Cai and Xu *Journal of Inequalities and Applications* (2017) 2017:140
DOI 10.1186/s13660-017-1413-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A basic problem of (p,q)-Bernstein-type operators

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Abstract

In this note, we give an elaboration of a basic problem on convergence theorem of (p,q)-analogue of Bernstein-type operators. By some classical analysis techniques, we derive an exact class of (p_n,q_n)-integer satisfying $\lim_{n\to\infty}[n]_{p_n,q_n}=\infty$ with $\lim_{n\to\infty}p_n=1$ and $\lim_{n\to\infty}q_n=1$ under $0<q_n<p_n\leq 1$. Our results provide an erratum to corresponding results on (p,q)-analogue of Bernstein-type operators that appeared in recent literature.

MSC: 41A50

Keywords: (p,q)-integer; Bernstein-type approximation; convergence theorem; equivalent condition

1 Introduction

During the last decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. A detailed review of the results on q-Bernstein polynomials along with an extensive bibliography is given in [1]. The q-Bernstein polynomials are shown to be closely related to the q-deformed binomial distribution [2]. It plays an important role in the q-boson theory giving a q-deformation of the quantum harmonic formalism [3]. The q-analogue of the boson operator calculus has proved to be a powerful tool in theoretical physics. It provides explicit expressions for the representations of the quantum group $SU_q(2)$ [4]. Meanwhile, the (p,q)-integers were introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [5].

Recently, (p,q)-integers have been introduced into classical linear positive operators to construct new approximation processes. A sequence of (p,q)-analogue of Bernstein operators was first introduced by Mursaleen [6, 7]. Besides, (p,q)-analogues of Szász-Mirakyan [8], Baskakov Kantorovich [9], Bleimann-Butzer-Hahn [10] and Kantorovich-type Bernstein-Stancu-Schurer [11] operators were also considered, see [12–15]. For further developments, one can also refer to [8, 16–18]. These operators are double parameters corresponding to p and q versus single parameter q-Bernstein-type operators [1, 19, 20]. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations (see, e.g., [21]).

For example, consider the (p,q)-analogue of the Bernstein operators proposed in [7]. Given $f \in C[0,1]$ and $0 < q < p \le 1$, operators $B_{n,p,q}$ are defined as follows:

$$B_{n,p,q}(f;x) := \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \quad n = 1,2,\ldots, \tag{1.1}$$

where, for any nonnegative integer k and $0 < q < p \le 1$, the (p,q)-integer $[k]_{p,q}$ is defined by

$$[k]_{p,q} := p^{k-1} + p^{k-2}q + \cdots + q^{k-1} = \frac{p^k - q^k}{p - q} \quad (k = 0,1,2,\ldots), \qquad [0]_{p,q} := 0,$$

and the (p,q)-factorial $[k]_{p,q}!$ is defined by

$$[k]_{p,q}! := [1]_{p,q}[2]_{p,q} \cdots [k]_{p,q} \quad (k = 1,2,\ldots), \qquad [0]_{p,q}! := 1.$$

For integers k, n with $0 \le k \le n$, the (p,q)-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

In general, we expect $B_{n,p,q}(f;x)$ to converge to $f(x)$ as $n \to \infty$. But we see that, for fixed value of p and q with $q \in (0,1)$ and $p \in (q,1]$,

$$[n]_{p,q} \to 0 \quad \text{or} \quad 1/(1-q) \quad \text{as } n \to \infty.$$

To obtain a sequence of generalized (p,q)-analogue Bernstein polynomials which converge, we let $q_n \in (0,1)$ and $p_n \in (q_n,1]$ depend on n. We then choose a sequence (p_n,q_n) such that $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, to ensure that $B_{n,p,q}(f;x)$ converge to $f(x)$.

The convergence theorems for (p,q)-analogue Bernstein-type operators were established in some recent papers (see [6], Theorem 3.1 (Remark 3.1), [7], Theorem 1, and further reading [10], Theorem 2.2, [14], Theorem 3.1, [12], Theorem 3, and [11], Remark 2.3, see also [9, 15]). For example, Mursaleen [7] gives the following.

