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An $\mathcal{S}$-type singular value inclusion set for rectangular tensors

Caili Sang*

*Correspondence: sangcl@126.com
College of Data Science and Information Engineering, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China

Abstract
An S-type singular value inclusion set for rectangular tensors is given. Based on the set, new upper and lower bounds for the largest singular value of nonnegative rectangular tensors are obtained and proved to be sharper than some existing results. Numerical examples are given to verify the theoretical results.

MSC: 15A18; 15A69

Keywords: rectangular tensors; nonnegative tensors; singular value; inclusion set

1 Introduction

Let $\mathbb{R}(\mathbb{C})$ be the real (complex) field, p, q, m, n be positive integers, $l = p + q$, $m, n \geq 2$ and $N = \{1, 2, \ldots, n\}$. We call $\mathcal{A} = (a_{i_1 \cdots i_p j_1 \cdots j_q})$ a real (p, q)th order $m \times n$ dimensional rectangular tensor, or simply a real rectangular tensor, denoted by $\mathcal{A} \in \mathbb{R}^{[p,q;m,n]}$, if

$$a_{i_1 \cdots i_p j_1 \cdots j_q} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_p \leq m, 1 \leq j_1, \ldots, j_q \leq n.$$

When $p = q = 1$, $\mathcal{A}$ is simply a real $m \times n$ rectangular matrix. This justifies the word 'rectangular'. We call $\mathcal{A}$ nonnegative, denoted by $\mathcal{A} \in \mathbb{R}_+^{[p,q;m,n]}$, if each of its entries $a_{i_1 \cdots i_p j_1 \cdots j_q} \geq 0$.

For any vectors $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$, $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ and any real number α, denote $x^{[\alpha]} = (x_1^\alpha, x_2^\alpha, \ldots, x_m^\alpha)^{\mathrm{T}}$ and $y^{[\alpha]} = (y_1^\alpha, y_2^\alpha, \ldots, y_n^\alpha)^{\mathrm{T}}$. Let $\mathcal{A}x^{p-1}y^q$ be a vector in $\mathbb{R}^m$ such that

$$\left(\mathcal{A}x^{p-1}y^q\right)_i = \sum_{i_2,\ldots,i_p=1}^{m} \sum_{j_1,\ldots,j_q=1}^{n} a_{i i_2 \cdots i_p j_1 \cdots j_q} x_{i_2} \cdots x_{i_p} y_{j_1} \cdots y_{j_q},$$

where $i = 1, \ldots, m$. Similarly, let $\mathcal{A}x^p y^{q-1}$ be a vector in $\mathbb{R}^n$ such that

$$\left(\mathcal{A}x^p y^{q-1}\right)_j = \sum_{i_1,\ldots,i_p=1}^{m} \sum_{j_2,\ldots,j_q=1}^{n} a_{i_1 \cdots i_p j j_2 \cdots j_q} x_{i_1} \cdots x_{i_p} y_{j_2} \cdots y_{j_q},$$

where $j = 1, \ldots, n$. If there are a number $\lambda \in \mathbb{C}$, vectors $x \in \mathbb{C}^m \backslash \{0\}$, and $y \in \mathbb{C}^n \backslash \{0\}$ such that

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[l-1]}, \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[l-1]}, \end{cases}$$

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then λ is called the singular value of $\mathcal{A}$, and (x, y) is a pair of left and right eigenvectors of $\mathcal{A}$, associated with λ, respectively. If $\lambda \in \mathbb{R}, x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$, then we say that λ is an H-singular value of $\mathcal{A}$, and (x, y) is a pair of left and right H-eigenvectors associated with λ, respectively. If a singular value is not an H-singular value, we call it an N-singular value of $\mathcal{A}$ [1]. We call

$$\lambda_0 = \max\{|\lambda| : \lambda \text{ is a singular value of } \mathcal{A}\}$$

the largest singular value [2].

Note here that the definition of singular values for tensors was first proposed by Lim in [3]. When l is even, the definition in [1] is the same as in [3]. When l is odd, the definition in [1] is slightly different from that in [3], but parallel to the definition of eigenvalues of square matrices [4]; see [1] for details.

When $m = n$, such real rectangular tensors have a sound application background. For example, the elasticity tensor is a tensor with $p = q = 2$ and $m = n = 2$ or 3; for details, see [1]. Due to the fact that singular values of rectangular tensors have a wide range of practical applications in the strong ellipticity condition problem in solid mechanics [5, 6] and the entanglement problem in quantum physics [7, 8], very recently, it has attracted attention of researchers [9–17]. Chang *et al.* [1] studied some properties of singular values of rectangular tensors, which include the Perron-Frobenius theorem of nonnegative irreducible tensors. Yang *et al.* [2] extended the Perron-Frobenius theorem of nonnegative irreducible tensors to nonnegative tensors, and gave the upper and lower bounds of the largest singular value of nonnegative rectangular tensors.

Our goal in this paper is to propose a singular value inclusion set for rectangular tensors and use the set to obtain new upper and lower bounds for the largest singular value of nonnegative rectangular tensors.

