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The approximation of Laplace-Stieltjes transforms with finite order

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Abstract

In this paper, we study the irregular growth of an entire function defined by the Laplace-Stieltjes transform of finite order convergent in the whole complex plane and obtain some results about λ-lower type. In addition, we also investigate the problem on the error in approximating entire functions defined by the Laplace-Stieltjes transforms. Some results about the irregular growth, the error, and the coefficients of Laplace-Stieltjes transforms are obtained; they are generalization and improvement of the previous conclusions given by Luo and Kong, Singhal and Srivastava.

MSC: 44A10; 30D15

Keywords: irregular growth; Laplace-Stieltjes transform; approximate; error

1 Introduction

Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_n s},\quad s=\sigma+it, \tag{1}$$

where

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\qquad \lambda_n\to\infty\quad \text{as } n\to\infty; \tag{2}$$

$s=\sigma+it$ (σ, t are real variables), a_n are nonzero complex numbers. When a_n, λ_n, n satisfy some conditions, the series (1) is convergent in the whole plane or the half-plane, that is, $f(s)$ is an analytic function or entire function in the whole plane or the half-plane. In the past few decades, many mathematicians studied the growth and value distribution of the analytic (entire) function defined by Dirichlet series and obtained lots of interesting results (see [1–9]).

As we know, Dirichlet series is regarded as a special example of the Laplace-Stieltjes transform. The Laplace-Stieltjes transform, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. For real-valued functions, it is the Laplace transform of a Stieltjes measure, however it is often defined for functions with values in a Banach space. It can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

For the Laplace-Stieltjes transforms,

$$G(s) = \int_0^{+\infty} e^{-sx}\,d\alpha(x), \quad s = \sigma + it, \tag{3}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y]$ $(0 < Y < +\infty)$, and σ and t are real variables. Let

$$B_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-ity}\,d\alpha(y) \right|,$$

where the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies (2) and

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty. \tag{4}$$

In 1963, Yu [10] proved the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace-Stieltjes.

Theorem A *Suppose that Laplace-Stieltjes transforms* (3) *satisfy* (2), (4) *and* $\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} < +\infty$, *then*

$$\limsup_{n\to+\infty} \frac{\log B_n^*}{\lambda_n} \leq \sigma_u^G \leq \limsup_{n\to+\infty} \frac{\log B_n^*}{\lambda_n} + \limsup_{n\to+\infty} \frac{\log n}{\lambda_n},$$

where σ_u^F *is called the abscissa of uniform convergence of* $F(s)$.

Moreover, Yu [10] first introduced the maximal molecule $M_u(\sigma, G)$, the maximal term $\mu(\sigma, G)$ and the Borel line, and the order of analytic functions represented by Laplace-Stieltjes transforms convergent in the complex plane. After his works, considerable attention has been paid to the growth and value distribution of the functions represented by the Laplace-Stieltjes transform convergent in the half-plane or the whole complex plane in the field of complex analysis (see [11–15]).

In 2012, Luo and Kong [16] studied the following form of Laplace-Stieltjes transform:

$$F(s) = \int_0^{+\infty} e^{sx}\,d\alpha(x), \quad s = \sigma + it, \tag{5}$$

where $\alpha(x)$ is stated as in (3), and $\{\lambda_n\}$ satisfies (2),(4). Set

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity}\,d\alpha(y) \right|.$$

By using the same argument as in [10], we can get a similar result about the abscissa of uniform convergence of $F(s)$ easily. If

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D < \infty, \qquad \limsup_{n\to+\infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \tag{6}$$

by (2), (4) and Theorem 1, one can get that $\sigma_u^F = +\infty$, *i.e.*, $F(s)$ is an entire function.

Set

$$M(\sigma,F)=\sup_{-\infty<t<+\infty}\left|F(\sigma+it)\right|,\qquad M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x e^{(\sigma+it)y}\,d\alpha(y)\right|$$

and

$$\mu(\sigma,F)=\max_{n\in N}\left\{A_n^*e^{\lambda_n\sigma}\right\}(\sigma<+\infty),\quad N(\sigma,F)=\max\left\{\lambda_n:\ A_n^*e^{\lambda_n\sigma}=\mu(\sigma,F)\right\}.$$

Since $M(\sigma,F)$ and $M_u(\sigma,F)$ tend to $+\infty$ as $\sigma\to+\infty$, in order to estimate the growth of $F(s)$ more precisely, we will adapt some concepts of order, lower order, type, lower type as follows.