Theorem 1.1 *Let* $0 < q_n < p_n \le 1$ *such that* $\lim_{n\to\infty} p_n = 1$ *and* $\lim_{n\to\infty} q_n = 1$. *Then, for each* $f \in C[0,1]$, $B_{n,p_n,q_n}(f;x)$ *converge uniformly to* f *on* $[0,1]$.

All linear positive operators mentioned in the articles cited above require that $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$; otherwise, these operators do not define approximation processes. However, the claim that both $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$ with $0 < q_n < p_n \le 1$ imply that $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$, in general, is not true. A counterexample is presented below.

Example 1.2 Let $p_n = 1 - 1/\sqrt{n}$, then $[n]_{p_n,q_n} \to 0$ for any sequence $\{q_n\}$ satisfying $0 < q_n < p_n$. Indeed,

$$\begin{aligned} 0 \le [n]_{p_n,q_n} &= p_n^{n-1} + p_n^{n-2}q_n + \cdots + p_n q_n^{n-2} + q_n^{n-1} \\ &\le n p_n^{n-1} = n(1 - 1/\sqrt{n})^{n-1} \sim n e^{-\sqrt{n}} \to 0, \quad n \to \infty. \end{aligned} \tag{1.2}$$

Later, the author [8] presented a more accurate assertion: Let q_n, p_n such that $0 < q_n < p_n \leq 1$ and $q_n \to 1$, $p_n \to 1$, $q_n^n \to a$, $p_n^n \to b$ $(a < b)$ as $n \to \infty$, then $\lim_{n\to\infty} [n]_{p_n,q_n} \to \infty$. It is natural to ask: What is the class of sequences (p_n, q_n) satisfying $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$ when $\lim_{n\to\infty} p_n = 1$ and $\lim_{n\to\infty} q_n = 1$ under $0 < q_n < p_n \leq 1$? Undoubtedly, this is an important problem. In this note, we will solve this problem in Section 2.

2 Main results

For $0 < q_n < p_n \leq 1$, set $q_n := 1 - \alpha_n$, $p_n := 1 - \beta_n$ such that $0 \leq \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$. In the sequel, we use notation $a_n \sim b_n$ $(a_n, b_n > 0) \Leftrightarrow \lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

First, let us present the following auxiliary proposition.

Lemma 2.1 *Let $n \in \mathbb{N}$, then as $n \to \infty$ we have*

$$[n]_{p_n,q_n} \to \infty \quad \Rightarrow \quad e^{n\beta_n}/n \to 0. \tag{2.1}$$

On the other hand,

$$e^{n\alpha_n}/n \to 0 \quad \Rightarrow \quad [n]_{p_n,q_n} \to \infty. \tag{2.2}$$

Proof We note that

$$\begin{aligned} [n]_{p_n,q_n} &= q_n^{n-1} + p_n q_n^{n-2} + \cdots + p_n^{n-2} q_n + p_n^{n-1} = q_n^{n-1}\bigl(1 + p_n/q_n + \cdots + (p_n/q_n)^{n-1}\bigr) \\ &> n q_n^{n-1} = n(1-\alpha_n)^{(-1/\alpha_n)(n-1)(-\alpha_n)} \sim n/e^{n\alpha_n}, \quad n \to \infty. \end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned} [n]_{p_n,q_n} &= q_n^{n-1} + p_n q_n^{n-2} + \cdots + p_n^{n-2} q_n + p_n^{n-1} = p_n^{n-1}\bigl(1 + q_n/p_n + \cdots + (q_n/p_n)^{n-1}\bigr) \\ &< n p_n^{n-1} = n(1-\beta_n)^{(-1/\beta_n)(n-1)(-\beta_n)} \sim n/e^{n\beta_n}, \quad n \to \infty. \end{aligned} \tag{2.4}$$

Therefore, from (2.3) and (2.4), for sufficiently large n, there exist two positive real numbers C_1 and C_2 satisfying

$$C_1 \cdot n/e^{n\alpha_n} < [n]_{p_n,q_n} < C_2 \cdot n/e^{n\beta_n}. \tag{2.5}$$

This yields the proof. □

The main result of this work is expressed by the next assertion.