2 Main results

In this section, we begin with some notation. Let $\mathcal{A} \in \mathbb{R}^{[p,q;n,n]}$. For $\forall i, j \in N, i \neq j$, denote

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_p,j_1,\ldots,j_q \in N} |a_{ii_2\cdots i_pj_1\cdots j_q}|,$$

$$r_i^j(\mathcal{A}) = \sum_{\delta_{ji_2\cdots i_pj_1\cdots j_q}=0} |a_{ii_2\cdots i_pj_1\cdots j_q}| = R_i(\mathcal{A}) - |a_{ij\cdots jj\cdots j}|,$$

$$C_j(\mathcal{A}) = \sum_{i_1,\ldots,i_p,j_2,\ldots,j_q \in N} |a_{i_1\cdots i_pjj_2\cdots j_q}|,$$

$$c_j^i(\mathcal{A}) = \sum_{\delta_{i_1\cdots i_pij_2\cdots j_q}=0} |a_{i_1\cdots i_pjj_2\cdots j_q}| = C_j(\mathcal{A}) - |a_{i\cdots iji\cdots i}|,$$

where

$$\delta_{i_1\cdots i_pj_1\cdots j_q} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_p = j_1 = \cdots = j_q, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 *Let $\mathcal{A} \in \mathbb{R}^{[p,q;n,n]}$, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon^{S}(\mathcal{A}) = \Bigl(\bigcup_{i\in S, j\in \bar{S}} \bigl(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A})\bigr)\Bigr) \cup \Bigl(\bigcup_{i\in \bar{S}, j\in S} \bigl(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A})\bigr)\Bigr),$$

where

$$\hat{\Upsilon}_{i,j}(\mathcal{A}) = \bigl\{z \in \mathbb{C} : \bigl(|z| - r_i^j(\mathcal{A})\bigr)|z| \le |a_{ij\cdots jj\cdots j}| \max\bigl\{R_j(\mathcal{A}), C_j(\mathcal{A})\bigr\}\bigr\},$$
$$\tilde{\Upsilon}_{i,j}(\mathcal{A}) = \bigl\{z \in \mathbb{C} : \bigl(|z| - c_i^j(\mathcal{A})\bigr)|z| \le |a_{j\cdots jij\cdots j}| \max\bigl\{R_j(\mathcal{A}), C_j(\mathcal{A})\bigr\}\bigr\}.$$

Proof For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n \backslash \{0\}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbb{C}^n \backslash \{0\}$ be the associated left and right eigenvectors, that is,

$$\begin{cases} \mathcal{A}x^{p-1}y^{q} = \lambda x^{[l-1]}, & (1)\\ \mathcal{A}x^{p}y^{q-1} = \lambda y^{[l-1]}. & (2)\end{cases}$$

Let

$$|x_s| = \max_{i\in S}\bigl\{|x_i|\bigr\}, \qquad |x_t| = \max_{i\in \bar{S}}\bigl\{|x_i|\bigr\}, \qquad |y_g| = \max_{i\in S}\bigl\{|y_i|\bigr\}, \qquad |y_h| = \max_{i\in \bar{S}}\bigl\{|y_i|\bigr\},$$
$$w_i = \max_{i\in N}\bigl\{|x_i|, |y_i|\bigr\}, \qquad w_S = \max_{i\in S}\{w_i\}, \qquad w_{\bar{S}} = \max_{i\in \bar{S}}\{w_i\}.$$

Then, at least one of $|x_s|$ and $|x_t|$ is nonzero, and at least one of $|y_g|$ and $|y_h|$ is nonzero. We divide the proof into four parts.

Case I: Suppose that $w_S = |x_s|, w_{\bar{S}} = |x_t|$, then $|x_s| \ge |y_s|, |x_t| \ge |y_t|$.

(i) If $|x_s| \ge |x_t|$, then $|x_s| = \max_{i\in N}\{w_i\}$. The sth equality in (1) is

$$\lambda x_s^{l-1} = \sum_{\delta_{ti_2\cdots i_p j_1\cdots j_q}=0} a_{si_2\cdots i_p j_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} + a_{st\cdots tt\cdots t} x_t^{p-1} y_t^{q}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned} |\lambda||x_s|^{l-1} &\le \sum_{\delta_{ti_2\cdots i_p j_1\cdots j_q}=0} |a_{si_2\cdots i_p j_1\cdots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots y_{j_q}| \\ &\quad + |a_{st\cdots tt\cdots t}||x_t|^{p-1}|y_t|^{q} \\ &\le \sum_{\delta_{ti_2\cdots i_p j_1\cdots j_q}=0} |a_{si_2\cdots i_p j_1\cdots j_q}||x_s|^{l-1} + |a_{st\cdots tt\cdots t}||x_t|^{l-1} \\ &= r_s^t(\mathcal{A})|x_s|^{l-1} + |a_{st\cdots tt\cdots t}||x_t|^{l-1}, \end{aligned}$$

i.e.,

$$\bigl(|\lambda| - r_s^t(\mathcal{A})\bigr)|x_s|^{l-1} \le |a_{st\cdots tt\cdots t}||x_t|^{l-1}. \tag{3}$$

If $|x_t| = 0$, then $|\lambda| - r_s^t(\mathcal{A}) \leq 0$ as $|x_s| > 0$, and it is obvious that

$$\big(|\lambda| - r_s^t(\mathcal{A})\big)|\lambda| \leq 0 \leq |a_{st\cdots tt\cdots t}|R_t(\mathcal{A}),$$

which implies that $\lambda \in \hat{\Upsilon}_{s,t}(\mathcal{A})$. Otherwise, $|x_t| > 0$. Moreover, from the tth equality in (1), we can get

$$\begin{aligned} |\lambda||x_t|^{l-1} &\leq \sum_{i_2,\ldots,i_p,j_1,\ldots,j_q \in N} |a_{ti_2\cdots i_p j_1\cdots j_q}||x_{i_2}|\cdots|x_{i_p}||y_{j_1}|\cdots|y_{j_q}| \\ &\leq R_t(\mathcal{A})|x_s|^{l-1}. \end{aligned} \tag{4}$$

Multiplying (3) by (4) and noting that $|x_s|^{l-1}|x_t|^{l-1} > 0$, we have

$$\big(|\lambda| - r_s^t(\mathcal{A})\big)|\lambda| \leq |a_{st\cdots tt\cdots t}|R_t(\mathcal{A}),$$

which also implies that $\lambda \in \hat{\Upsilon}_{s,t}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in \bar{S}} \hat{\Upsilon}_{i,j}(\mathcal{A})$.