Definition 1.1 If Laplace-Stieltjes transform (5) satisfies $\sigma_u^F=+\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)) and

$$\limsup_{\sigma\to+\infty}\frac{\log^+\log^+M_u(\sigma,F)}{\sigma}=\rho,$$

we call $F(s)$ of order ρ in the whole plane, where $\log^+x=\max\{\log x,0\}$. If $\rho\in(0,+\infty)$, we say that $F(s)$ is an entire function of finite order in the whole plane. Moreover, the lower order of $F(s)$ is defined by

$$\liminf_{\sigma\to+\infty}\frac{\log^+\log^+M_u(\sigma,F)}{\sigma}=\lambda.$$

Remark 1.1 We say that $F(s)$ is of the regular growth, when $\rho=\lambda$, and $F(s)$ is of the irregular growth, when $\rho\neq\lambda$.

Definition 1.2 If Laplace-Stieltjes transform (5) satisfies $\sigma_u^F=+\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)) and is of order ρ $(0<\rho<\infty)$, then we define the type and lower type of L-S transform $F(s)$ as follows:

$$\limsup_{\sigma\to+\infty}\frac{\log^+M_u(\sigma,F)}{e^{\sigma\rho}}=T,\qquad\liminf_{\sigma\to+\infty}\frac{\log^+M_u(\sigma,F)}{e^{\sigma\rho}}=\tau.$$

Remark 1.2 The purpose of the definition of type is to compare the growth of class functions which all have the same order. For example, let $f(s)=e^{e^s}$, $g(s)=e^{e^{2s}}$, by a simple computation, we have $\rho(f)=1=\rho(g)$, but $T(f)=1$ and $T(g)=\infty$. Thus, we can see that the growth of $g(s)$ is faster than $f(s)$ as $|s|\to+\infty$.

2 Results and discussion

Recently, many people studied some problems on analytic functions defined by the Laplace-Stieltjes transforms and obtained a number of interesting results. Kong, Sun, Huo and Xu investigated the growth of analytic functions with kinds of order defined by the Laplace-Stieltjes transforms (see [16–22]), and Shang, Gao, and Sun investigated the value distribution of such functions (see [23–26]). From these references, we get the following results.

Theorem 2.1 *If Laplace-Stieltjes transform* (5) *satisfies* $\sigma_u^F = +\infty$ (*the sequence* $\{\lambda_n\}$ *satisfies* (2), (4), *and* (6)), *and is of order* ρ $(0 < \rho < \infty)$ *and of type* T, *then*

$$\rho = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, \qquad T = \limsup_{n\to+\infty} \frac{\lambda_n}{\rho e}\left(A_n^*\right)^{\frac{\rho}{\lambda_n}}.$$

Furthermore, if $F(s)$ *is of the lower order* λ *and the lower type* τ, *and* $\lambda_n \sim \lambda_{n+1}$ *and the function*

$$\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}$$

forms a non-decreasing function of n for $n > n_0$, *then we have*

$$\lambda = \liminf_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, \qquad \tau = \liminf_{n\to+\infty} \frac{\lambda_n}{\rho e}\left(A_n^*\right)^{\frac{\rho}{\lambda_n}}.$$

From Definition 1.2, a natural question to ask is: What happened if $e^{\sigma\rho}$ is replaced by $e^{\lambda\sigma}$ in the definition of lower type when $\rho \neq \lambda$? We are going to consider this question.

Definition 2.1 If Laplace-Stieltjes transform (5) satisfies $\sigma_u^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)), and is of order ρ $(0 < \rho < \infty)$ and of the lower order λ $(0 < \lambda < \infty)$, if $\lambda \neq \rho$, and

$$\liminf_{\sigma\to+\infty} \frac{\log^+ M_u(\sigma, F)}{e^{\sigma\lambda}} = \tau_\lambda,$$

we say that τ_λ is the λ-type of $F(s)$.

Remark 2.1 Obviously, $\tau_\lambda \geq \tau$ and $\tau_\lambda = \tau$ as $\rho = \lambda$. But we cannot confirm whether $\tau_\lambda \geq T$ or $\tau_\lambda \leq T$.

The following results are the main theorems of this paper.

Theorem 2.2 *If Laplace-Stieltjes transform* (5) *satisfies* $\sigma_u^F = +\infty$ (*the sequence* $\{\lambda_n\}$ *satisfies* (2), (4), *and* (6)), *and is of order* ρ *and of the lower order* λ, $0 \leq \lambda \neq \rho < \infty$, *then we have*

$$\liminf_{\sigma\to\infty} \frac{\log M(\sigma, F)}{e^{\rho\sigma}} = \liminf_{\sigma\to\infty} \frac{\log \mu(\sigma, F)}{e^{\rho\sigma}} = 0, \tag{7}$$

and

$$\liminf_{\sigma\to\infty} \frac{N(\sigma, F)}{e^{\rho\sigma}} = 0. \tag{8}$$

Theorem 2.3 *If Laplace-Stieltjes transform* (5) *satisfies* $\sigma_u^F = +\infty$ (*the sequence* $\{\lambda_n\}$ *satisfies* (2), (4), *and* (6)), *and is of order* ρ *and of the lower order* λ, $0 < \lambda \neq \rho < \infty$, *type* T, λ-*type* τ_λ,

