \left| T_n^{\alpha,\beta} \right|^k < \infty. \tag{31}$$

Hence the proof of the theorem is completed.

6 Corollaries

Corollary 6.1 *Let X_n be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$ and there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying* (13)-(17) *and the following condition*:

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha,\beta})^k}{n^{1-\delta k}} = O(X_m) \quad \text{as } m \to \infty, \tag{32}$$

then the series $\sum a_n \lambda_n$ is $|C,\alpha,\beta;\delta|_k$ summable, $\alpha + \beta > \delta$, $0 < \alpha \leq 1$, $\beta > -1$, $\delta \geq 0$, $k \geq 1$, where $w_n^{\alpha,\beta}$ is given by (20).

Proof On putting $\varphi_n = n$ in Theorem 3.1, we will get (32) and the following condition:

$$\sum_{n=v}^{m} \frac{1}{n^{1+k(\alpha+\beta-\delta)}} = O\left(\frac{1}{v^{(\alpha+\beta-\delta)k}}\right). \tag{33}$$

Here, condition (33) always holds. We omit the details as the proof is similar to that of Theorem 3.1 using the conditions (33) and (32) instead of (18) and (19). □

Corollary 6.2 *Let X_n be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$ and there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying* (13)-(17) *and the following conditions*:

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k(1+\alpha+\beta)}} = \frac{\varphi_v^{k-1}}{v^{k(1+\alpha+\beta)-1}}, \tag{34}$$

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha,\beta})^k \varphi_n^{k-1}}{n^k} = O(X_m) \quad \text{as } m \to \infty, \tag{35}$$

then the series $\sum a_n \lambda_n$ is $\varphi - |C,\alpha,\beta|_k$ summable, $\alpha + \beta > 0$, $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, where $w_n^{\alpha,\beta}$ is given by (20).

Proof On putting $\delta = 0$ in Theorem 3.1, we will get (34) and (35). We omit the details as the proof is similar to that of Theorem 3.1 using the conditions (34) and (35) instead of (18) and (19). □

Corollary 6.3 ([14]) *Let X_n be a quasi-f-power increasing sequence for some η $(0 < \eta < 1)$ and there exists a sequence of numbers (A_n) such that it is ξ-quasi-monotone satisfying* (13)-(17) *and the following conditions*:

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} = O(X_m) \quad \text{as } m \to \infty, \tag{36}$$

then the series $\sum a_n\lambda_n$ *is* $|C,\alpha|_k$ *summable,* $0 < \alpha \le 1$, $k \ge 1$, *where* w_n^α *is given by*

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\le v\le n} |t_v^\alpha|, & 0 < \alpha < 1. \end{cases} \tag{37}$$

Proof On putting $\varphi_n = n$, $\delta = 0$ and $\beta = 0$ in Theorem 3.1, we will get (36) and the following condition:

$$\sum_{n=v}^{m} \frac{1}{n^{1+k\alpha}} = O\left(\frac{1}{v^{k\alpha}}\right). \tag{38}$$

Here, condition (38) always holds. We omit the details as the proof is similar to that of Theorem 3.1 using the conditions (38) and (36) instead of (18) and (19). □

7 Conclusion

The aim of our paper is to obtain the minimal set of sufficient conditions for an infinite series to be absolute Cesáro $\varphi - |C,\alpha,\beta;\delta|_k$ summable. Through the investigation, we may conclude that our theorem is a generalized version which can be reduced for several well-known summabilities as shown in the corollaries. Further, our theorem has been validated through Corollary 6.3, which is a result of Bor [14].

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Authors' contributions
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