(ii) If $|x_t| \geq |x_s|$, then $|x_t| = \max_{i\in N}\{w_i\}$. Similarly, we can get

$$\big(|\lambda| - r_t^s(\mathcal{A})\big)|\lambda| \leq |a_{ts\cdots ss\cdots s}|R_s(\mathcal{A}),$$

and $\lambda \in \hat{\Upsilon}_{t,s}(\mathcal{A}) \subseteq \bigcup_{i\in \bar{S}, j\in S} \hat{\Upsilon}_{i,j}(\mathcal{A})$.

Case II: Suppose that $w_S = |y_g|$, $w_{\bar{S}} = |y_h|$, then $|y_g| \geq |x_g|$, $|y_h| \geq |x_h|$.

(i) If $|y_g| \geq |y_h|$, then $|y_g| = \max_{i\in N}\{w_i\}$. The gth equality in (2) is

$$\lambda y_g^{l-1} = \sum_{\delta_{i_1\cdots i_p h j_2\cdots j_q}=0} a_{i_1\cdots i_p g j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q} + a_{h\cdots hgh\cdots h} x_h^p y_h^{q-1}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned} |\lambda||y_g|^{l-1} &\leq \sum_{\delta_{i_1\cdots i_p h j_2\cdots j_q}=0} |a_{i_1\cdots i_p g j_2\cdots j_q}||x_{i_1}|\cdots|x_{i_p}||y_{j_2}|\cdots|y_{j_q}| \\ &\quad + |a_{h\cdots hgh\cdots h}||x_h|^p|y_h|^{q-1} \\ &\leq \sum_{\delta_{i_1\cdots i_p h j_2\cdots j_q}=0} |a_{i_1\cdots i_p g j_2\cdots j_q}||y_g|^{l-1} + |a_{h\cdots hgh\cdots h}||y_h|^{l-1} \\ &= c_g^h(\mathcal{A})|y_g|^{l-1} + |a_{h\cdots hgh\cdots h}||y_h|^{l-1}, \end{aligned}$$

i.e.,

$$\big(|\lambda| - c_g^h(\mathcal{A})\big)|y_g|^{l-1} \leq |a_{h\cdots hgh\cdots h}||y_h|^{l-1}. \tag{5}$$

If $|y_h| = 0$, then $|\lambda| - c_g^h(\mathcal{A}) \leq 0$ as $|y_g| > 0$, and furthermore

$$\big(|\lambda| - c_g^h(\mathcal{A})\big)|\lambda| \leq 0 \leq |a_{h\cdots hgh\cdots h}|C_h(\mathcal{A}),$$

which implies that $\lambda \in \tilde{\Upsilon}_{g,h}(\mathcal{A})$. Otherwise, $|y_h| > 0$. Moreover, from the hth equality in (2), we can get

$$\begin{aligned}|\lambda||y_h|^{l-1} &\leq \sum_{i_1,\ldots,i_p,j_2,\ldots,j_q \in N} |a_{i_1\cdots i_p h j_2\cdots j_q}||x_{i_1}|\cdots|x_{i_p}||y_{j_2}|\cdots|y_{j_q}| \\ &\leq C_h(\mathcal{A})|y_g|^{l-1}. \end{aligned} \tag{6}$$

Multiplying (5) by (6) and noting that $|y_g|^{l-1}|y_h|^{l-1} > 0$, we have

$$\big(|\lambda| - c_g^h(\mathcal{A})\big)|\lambda| \leq |a_{h\cdots hgh\cdots h}|C_h(\mathcal{A}),$$

which also implies that $\lambda \in \tilde{\Upsilon}_{g,h}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar{S}} \tilde{\Upsilon}_{i,j}(\mathcal{A})$.

(ii) If $|y_h| \geq |y_g|$, then $|y_h| = \max_{i\in N}\{w_i\}$. Similarly, we can get

$$\big(|\lambda| - c_h^g(\mathcal{A})\big)|\lambda| \leq |a_{g\cdots ghg\cdots g}|C_g(\mathcal{A}),$$

and $\lambda \in \tilde{\Upsilon}_{h,g}(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S}, j\in S} \tilde{\Upsilon}_{i,j}(\mathcal{A})$.

Case III: Suppose that $w_S = |x_s|, w_{\bar{S}} = |y_h|$, then $|x_s| \geq |y_s|, |y_h| \geq |x_h|$. If $|x_s| \geq |y_h|$, then $|x_s| = \max_{i\in N}\{w_i\}$. Similar to the proof of (3) and (6), we have

$$\big(|\lambda| - r_s^h(\mathcal{A})\big)|x_s|^{l-1} \leq |a_{sh\cdots hh\cdots h}||y_h|^{l-1}$$

and

$$|\lambda||y_h|^{l-1} \leq C_h(\mathcal{A})|x_s|^{l-1}.$$

Furthermore, we have

$$\big(|\lambda| - r_s^h(\mathcal{A})\big)|\lambda| \leq |a_{sh\cdots hh\cdots h}|C_h(\mathcal{A}),$$

which implies that $\lambda \in \hat{\Upsilon}_{s,h}(\mathcal{A}) \subseteq \bigcup_{i\in S, j\in\bar{S}} \hat{\Upsilon}_{i,j}(\mathcal{A})$. And if $|y_h| \geq |x_s|$, then $|y_h| = \max_{i\in N}\{w_i\}$. Similarly, we can get

$$\big(|\lambda| - c_h^s(\mathcal{A})\big)|\lambda| \leq |a_{s\cdots shs\cdots s}|R_s(\mathcal{A}),$$

which implies that $\lambda \in \tilde{\Upsilon}_{h,s}(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S}, j\in S} \tilde{\Upsilon}_{i,j}(\mathcal{A})$.