$$\limsup_{\sigma\to+\infty} \frac{N(\sigma, F)}{e^{\rho\sigma}} = H, \qquad \liminf_{\sigma\to+\infty} \frac{N(\sigma, F)}{e^{\rho\sigma}} = h,$$

and let

$$T_\rho(\sigma,F)=\frac{\log\mu(\sigma,F)}{\exp(\rho\sigma)},\qquad T_\lambda(\sigma,F)=\frac{\log\mu(\sigma,F)}{\exp(\lambda\sigma)},$$

then we have

$$H-\rho T\le\limsup_{\sigma\to+\infty}T'_\rho(\sigma,F)\le H, \tag{9}$$

$$-\infty\le\liminf_{\sigma\to+\infty}T'_\lambda(\sigma,F)\le h-\lambda\tau_\lambda \tag{10}$$

for almost all values of $\sigma>\sigma_0$, *where* $T'_\rho(\sigma)$ *and* $T'_\lambda(\sigma)$ *are the derivatives of* $T_\rho(\sigma)$ *and* $T_\lambda(\sigma)$ *with respect to* σ.

Theorem 2.4 *If Laplace-Stieltjes transform* (5) *satisfies* $\sigma_u^F=+\infty$ (*the sequence* $\{\lambda_n\}$ *satisfies* (2), (4), *and* (6)), *and is of the lower order* λ $(0\le\lambda\ne\rho<\infty)$, *if* $\lambda_n\sim\lambda_{n+1}$, *then*

$$\tau_\lambda\ge\liminf_{n\to\infty}\left(\frac{\lambda_n}{e\lambda}\right)\left(A_n^*\right)^{\frac{\lambda}{\lambda_n}}\quad(0\le\tau_\lambda\le\infty). \tag{11}$$

Furthermore, there exists a positive integer n_0 *such that*

$$\psi(n)=\frac{\log A_n^*-\log A_{n+1}^*}{\lambda_{n+1}-\lambda_n}$$

forms a non-decreasing function of n for $n>n_0$, *then we have*

$$\tau_\lambda=\liminf_{n\to\infty}\left(\frac{\lambda_n}{e\lambda}\right)\left(A_n^*\right)^{\frac{\lambda}{\lambda_n}}\quad(0\le\tau_\lambda\le\infty). \tag{12}$$

We denote by $\overline{L}_\beta$ the class of all the functions $F(s)$ of the form (5) which are analytic in the half-plane $\Re s<\beta$ $(-\infty<\beta<\infty)$ and the sequence $\{\lambda_n\}$ satisfies (2) and (4); and we denote by L_∞ the class of all the functions $F(s)$ of the form (5) which are analytic in the half-plane $\Re s<+\infty$ and the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6). Thus, if $-\infty<\beta<+\infty$ and $F(s)\in\overline{L}_\beta$, then $F(s)\in L_\infty$. If Laplace-Stieltjes transform (5) $A_n^*=0$ for $n\ge k+1$ and $A_n^*\ne0$, then $F(s)$ will be called an exponential polynomial of degree k usually denoted by p_k, *i.e.*, $p_k(s)=\int_0^{\lambda_k}\exp(sy)\,d\alpha(y)$. When we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i\exp(s\lambda_i)$.

For $F(s)\in\overline{L}_\beta$, $-\infty<\beta<+\infty$, we denote by $E_n(F,\beta)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F,\beta)=\inf_{p\in\Pi_n}\|F-p\|_\beta,\quad n=1,2,\ldots,$$

where

$$\|F-p\|_\beta=\max_{-\infty<t<+\infty}\left|F(\beta+it)-p(\beta+it)\right|.$$

In this paper, we will further investigate the relation between $E_n(F,\beta)$ and the growth of an entire function defined by the L-S transform with irregular growth. It seems that this problem has never been treated before. Our main result is as follows.

Theorem 2.5 *If the Laplace-Stieltjes transform $F(s) \in L_\infty$ and is of lower order λ $(0 \leq \lambda \neq \rho < \infty)$, if $\lambda_n \sim \lambda_{n+1}$, then for any real number $-\infty < \beta < +\infty$, we have*

$$\tau_\lambda \geq \liminf_{n\to\infty} \left(\frac{\lambda_n}{e\lambda}\right)\bigl(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\bigr)^{\frac{\lambda}{\lambda_n}} \quad (0 \leq \tau_\lambda \leq \infty). \tag{13}$$

Furthermore, there exists a positive integer n_0 such that

$$\psi_1(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}$$

forms a non-decreasing function of n for $n > n_0$, then we have

$$\tau_\lambda = \liminf_{n\to\infty} \left(\frac{\lambda_n}{e\lambda}\right)\bigl(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\bigr)^{\frac{\lambda}{\lambda_n}} \quad (0 \leq \tau_\lambda \leq \infty),$$

i.e.,

$$\exp(\beta\lambda)e\lambda\tau_\lambda = \liminf_{n\to\infty} \lambda_n \bigl(E_{n-1}(F,\beta)\bigr)^{\frac{\lambda}{\lambda_n}}. \tag{14}$$