Case IV: Suppose that $w_S = |y_g|, w_{\bar{S}} = |x_t|$, then $|y_g| \geq |x_g|, |x_t| \geq |y_t|$. If $|y_g| \geq |x_t|$, then $|y_g| = \max_{i\in N}\{w_i\}$. Similar to the proof of (5) and (4), we have

$$\big(|\lambda| - c_g^t(\mathcal{A})\big)|y_g|^{l-1} \leq |a_{t\cdots tgt\cdots t}||x_t|^{l-1}$$

and

$$|\lambda||x_t|^{l-1} \leq R_t(\mathcal{A})|y_g|^{l-1}.$$

Furthermore, we have

$$\big(|\lambda| - c_g^t(\mathcal{A})\big)|\lambda| \le |a_{t\cdots tgt\cdots t}|R_t(\mathcal{A}),$$

which implies that $\lambda \in \tilde{\Upsilon}_{g,t}(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S},j\in S} \tilde{\Upsilon}_{i,j}(\mathcal{A})$. And if $|x_t| \ge |y_g|$, then $|x_t| = \max_{i\in N}\{w_i\}$. Similarly, we can get

$$\big(|\lambda| - r_t^g(\mathcal{A})\big)|\lambda| \le |a_{tg\cdots gg\cdots g}|C_g(\mathcal{A}),$$

which implies that $\lambda \in \hat{\Upsilon}_{t,g}(\mathcal{A}) \subseteq \bigcup_{i\in\bar{S},j\in S} \hat{\Upsilon}_{i,j}(\mathcal{A})$. The proof is completed. □

Based on Theorem 1, bounds for the largest singular value of nonnegative rectangular tensors are given.

Theorem 2 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}_+^{[p,q;n,n]}$, S be a nonempty proper subset of N, $\bar{S}$ be the complement of S in N. Then*

$$L^S(\mathcal{A}) \le \lambda_0 \le U^S(\mathcal{A}), \tag{7}$$

where

$$L^S(\mathcal{A}) = \min\big\{\hat{L}^S(\mathcal{A}), \hat{L}^{\bar{S}}(\mathcal{A}), \tilde{L}^S(\mathcal{A}), \tilde{L}^{\bar{S}}(\mathcal{A})\big\},$$
$$U^S(\mathcal{A}) = \max\big\{\hat{U}^S(\mathcal{A}), \hat{U}^{\bar{S}}(\mathcal{A}), \tilde{U}^S(\mathcal{A}), \tilde{U}^{\bar{S}}(\mathcal{A})\big\}$$

and

$$\hat{L}^S(\mathcal{A}) = \min_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}\min\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\},$$
$$\tilde{L}^S(\mathcal{A}) = \min_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}\min\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\},$$
$$\hat{U}^S(\mathcal{A}) = \max_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\},$$
$$\tilde{U}^S(\mathcal{A}) = \max_{i\in S,j\in\bar{S}} \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}.$$

Proof First, we prove that the second inequality in (7) holds. By Theorem 2 in [2], we know that λ_0 is a singular value of $\mathcal{A}$. Hence, by Theorem 1, $\lambda_0 \in \Upsilon^S(\mathcal{A})$, that is,

$$\lambda_0 \in \bigcup_{i\in S,j\in\bar{S}} \big(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A})\big) \quad \text{or}$$
$$\lambda_0 \in \bigcup_{i\in\bar{S},j\in S} \big(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A})\big).$$

If $\lambda_0 \in \bigcup_{i\in S,j\in\bar{S}}(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A}))$, then there are $i \in S, j \in \bar{S}$ such that $\lambda_0 \in \hat{\Upsilon}_{i,j}(\mathcal{A})$ or $\lambda_0 \in \tilde{\Upsilon}_{i,j}(\mathcal{A})$. When $\lambda_0 \in \hat{\Upsilon}_{i,j}(\mathcal{A})$, i.e., $(\lambda_0 - r_i^j(\mathcal{A}))\lambda_0 \le a_{ij\cdots jj\cdots j}\max\{R_j(\mathcal{A}), C_j(\mathcal{A})\}$, then solving λ_0

gives

$$\begin{aligned}\lambda_0 &\le \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}\\ &\le \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}\\ &= \hat{U}^S(\mathcal{A}).\end{aligned}$$

When $\lambda_0 \in \tilde{\Upsilon}_{i,j}(\mathcal{A})$, i.e., $(\lambda_0 - c_i^j(\mathcal{A}))\lambda_0 \le a_{j\cdots jij\cdots j}\max\{R_j(\mathcal{A}), C_j(\mathcal{A})\}$, then solving λ_0 gives

$$\begin{aligned}\lambda_0 &\le \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}\\ &\le \max_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}\max\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}\\ &= \tilde{U}^S(\mathcal{A}).\end{aligned}$$

And if $\lambda_0 \in \bigcup_{i\in\bar{S}, j\in S}(\hat{\Upsilon}_{i,j}(\mathcal{A}) \cup \tilde{\Upsilon}_{i,j}(\mathcal{A}))$, similarly, we can obtain that $\lambda_0 \le \hat{U}^{\bar{S}}(\mathcal{A})$ and $\lambda_0 \le \tilde{U}^{\bar{S}}(\mathcal{A})$.

Second, we prove that the first inequality in (7) holds. Assume that $\mathcal{A}$ is an irreducible nonnegative rectangular tensor, by Theorem 6 of [1], then $\lambda_0 > 0$ with two positive left and right associated eigenvectors $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$. Let

$$\begin{aligned}&x_s = \min_{i\in S}\{x_i\}, \qquad x_t = \min_{i\in\bar{S}}\{x_i\}, \qquad y_g = \min_{i\in S}\{y_i\}, \qquad y_h = \min_{i\in\bar{S}}\{y_i\},\\ &w_i = \min_{i\in N}\{x_i, y_i\}, \qquad w_S = \min_{i\in S}\{w_i\}, \qquad w_{\bar{S}} = \min_{i\in\bar{S}}\{w_i\}.\end{aligned}$$

We divide the proof into four parts.

Case I: Suppose that $w_S = x_s, w_{\bar{S}} = x_t$, then $y_s \ge x_s, y_t \ge x_t$.

(i) If $x_t \ge x_s$, then $x_s = \min_{i\in N}\{w_i\}$. From the sth equality in (1), we have

$$\begin{aligned}\lambda_0 x_s^{l-1} &= \sum_{\delta_{ti_2\cdots i_pj_1\cdots j_q}=0} a_{si_2\cdots i_pj_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} + a_{st\cdots tt\cdots t} x_t^{p-1} y_t^q\\ &\ge \sum_{\delta_{ti_2\cdots i_pj_1\cdots j_q}=0} a_{si_2\cdots i_pj_1\cdots j_q} x_s^{l-1} + a_{st\cdots tt\cdots t} x_t^{l-1}\\ &= r_s^t(\mathcal{A}) x_s^{l-1} + a_{st\cdots tt\cdots t} x_t^{l-1},\end{aligned}$$

i.e.,

$$\big(\lambda_0 - r_s^t(\mathcal{A})\big)x_s^{l-1} \ge a_{st\cdots tt\cdots t} x_t^{l-1}. \tag{8}$$

Moreover, from the tth equality in (1), we can get

$$\lambda_0 x_t^{l-1} = \sum_{i_2,\ldots,i_p,j_1,\ldots,j_q\in N} a_{ti_2\cdots i_pj_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q} \ge R_t(\mathcal{A}) x_s^{l-1}. \tag{9}$$

Multiplying (8) by (9) and noting that $x_s^{l-1}x_t^{l-1} > 0$, we have

$$\big(\lambda_0 - r_s^t(\mathcal{A})\big)\lambda_0 \ge a_{st\cdots tt\cdots t}R_t(\mathcal{A}).$$

Then solving for λ_0 gives

$$\begin{aligned}\lambda_0(\mathcal{A}) &\ge \frac{1}{2}\Big\{r_s^t(\mathcal{A}) + \big[\big(r_s^t(\mathcal{A})\big)^2 + 4a_{st\cdots tt\cdots t}R_t(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \\ &\ge \min_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}R_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \ge \hat{L}^S(\mathcal{A}).\end{aligned}$$

(ii) If $x_s \ge x_t$, then $x_t = \min_{i\in N}\{w_i\}$. Similarly, we can get

$$\begin{aligned}\lambda_0(\mathcal{A}) &\ge \frac{1}{2}\Big\{r_t^s(\mathcal{A}) + \big[\big(r_t^s(\mathcal{A})\big)^2 + 4a_{ts\cdots ss\cdots s}R_s(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \\ &\ge \min_{i\in \bar{S}, j\in S} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}R_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \ge \hat{L}^{\bar{S}}(\mathcal{A}).\end{aligned}$$

Case II: Suppose that $w_S = y_g, w_{\bar{S}} = y_h$, then $x_g \ge y_g, x_h \ge y_h$.
(i) If $y_h \ge y_g$, then $y_g = \min_{i\in N}\{w_i\}$. From the gth equality in (2), we have

$$\begin{aligned}\lambda_0 y_g^{l-1} &= \sum_{\delta_{i_1\cdots i_p h j_2\cdots j_q}=0} a_{i_1\cdots i_p g j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q} + a_{h\cdots hgh\cdots h} x_h^p y_h^{q-1} \\ &\ge \sum_{\delta_{i_1\cdots i_p h j_2\cdots j_q}=0} a_{i_1\cdots i_p g j_2\cdots j_q} y_g^{l-1} + a_{h\cdots hgh\cdots h} y_h^{l-1} \\ &= c_g^h(\mathcal{A}) y_g^{l-1} + a_{h\cdots hgh\cdots h} y_h^{l-1},\end{aligned}$$

i.e.,

$$\big(\lambda_0 - c_g^h(\mathcal{A})\big) y_g^{l-1} \ge a_{h\cdots hgh\cdots h} y_h^{l-1}. \tag{10}$$

Moreover, from the hth equality in (2), we can get

$$\lambda_0 y_h^{l-1} = \sum_{i_1,\ldots,i_p,j_2,\ldots,j_q\in N} a_{i_1\cdots i_p h j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q} \ge C_h(\mathcal{A}) y_g^{l-1}. \tag{11}$$

Multiplying (10) by (11) and noting that $y_g^{l-1}y_h^{l-1} > 0$, we have

$$\big(\lambda_0 - c_g^h(\mathcal{A})\big)\lambda_0 \ge a_{h\cdots hgh\cdots h} C_h(\mathcal{A}),$$

which gives

$$\begin{aligned}\lambda_0 &\ge \frac{1}{2}\Big\{c_g^h(\mathcal{A}) + \big[\big(c_g^h(\mathcal{A})\big)^2 + 4a_{h\cdots hgh\cdots h}C_h(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \\ &\ge \min_{i\in S, j\in\bar{S}} \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots j}C_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \\ &\ge \tilde{L}^S(\mathcal{A}).\end{aligned}$$

(ii) If $y_g \geq y_h$, then $y_h = \min_{i\in N}\{w_i\}$. Similarly, we can get

$$\begin{aligned}\lambda_0 &\geq \frac{1}{2}\Bigl\{c_h^g(\mathcal{A}) + \bigl[\bigl(c_h^g(\mathcal{A})\bigr)^2 + 4a_{g\cdots ghg\cdots g}C_g(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \min_{i\in\bar{S},j\in S}\frac{1}{2}\Bigl\{c_i^j(\mathcal{A}) + \bigl[\bigl(c_i^j(\mathcal{A})\bigr)^2 + 4a_{j\cdots jij\cdots j}C_j(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \tilde{L}^{\bar{S}}(\mathcal{A}).\end{aligned}$$