3 Conclusions

From Theorems 2.2-2.5, we can see that the growth of Laplace-Stieltjes transforms is investigated under the assumption $\rho \neq \lambda$, and that some theorems about the λ-lower type τ_λ, λ_n, A_n^*, and λ are obtained. In addition, we also study the problem on the error in approximating entire functions defined by the Laplace-Stieltjes transforms. This project is a new issue of Laplace-Stieltjes transforms in the field of complex analysis. Our results are generalization and improvement of the previous conclusions given by Luo and Kong [16, 27], Singhal and Srivastava [28].

4 Methods

4.1 Proofs of Theorems 2.2 and 2.3

To prove the above theorems, we require the following lemmas.

Lemma 4.1 (see [27], Lemma 2.1) *If the L-S transform $F(s) \in L_\infty$, for any σ $(-\infty < \sigma < +\infty)$ and $\varepsilon(> 0)$, we have*

$$\frac{1}{2}\mu(\sigma,F) \leq M_u(\sigma,F) \leq C\mu\bigl((1+2\varepsilon)\sigma,F\bigr),$$

where C is a constant.

Lemma 4.2 (see [16], Lemma 2.2) *If the L-S transform $F(s) \in L_\infty$, then we have*

$$\log\mu(\sigma,F) = \log\mu(\sigma_0,F) + \int_{\sigma_0}^{\sigma} N(t,F)\,dt$$

for $\sigma_0 > 0$.

4.1.1 The proof of Theorem 2.2

Since $\rho > \lambda > 0$ and $F(s)$ is of the lower order λ, that is,

$$\lambda = \liminf_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma}, \tag{15}$$

for any small $\varepsilon (0 < \varepsilon < \rho - \lambda)$, it follows from (15) that there exists a constant σ_0 such that, for $\sigma > \sigma_0$,

$$\log M_u(\sigma, F) > \exp\{(\lambda - \varepsilon)\sigma\}, \tag{16}$$

and there exists a sequence $\{\sigma_k\}$ tending to $+\infty$ such that

$$\log M_u(\sigma_k, F) < \exp\{(\lambda + \varepsilon)\sigma_k\}. \tag{17}$$

Since $0 < \varepsilon < \rho - \lambda$, it follows from (16) and (17) that

$$\liminf_{\sigma \to +\infty} \frac{\log M_u(\sigma, F)}{\exp(\rho\sigma)} = 0. \tag{18}$$

From Lemmas 4.1 and 4.2, we have

$$\rho = \limsup_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log \log \mu(\sigma, F)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log N(\sigma, F)}{\sigma}$$

and

$$\lambda = \liminf_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma} = \liminf_{\sigma \to +\infty} \frac{\log \log \mu(\sigma, F)}{\sigma} = \liminf_{\sigma \to +\infty} \frac{\log N(\sigma, F)}{\sigma}.$$

Thus, similar to the process of (18), we can easily prove

$$\liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{\exp(\rho\sigma)} = \liminf_{\sigma \to +\infty} \frac{N(\sigma, F)}{\exp(\rho\sigma)} = 0.$$

Hence, this completes the proof of Theorem 2.2.

4.1.2 The proof of Theorem 2.3

From Lemma 4.2, it follows that

$$\limsup_{\sigma \to +\infty} \frac{\int_{\sigma_0}^{\sigma} N(t, F)\, dt}{e^{\rho\sigma}} = \limsup_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{e^{\rho\sigma}} = \limsup_{\sigma \to +\infty} T_\rho(\sigma, F) = T \tag{19}$$

and

$$\liminf_{\sigma \to +\infty} \frac{\int_{\sigma_0}^{\sigma} N(t, F)\, dt}{e^{\lambda\sigma}} = \liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{e^{\lambda\sigma}} = \liminf_{\sigma \to +\infty} T_\lambda(\sigma, F) = \tau_\lambda. \tag{20}$$

Dividing two sides of the equality in Lemma 4.2 by $e^{\rho\sigma}$ and differentiating it with respect to σ, for almost all values $\sigma > \sigma_0$, we have

$$T'_\rho(\sigma, F) = -\rho \frac{\log \mu(\sigma_0, F)}{e^{\rho\sigma}} - \frac{\rho}{e^{\rho\sigma}} \int_{\sigma_0}^{\sigma} N(t, F)\, dt + \frac{N(\sigma, F)}{e^{\rho\sigma}}. \tag{21}$$

On the basis of the assumptions of Theorem 2.3, taking lim sup in (21) when $\sigma \to +\infty$, from Theorem 2.2 and (19), we get (9) easily.

Similarly, dividing two sides of the equality in Lemma 4.2 by $e^{\lambda\sigma}$ and differentiating it with respect to σ, for almost all values $\sigma > \sigma_0$,

$$T'_\lambda(\sigma,F) = -\lambda\frac{\log\mu(\sigma_0,F)}{e^{\lambda\sigma}} - \frac{\lambda}{e^{\lambda\sigma}}\int_{\sigma_0}^{\sigma} N(t,F)\,dt + \frac{N(\sigma,F)}{e^{\lambda\sigma}}. \tag{22}$$

On the basis of the assumptions of Theorem 2.3, taking lim inf in (22) when $\sigma \to +\infty$, from Theorem 2.1 and (20), we get (10) easily.

Thus, this completes the proof of Theorem 2.3.

4.2 The proof of Theorem 2.4

Let

$$\vartheta = \liminf_{n\to+\infty}\frac{\lambda_n}{e\lambda}\left(A_n^*\right)^{\frac{\lambda}{\lambda_n}} \quad (0 < \vartheta < +\infty).$$

Thus, for any $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that, for $n > n_0(\varepsilon)$,

$$\lambda_n\left(A_n^*\right)^{\frac{\lambda}{\lambda_n}} > (\vartheta - \varepsilon)e\lambda. \tag{23}$$

By Lemma 4.1, it follows from (23) that for $n > n_0(\varepsilon)$

$$\begin{aligned}\frac{\log M_u(\sigma,F)}{e^{\lambda\sigma}} &\geq \frac{\log A_n^* + \lambda_n\sigma - \log 2}{e^{\lambda\sigma}} \\ &> e^{-\lambda\sigma}\left(\lambda_n\sigma + \frac{\lambda_n}{\lambda}\log\left[(\vartheta-\varepsilon)e\lambda\right] - \frac{\lambda_n}{\lambda}\log\lambda_n - \log 2\right).\end{aligned} \tag{24}$$

Let

$$\left(\frac{\lambda_n}{\lambda\vartheta}\right)^{\frac{1}{\lambda}} \leq e^{\sigma} < \left(\frac{\lambda_{n+1}}{\lambda\vartheta}\right)^{\frac{1}{\lambda}},$$

and take

$$\sigma = \frac{1}{\lambda}\log\left(\frac{\lambda_n}{\lambda\vartheta}\right) + o\left(\frac{1}{\lambda_n}\right).$$

Then from (24) it follows

$$\frac{\log M_u(\sigma,F)}{e^{\lambda\sigma}} \geq \frac{\lambda\vartheta}{\lambda_{n+1}}\left(\frac{\lambda_n}{\lambda}\log\frac{1}{\lambda\vartheta} + \frac{\lambda_n}{\lambda}\log\bigl((\vartheta-\varepsilon)e\lambda\bigr) - \log 2 + o(1)\right). \tag{25}$$

Since $\lambda_n \sim \lambda_{n+1}$ and $\lambda_n \to +\infty$ as $n \to +\infty$, thus by a simple computation, from (25) we have $\tau_\lambda \geq \vartheta$. When $\vartheta = 0$, $\tau_\lambda \geq \vartheta$ is obvious; if $\vartheta = \infty$, we also prove that $\tau_\lambda \geq \vartheta$ by using the same argument as above. Hence we prove that (11) holds.

Let $\mu(\sigma,F)$ denote the maximum term for $\Re s = \sigma$, $-\infty < t < +\infty$. Since

$$\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}$$

forms a non-decreasing function of n for $n > n_0$, then for $\psi(n-1) \leq \sigma < \psi(n)$

$$\log \mu(\sigma, F) = \log A_n^* + \lambda_n \sigma.$$

Since $\tau_\lambda < \infty$, for any small $\varepsilon > 0$, it follows from (20) that

$$\log \mu(\sigma, F) = \log A_n^* + \lambda_n \sigma \geq (\tau_\lambda - \varepsilon) \exp(\lambda\sigma) \tag{26}$$

for $\sigma > \sigma_0$ and all n such that $\psi(n-1) \leq \sigma < \psi(n)$.

Let $\Re s = \sigma > \sigma_0$ and $A_{n_1}^* \exp(\sigma\lambda_{n_1})$ and $A_{n_2}^* \exp(\sigma\lambda_{n_2})$ $(n_1 > n_0, \psi(n-1) > \sigma_0)$ be two consecutive maximum terms such that $n_2 - 1 \geq n_1$, it follows from (26) that

$$\log A_{n_2}^* + \lambda_{n_2} \sigma \geq (\tau_\lambda - \varepsilon) \exp(\lambda\sigma)$$

for all $\sigma > \sigma_0$ satisfying $\psi(n_2 - 1) \leq \sigma < \psi(n_2)$. Let $n_1 \leq n \leq n_2 - 1$, then

$$\psi(n_1) = \psi(n_1 + 1) = \cdots = \psi(n) = \cdots = \psi(n_2 - 1)$$

and $A_n^* \exp(\lambda_n \sigma) = A_{n_2}^* \exp(\lambda_{n_2} \sigma)$ for $\sigma = \psi(n)$. Then there exists a positive integer n_1 such that, for $n > n_1$ and $\sigma > \sigma_0$,

$$\log A_n^* > (\tau_\lambda - \varepsilon) e^{\lambda\sigma} - \lambda_n \sigma.$$