Case III: Suppose that $w_S = x_s, w_{\bar{S}} = y_h$, then $y_s \geq x_s, x_h \geq y_h$. If $y_h \geq x_s$, then $x_s = \min_{i\in N}\{w_i\}$. Similar to the proof of (8) and (11), we have

$$\bigl(\lambda_0 - r_s^h(\mathcal{A})\bigr)x_s^{l-1} \geq a_{sh\cdots hh\cdots h}y_h^{l-1}$$

and

$$\lambda_0 y_h^{l-1} \geq C_h(\mathcal{A})x_s^{l-1}.$$

Furthermore, we have

$$\bigl(\lambda_0 - r_s^h(\mathcal{A})\bigr)\lambda_0 \geq a_{sh\cdots hh\cdots h}C_h(\mathcal{A})$$

and

$$\begin{aligned}\lambda_0 &\geq \frac{1}{2}\Bigl\{r_s^h(\mathcal{A}) + \bigl[\bigl(r_s^h(\mathcal{A})\bigr)^2 + 4a_{sh\cdots hh\cdots h}C_h(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \min_{i\in S,j\in\bar{S}}\frac{1}{2}\Bigl\{r_i^j(\mathcal{A}) + \bigl[\bigl(r_i^j(\mathcal{A})\bigr)^2 + 4a_{ij\cdots jj\cdots j}C_j(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \hat{L}^S(\mathcal{A}).\end{aligned}$$

And if $x_s \geq y_h$, then $y_h = \min_{i\in N}\{w_i\}$. Similarly, we have

$$\begin{aligned}\lambda_0 &\geq \frac{1}{2}\Bigl\{c_h^s(\mathcal{A}) + \bigl[\bigl(c_h^s(\mathcal{A})\bigr)^2 + 4a_{s\cdots shs\cdots s}R_s(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \min_{i\in\bar{S},j\in S}\frac{1}{2}\Bigl\{c_i^j(\mathcal{A}) + \bigl[\bigl(c_i^j(\mathcal{A})\bigr)^2 + 4a_{j\cdots jij\cdots j}R_j(\mathcal{A})\bigr]^{\frac{1}{2}}\Bigr\}\\ &\geq \tilde{L}^{\bar{S}}(\mathcal{A}).\end{aligned}$$

Case IV: Suppose that $w_S = y_g, w_{\bar{S}} = x_t$, then $x_g \geq y_g, y_t \geq x_t$. If $x_t \geq y_g$, then $y_g = \min_{i\in N}\{w_i\}$. Similar to the proof of (10) and (9), we have

$$\bigl(\lambda_0 - c_g^t(\mathcal{A})\bigr)y_g^{l-1} \geq a_{t\cdots tgt\cdots t}x_t^{l-1}$$

and

$$\lambda_0 x_t^{l-1} \geq R_t(\mathcal{A})y_g^{l-1}.$$

Furthermore, we have

$$\big(\lambda_0 - c_g^t(\mathcal{A})\big)\lambda_0 \geq a_{t\cdots tgt\cdots t}R_t(\mathcal{A})$$

and

$$\begin{aligned}\lambda_0 &\geq \frac{1}{2}\Big\{c_g^t(\mathcal{A}) + \big[\big(c_g^t(\mathcal{A})\big)^2 + 4a_{t\cdots tgt\cdots t}R_t(\mathcal{A})\big]^{\frac{1}{2}}\Big\}\\ &\geq \min_{i\in S, j\in \bar{S}} \frac{1}{2}\Big\{c_i^j(\mathcal{A}) + \big[\big(c_i^j(\mathcal{A})\big)^2 + 4a_{j\cdots jij\cdots t}R_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \geq \tilde{L}^S(\mathcal{A}).\end{aligned}$$

And if $y_g \geq x_t$, then $x_t = \min_{i\in N}\{w_i\}$. Similarly, we have

$$\begin{aligned}\lambda_0 &\geq \frac{1}{2}\Big\{r_t^g(\mathcal{A}) + \big[\big(r_t^g(\mathcal{A})\big)^2 + 4a_{tg\cdots gg\cdots g}C_g(\mathcal{A})\big]^{\frac{1}{2}}\Big\}\\ &\geq \min_{i\in \bar{S}, j\in S} \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}C_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\} \geq \hat{L}^{\bar{S}}(\mathcal{A}).\end{aligned}$$

Assume that $\mathcal{A}$ is a nonnegative rectangular tensor, then by Lemma 3 of [2] and similar to the proof of Theorem 2 of [2], we can prove that the first inequality in (7) holds. The conclusion follows from what we have proved. □

Next, a comparison theorem for these bounds in Theorem 2 and Theorem 4 of [2] is given.

Theorem 3 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}_+^{[p,q;n,n]}$, *S be a nonempty proper subset of N. Then the bounds in Theorem* 2 *are better than those in Theorem* 4 *of* [2], *that is*,

$$\min_{1\leq i,j\leq n}\big\{R_i(\mathcal{A}), C_j(\mathcal{A})\big\} \leq L^S(\mathcal{A}) \leq U^S(\mathcal{A}) \leq \max_{1\leq i,j\leq n}\big\{R_i(\mathcal{A}), C_j(\mathcal{A})\big\}.$$

Proof Here, only $L^S(\mathcal{A}) = \min\{\hat{L}^S(\mathcal{A}), \hat{L}^{\bar{S}}(\mathcal{A}), \tilde{L}^S(\mathcal{A}), \tilde{L}^{\bar{S}}(\mathcal{A})\} \geq \min_{1\leq i,j\leq n}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$ is proved. Similarly, we can also prove that $U^S(\mathcal{A}) \leq \max_{1\leq i,j\leq n}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$. Without loss of generality, assume that $L^S(\mathcal{A}) = \hat{L}^S(\mathcal{A})$, that is, there are two indexes $i \in S, j \in \bar{S}$ such that

$$L^S(\mathcal{A}) = \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}\min\big\{R_j(\mathcal{A}), C_j(\mathcal{A})\big\}\big]^{\frac{1}{2}}\Big\}$$

(we can prove it similarly if $L^S(\mathcal{A}) = \hat{L}^{\bar{S}}(\mathcal{A}), \tilde{L}^S(\mathcal{A}), \tilde{L}^{\bar{S}}(\mathcal{A})$, respectively). Now, we divide the proof into two cases as follows.