Since $e^x \geq ex$ for any x, so it follows

$$\lambda_n \bigl(A_n^*\bigr)^{\frac{\lambda}{\lambda_n}} > \frac{\lambda_n}{e^{\lambda\sigma}} \exp\left\{\frac{\lambda(\tau_\lambda - \varepsilon)}{\lambda_n} e^{\lambda\sigma}\right\} > \frac{\lambda_n}{e^{\lambda\sigma}} \frac{e(\tau_\lambda - \varepsilon)\lambda}{\lambda_n} e^{\lambda\sigma} = e(\tau_\lambda - \varepsilon)\lambda. \tag{27}$$

Thus, for $\varepsilon \to 0$ and $n \to +\infty$, from (27) it follows

$$\vartheta = \liminf_{n \to +\infty} \frac{\lambda_n}{e\lambda} \bigl(A_n^*\bigr)^{\frac{\lambda}{\lambda_n}} \geq \tau_\lambda. \tag{28}$$

Hence, this proves that (12) holds.

4.3 The proof of Theorem 2.5

To prove this theorem, we require the following lemma.

Lemma 4.3 *If the abscissa $\sigma_u^F = +\infty$ of uniform convergence of the Laplace-Stieltjes transformation $F(s)$ and sequence* (2) *satisfies* (4), (6), *then for any real number β, we have*

$$\left| \int_{\lambda_k}^{\infty} \exp\bigl\{(\beta + it)y\bigr\}\, d\alpha(y) \right| \leq 2 \sum_{n=k}^{+\infty} A_n^* \exp\{\beta\lambda_{n+1}\},$$

where

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity}\, d\alpha(y) \right|.$$

Proof Set

$$I(x;it)=\int_0^x \exp\{ity\}\,d\alpha(y).$$

For any real number β, since

$$\left|\int_{\lambda_k}^{\infty}\exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right|=\lim_{b\to+\infty}\left|\int_{\lambda_k}^{b}\exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right|.$$

Set $I_{j+k}(b;it)=\int_{\lambda_{j+k}}^{b}\exp\{ity\}\,d\alpha(y)$, $(\lambda_{j+k}<b\le\lambda_{j+k+1})$, then we have $|I_{j+k}(b;it)|\le A^*_{j+k}$. Thus, it follows

$$\begin{aligned}
&\left|\int_{\lambda_k}^{b}\exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right|\\
&\quad=\left|\sum_{j=k}^{n+k-1}\int_{\lambda_j}^{\lambda_{j+1}}\exp\{\beta y\}d_yI_j(y;it)+\int_{\lambda_{n+k}}^{b}\exp\{\beta y\}d_yI_{n+k}(y;it)\right|\\
&\quad=\left|\left[\sum_{j=k}^{n+k-1}e^{\lambda_{j+1}\beta}I_j(\lambda_{j+1};it)-\beta\int_{\lambda_j}^{\lambda_{j+1}}e^{\beta y}I_j(y;it)\,dy\right]\right.\\
&\qquad\left.+e^{\beta b}I_{n+k}(b;it)-\beta\int_{\lambda_{n+k}}^{b}e^{\beta y}I_j(y;it)\,dy\right|\\
&\quad\le\sum_{j=k}^{n+k-1}\left[A^*_je^{\lambda_{j+1}\beta}+A^*_j\bigl(e^{\lambda_{j+1}\beta}-e^{\lambda_j\beta}\bigr)\right]+2e^{\beta\lambda_{n+k+1}}A^*_{n+k}-e^{\beta\lambda_{n+k}}A^*_{n+k}\\
&\quad\le 2\sum_{j=k}^{n+k}A^*_ne^{\lambda_{n+1}\beta}.
\end{aligned}$$

When $n\to+\infty$, we have $b\to+\infty$, thus we have

$$\left|\int_{\lambda_k}^{\infty}\exp\bigl\{(\beta+it)y\bigr\}\,d\alpha(y)\right|\le 2\sum_{n=k}^{+\infty}A^*_n\exp\{\beta\lambda_{n+1}\}.$$ □

Now, we are going to prove Theorem 2.5.