Case I: Assume that

$$L^S(\mathcal{A}) = \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4a_{ij\cdots jj\cdots j}R_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\}.$$

(i) If $R_i(\mathcal{A}) \geq R_j(\mathcal{A})$, then $a_{ij\cdots jj\cdots j} \geq R_j(\mathcal{A}) - r_i^j(\mathcal{A})$. When $R_j(\mathcal{A}) - r_i^j(\mathcal{A}) > 0$, we have

$$\begin{aligned}L^S(\mathcal{A}) &\geq \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(r_i^j(\mathcal{A})\big)^2 + 4\big(R_j(\mathcal{A}) - r_i^j(\mathcal{A})\big)R_j(\mathcal{A})\big]^{\frac{1}{2}}\Big\}\\ &= \frac{1}{2}\Big\{r_i^j(\mathcal{A}) + \big[\big(2R_j(\mathcal{A}) - r_i^j(\mathcal{A})\big)^2\big]^{\frac{1}{2}}\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{r_i^j(\mathcal{A}) + 2R_j(\mathcal{A}) - r_i^j(\mathcal{A})\right\} \\
&= R_j(\mathcal{A}) \\
&\geq \min_{j\in\bar{S}} R_j(\mathcal{A}) \\
&\geq \min_{1\leq i,j\leq n}\left\{R_i(\mathcal{A}), C_j(\mathcal{A})\right\}.
\end{aligned}
$$

And when $R_j(\mathcal{A}) - r_i^j(\mathcal{A}) \leq 0$, i.e., $r_i^j(\mathcal{A}) \geq R_j(\mathcal{A})$, we have

$$
\begin{aligned}
L^S(\mathcal{A}) &\geq \frac{1}{2}\left\{r_i^j(\mathcal{A}) + \left[\left(r_i^j(\mathcal{A})\right)^2\right]^{\frac{1}{2}}\right\} = r_i^j(\mathcal{A}) \geq R_j(\mathcal{A}) \geq \min_{j\in\bar{S}} R_j(\mathcal{A}) \\
&\geq \min_{1\leq i,j\leq n}\left\{R_i(\mathcal{A}), C_j(\mathcal{A})\right\}.
\end{aligned}
$$

(ii) If $R_i(\mathcal{A}) < R_j(\mathcal{A})$, then

$$
\begin{aligned}
L^S(\mathcal{A}) &\geq \frac{1}{2}\left\{r_i^j(\mathcal{A}) + \left[\left(r_i^j(\mathcal{A})\right)^2 + 4a_{ij\cdots jj\cdots j}R_i(\mathcal{A})\right]^{\frac{1}{2}}\right\} \\
&= \frac{1}{2}\left\{r_i^j(\mathcal{A}) + \left[\left(r_i^j(\mathcal{A})\right)^2 + 4a_{ij\cdots jj\cdots j}\left(r_i^j(\mathcal{A}) + a_{ij\cdots jj\cdots j}\right)\right]^{\frac{1}{2}}\right\} \\
&= \frac{1}{2}\left\{r_i^j(\mathcal{A}) + \left[\left(r_i^j(\mathcal{A}) + 2a_{ij\cdots jj\cdots j}\right)^2\right]^{\frac{1}{2}}\right\} \\
&= r_i^j(\mathcal{A}) + a_{ij\cdots jj\cdots j} \\
&= R_i(\mathcal{A}) \\
&\geq \min_{i\in S} R_i(\mathcal{A}) \\
&\geq \min_{1\leq i,j\leq n}\left\{R_i(\mathcal{A}), C_j(\mathcal{A})\right\}.
\end{aligned}
$$

Case II: Assume that

$$
L^S(\mathcal{A}) = \frac{1}{2}\left\{r_i^j(\mathcal{A}) + \left[\left(r_i^j(\mathcal{A})\right)^2 + 4a_{ij\cdots jj\cdots j}C_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.
$$

Similar to the proof of Case I, we have $L^S(\mathcal{A}) \geq \min_{1\leq i,j\leq n}\{R_i(\mathcal{A}), C_j(\mathcal{A})\}$. The conclusion follows from what we have proved. □

3 Numerical examples

In the following, two numerical examples are given to verify the theoretical results.

Example 1 Let $\mathcal{A} \in \mathbb{R}_+^{[2,2;3,3]}$ with entries defined as follows:

$$
\mathcal{A}(:,:,1,1) = \begin{bmatrix} 0 & 0 & 0 \\ 11 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A}(:,:,2,1) = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 6 & 3 \\ 10 & 0 & 3 \end{bmatrix},
$$

$$
\mathcal{A}(:,:,3,1) \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 7 & 2 & 2 \end{bmatrix}, \qquad \mathcal{A}(:,:,1,2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
$$

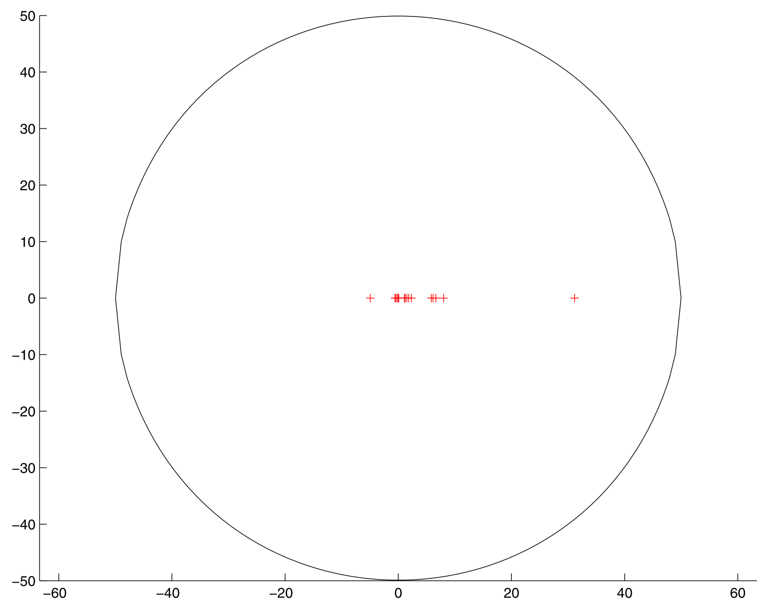

Figure 1 The singular value inclusion set $\Upsilon^S(\mathcal{A})$ and the exact singular values.