4.4 The proof of Theorem 2.5

Let

$$\vartheta_1=\liminf_{n\to\infty}\left(\frac{\lambda_n}{e\lambda}\right)\bigl(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\bigr)^{\frac{\lambda}{\lambda_n}}\quad(0<\vartheta_1<+\infty).$$

Then, for any small $\varepsilon>0$, there exists an integer $n_0(\varepsilon)$ such that, for any $n>n_0(\varepsilon)$,

$$\log\bigl(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\bigr)>\frac{\lambda_n}{\lambda}\log\frac{(\vartheta_1-\varepsilon)e\lambda}{\lambda_n}.\tag{29}$$

Since $F(s) \in L_\infty$, thus for any constant β ($-\infty < \beta < +\infty$), we have $F(s) \in \overline{L}_\beta$. For $\beta < \sigma < +\infty$. It follows from the definitions of $E_n(F,\beta)$ and p_n that

$$\begin{aligned} E_n(F,\beta) &\le \|F - p_n\|_\beta \le \left|F(\beta + it) - p_n(\beta + it)\right| \\ &\le \left|\int_0^{+\infty} \exp\{(\beta + it)y\}\, d\alpha(y) - \int_0^{\lambda_n} \exp\{(\beta + it)y\}\, d\alpha(y)\right| \\ &= \left|\int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\}\, d\alpha(y)\right|. \end{aligned} \tag{30}$$

Thus, from the definition of A_n^* and $M_u(\sigma,F)$, and by Lemma 4.1, we have $A_n^* \le 2M_u(\sigma,F)e^{-\sigma\lambda_n}$ for any σ ($\beta < \sigma < +\infty$). It follows from (30) and Lemma 4.3 that

$$E_n(F,\beta) \le 2\sum_{k=n+1}^{\infty} A_{k-1}^* \exp\{\beta\lambda_k\} \le 4M_u(\sigma,F)\sum_{k=n+1}^{\infty} \exp\{(\beta - \sigma)\lambda_k\}. \tag{31}$$

From (4), take h' ($0 < h' < h$) such that $(\lambda_{n+1} - \lambda_n) \ge h'$ for $n \ge 0$. Then, for $\sigma \ge \frac{\beta}{2}$, it follows from (31) that

$$\begin{aligned} E_n(F,\beta) &\le 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\sum_{k=n+1}^{\infty} \exp\{(\lambda_k - \lambda_{n+1})(\beta - \sigma)\} \\ &\le 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\exp\left\{-\frac{\beta}{2}h'(n+1)\right\}\sum_{k=n+1}^{\infty}\left(\exp\left\{\frac{\beta}{2}h'k\right\}\right) \\ &= 4M_u(\sigma,F)\exp\{\lambda_{n+1}(\beta - \sigma)\}\left(1 - \exp\left\{\frac{\beta}{2}h'\right\}\right)^{-1}, \end{aligned}$$

that is,

$$E_{n-1}(F,\beta) \le KM_u(\sigma,F)\exp\{\lambda_n(\beta - \sigma)\}, \tag{32}$$

where K is a constant. Let

$$\gamma_n = E_{n-1}(F,\beta)\exp(-\beta\lambda_n) \quad (n = 1,2,\ldots).$$

Thus, from (29) and (32), it follows that for $n > n_0(\varepsilon)$

$$\begin{aligned} \frac{\log M_u(\sigma,F)}{e^{\lambda\sigma}} &\ge \frac{\log\gamma_n + \lambda_n\sigma - \log K}{e^{\lambda\sigma}} \\ &> e^{-\lambda\sigma}\left(\lambda_n\sigma + \frac{\lambda_n}{\lambda}\log\left[(\vartheta_1 - \varepsilon)e\lambda\right] - \frac{\lambda_n}{\lambda}\log\lambda_n - \log K\right). \end{aligned} \tag{33}$$

By using the same argument as in Theorem 2.4, we can easily prove that $\tau_\lambda \ge \vartheta_1$.

From the proof of Theorem 2.4, we have that there exists a positive integer n_1 such that

$$\log A_n^* > (\tau_\lambda - \varepsilon)e^{\lambda\sigma} - \lambda_n\sigma$$

for $n > n_1$ and $\sigma > \sigma_0$. Since for any $\beta < +\infty$, from the definition of $E_k(F,\beta)$, there exists $p_1 \in \Pi_{n-1}$ such that

$$\|F - p_1\| \le 2E_{n-1}(F,\beta). \tag{34}$$

And since

$$\begin{aligned}
A_n^* \exp\{\beta\lambda_n\} &= \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} \exp\{ity\}\, d\alpha(y) \right| \exp\{\beta\lambda_n\} \\
&\le \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} \exp\{(\beta + it)y\}\, d\alpha(y) \right| \\
&\le \sup_{-\infty < t < +\infty} \left| \int_{\lambda_n}^{\infty} \exp\{(\beta + it)y\}\, d\alpha(y) \right|,
\end{aligned}$$

thus for any $p \in \Pi_{n-1}$, it follows

$$A_n^* \exp\{\beta\lambda_n\} \le \left| F(\beta + it) - p(\beta + it) \right| \le \|F - p\|_\beta. \tag{35}$$