$$\mathcal{A}(:,:,2,2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}, \qquad \mathcal{A}(:,:,3,2) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 6 & 2 & 1 \end{bmatrix},$$

$$\mathcal{A}(:,:,1,3) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{A}(:,:,2,3) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

$$\mathcal{A}(:,:,3,3) \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}.$$

By computation, we get that all different singular values of $\mathcal{A}$ are $-4.9395, -0.5833, -0.4341, -0.1977, 0, 0.0094, 0.0907, 1.0825, 1.2405, 1.5334, 1.8418, 2.3125, 5.8540, 6.1494, 6.6525, 8.0225$ and 31.1680.

(i) An S-type singular value inclusion set.

Let $S = \{1\}$. Obviously, $\bar{S} = \{2, 3\}$. By Theorem 1, the S-type singular inclusion set is

$$\Upsilon^S(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 49.9629\}.$$

The singular value inclusion set $\Upsilon^S(\mathcal{A})$ and the exact singular values are drawn in Figure 1, where $\Upsilon^S(\mathcal{A})$ is represented by black solid boundary and the exact singular values are plotted by red '+'. It is easy to see that $\Upsilon^S(\mathcal{A})$ can capture all singular values of $\mathcal{A}$ from Figure 1.

(ii) The bounds of the largest singular value.

By Theorem 4 of [2], we have

$$5 \leq \lambda_0 \leq 57.$$

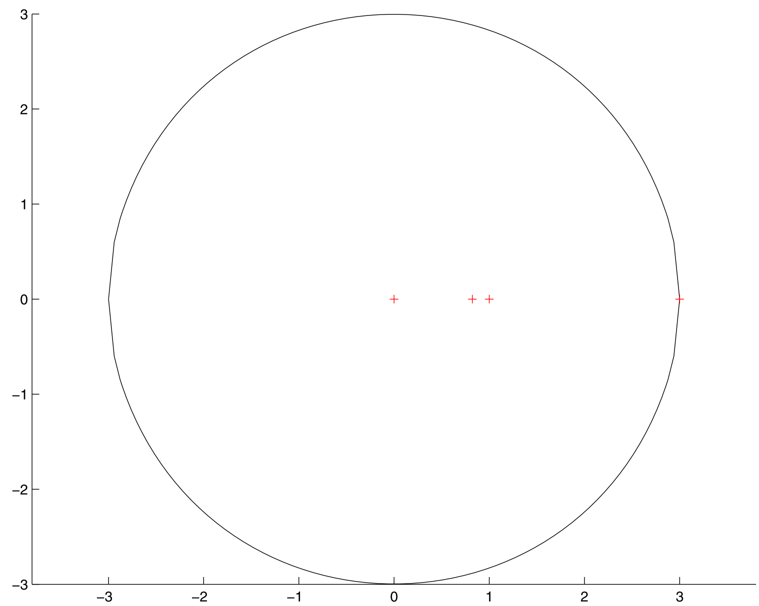

Figure 2 The singular value inclusion set $\Upsilon^S(\mathcal{A})$ and the exact singular values.

Let $S = \{1\}, \bar{S} = \{2, 3\}$. By Theorem 2, we have

$$9.0711 \leq \lambda_0 \leq 49.9629.$$

In fact, $\lambda_0 = 31.1680$. This example shows that the bounds in Theorem 2 are better than those in Theorem 4 of [2].

Example 2 Let $\mathcal{A} \in \mathbb{R}_+^{[2,2;2,2]}$ with entries defined as follows:

$$a_{1111} = a_{1112} = a_{1222} = a_{2112} = a_{2121} = a_{2221} = 1,$$

other $a_{ijkl} = 0$. By computation, we get that all different singular values of $\mathcal{A}$ are $0, 0.8226, 1, 3$.

(i) An S-type singular value inclusion set.

Let $S = \{1\}$. Obviously, $\bar{S} = \{2, 3\}$. By Theorem 1, the S-type singular inclusion set is

$$\Upsilon^S(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 3\}.$$

The singular value inclusion set $\Upsilon^S(\mathcal{A})$ and the exact singular values are drawn in Figure 2, where $\Upsilon^S(\mathcal{A})$ is represented by black solid boundary and the exact singular values are plotted by red '+'. It is easy to see that $\Upsilon^S(\mathcal{A})$ captures exactly all singular values of $\mathcal{A}$ from Figure 2.

(ii) The bounds of the largest singular value.

By Theorem 2, we have

$$3 \leq \lambda_0 \leq 3.$$

In fact, $\lambda_0 = 3$. This example shows that the bounds in Theorem 2 are sharp.

4 Conclusions

In this paper, we give an S-type singular value inclusion set $\Upsilon^S(\mathcal{A})$ for rectangular tensors. As an application of this set, an S-type upper bound $U^S(\mathcal{A})$ and an S-type lower bound $L^S(\mathcal{A})$ for the largest singular value λ_0 of a nonnegative rectangular tensor $\mathcal{A}$ are obtained and proved to be sharper than those in [2]. Then, an interesting problem is how to pick S to make $\Upsilon^S(\mathcal{A})$ as tight as possible. But it is difficult when the dimension of the tensor $\mathcal{A}$ is large. We will continue to study this problem in the future.

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Competing interests

The author declares that they have no competing interests.

Author's contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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