Hence from (34) and (35), for any $\beta < +\infty$ and $F(s) \in L_\infty$, we have

$$A_n^* \exp\{\beta\lambda_n\} \le 2E_{n-1}(F,\beta).$$

Since $e^x \ge ex$ for any x, so it follows

$$\begin{aligned}
\lambda_n(\gamma_n)^{\frac{\lambda}{\lambda_n}} &> \frac{\lambda_n}{e^{\lambda\sigma}} \exp\left\{ \frac{\lambda(\tau_\lambda - \varepsilon)}{\lambda_n} e^{\lambda\sigma} - \frac{\lambda \log 2}{\lambda_n} \right\} \\
&> \frac{\lambda_n}{e^{\lambda\sigma}} \left(\frac{e(\tau_\lambda - \varepsilon)\lambda}{\lambda_n} e^{\lambda\sigma} \exp\{o(1)\} \right) = e(\tau_\lambda - \varepsilon)\lambda.
\end{aligned} \tag{36}$$

Thus, for $\varepsilon \to 0$ and $n \to +\infty$, from (36) it follows

$$\vartheta_1 = \liminf_{n\to\infty} \frac{\lambda_n}{e\lambda} (\gamma_n)^{\frac{\lambda}{\lambda_n}} \ge \tau_\lambda.$$

Since $[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)]^{\frac{\lambda}{\lambda_n}} = [E_{n-1}(F,\beta)]^{\frac{\lambda}{\lambda_n}} \exp(-\beta\lambda)$, then (14) follows.
Therefore, we complete the proof of Theorem 2.5.

4.5 Remarks

From the proof of Theorem 2.5, and combining those results of the Laplace-Stieltjes transforms in Ref. [14, 16, 27], we can obtain the following results on the approximation of Laplace-Stieltjes transforms, which can be found partly in [28].

Theorem 4.1 *If the L-S transform $F(s) \in L_\infty$ and is of order ρ $(0 < \rho < \infty)$ and of type T, then for any real number $-\infty < \beta < +\infty$, we have*

$$\rho = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F,\beta)\exp(-\beta\lambda_n)} = \limsup_{n\to+\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F,\beta)}$$

and

$$\begin{aligned} T &= \limsup_{n\to+\infty} \frac{\lambda_n}{\rho e}\big(E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\big)^{\frac{\rho}{\lambda_n}} \\ &= \limsup_{n\to+\infty} \frac{\lambda_n}{\rho\exp(\rho\beta+1)}\big(E_{n-1}(F,\beta)\big)^{\frac{\rho}{\lambda_n}}. \end{aligned}$$

Furthermore, if $F(s)$ is of the lower order λ and the lower type τ, and $\lambda_n \sim \lambda_{n+1}$ and the function

$$\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}$$

forms a non-decreasing function of n for $n > n_0$, then we have

$$\lambda = \liminf_{n\to+\infty} \frac{\lambda_n \log\lambda_n}{-\log E_{n-1}(F,\beta)}, \qquad \tau = \liminf_{n\to+\infty} \frac{\lambda_n}{\rho\exp(\rho\beta+1)}\big(E_{n-1}(F,\beta)\big)^{\frac{\rho}{\lambda_n}}.$$

Theorem 4.2 *If the L-S transform $F(s) \in L_\infty$, then for any real number $-\infty < \beta < +\infty$. For $p = 1$, we have*

$$\limsup_{\sigma\to+\infty} \frac{h(\log M_u(\sigma,F))}{h(\sigma)} - 1 = \limsup_{n\to+\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)])},$$

and for $p = 2, 3, \ldots$, we have

$$\begin{aligned} &\limsup_{n\to+\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)])} \\ &\quad \le \limsup_{\sigma\to+\infty} \frac{h(\log M_u(\sigma,F))}{h(\sigma)} \\ &\quad \le \limsup_{n\to+\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n}\log[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)])} + 1, \end{aligned}$$

where $h(x)$ satisfies the following conditions:

(i) *$h(x)$ is defined on $[a,+\infty)$ and is positive, strictly increasing, differentiable and tends to $+\infty$ as $x \to +\infty$;*

(ii) $\lim_{x\to+\infty} \frac{d(h(x))}{d(\log^{[p]} x)} = k \in (0,+\infty)$, $p \ge 1$, $p \in \mathbb{N}^+$, *where* $\log^{[0]} x = x$, $\log^{[1]} x = \log x$ *and* $\log^{[p]} x = \log(\log^{[p-1]} x)$.

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Competing interests
The authors declare that none of the authors has any competing interests in the manuscript.

Authors' contributions
HYX and SYL completed the main part of this article, HYX and SYL corrected the main theorems. All authors read and approved the final manuscript.

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