Theorem 2.1 *The following statements are true*:

($\mathcal{A}$) *If* $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $e^{n\beta_n}/n \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.
($\mathcal{B}$) *If* $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ *and* $e^{n\beta_n}(\alpha_n - \beta_n) \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.
($\mathcal{C}$) *If* $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n - \beta_n)\} \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

Conversely,

($\mathcal{B}'$) *If* $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ *and* $[n]_{p_n,q_n} \to \infty$, *then* $e^{n\beta_n}(\alpha_n - \beta_n) \to 0$.
($\mathcal{C}'$) *If* $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $[n]_{p_n,q_n} \to \infty$, *then* $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n - \beta_n)\} \to 0$.

Proof Case ($\mathcal{A}$):

If $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$, since $\lim_{n\to\infty} e^{n\alpha_n}/n = \lim_{n\to\infty} e^{n\beta_n}/n/e^{n(\beta_n-\alpha_n)} \to 0$ and combined with (2.2) imply $[n]_{p_n,q_n} \to \infty$ (see Remark 2.1).

Case ($\mathcal{B}$) and Case ($\mathcal{B}'$):

If $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$. Note that, for sufficiently large n,

$$[n]_{p_n,q_n} = p_n^{n-1}\frac{1-(q_n/p_n)^n}{1-q_n/p_n} \sim \frac{1}{e^{n\beta_n}}\frac{1-(1-(\alpha_n-\beta_n)/(1-\beta_n))^n}{(\alpha_n-\beta_n)/(1-\beta_n)}. \tag{2.6}$$

Since

$$\frac{\alpha_n-\beta_n}{1-\beta_n} \to 0, \tag{2.7}$$

it is not difficult to obtain from (2.7) and $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ that, for sufficiently large n, there exists $c \in (0,1)$ such that

$$\left(1-\frac{\alpha_n-\beta_n}{1-\beta_n}\right)^n \sim e^{n(\beta_n-\alpha_n)} < c. \tag{2.8}$$

Set $d_n := 1-(1-\frac{\alpha_n-\beta_n}{1-\beta_n})^n$, then for sufficiently large n, $d_n > 1-c$. Thus, from (2.6), (2.7) and (2.8), we have

$$[n]_{p_n,q_n} \sim \frac{1}{e^{n\beta_n}} \cdot \frac{d_n}{\alpha_n-\beta_n}, \tag{2.9}$$

which entails that

$$[n]_{p_n,q_n} \to \infty \quad \Leftrightarrow \quad e^{n\beta_n}(\alpha_n-\beta_n) \to 0. \tag{2.10}$$

This yields the proof of Case ($\mathcal{B}$) and Case ($\mathcal{B}'$).

Case ($\mathcal{C}$) and Case ($\mathcal{C}'$):

If $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$. Since the sequence $0 < x_n := e^{n(\beta_n-\alpha_n)} < 1$ is bounded, set $E := \{x | x \text{ is a limit point of } \{x_n\}, n \geq 1\}$, then $\sup E = 1$ from $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$. Now, we are going to extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

(a) $\lim_{k\to\infty} x_{n_k} = 1$, and

(b) $E^1 := \{x | x \text{ is a limit point of } \{x_n\} \setminus \{x_{n_k}\}\}$, with $\sup E^1 < 1$.

We verify that it is possible to extract such a subsequence $\{x_{n_k}\}$. Since 1 is a limit point of $A := \{x_n, n \geq 1\}$, take a subsequence $\{x_{n_k^{(1)}}\}$ of A such that $\lim_{k\to\infty} x_{n_k^{(1)}} = 1$, set $A^{(1)} := \{x_{n_k^{(1)}}, k \geq 1\}$ and let $A_1 := A \setminus A^{(1)}$. If 1 is also a limit point of A_1, take a subsequence $\{x_{n_k^{(2)}}\}$ of A_1 such that $\lim_{k\to\infty} x_{n_k^{(2)}} = 1$, set $A^{(2)} := \{x_{n_k^{(2)}}, k \geq 1\}$ and let $A_2 := A_1 \setminus A^{(2)}$. Continuing this process, we obtain a series of sequences, i.e., $A^{(1)}, A^{(2)}, \ldots$, set $A_f := A \setminus \bigcup_{s\in\mathbb{I}\subseteq\mathbb{N}} A^{(s)}$. Since A is a countable set, this process will stop until 1 is not a limit point of A_f after finite or countable steps; otherwise, we will see that 1 is the only limit point of A, which contradicts to the assumptions of Case ($\mathcal{C}$). Then we can take the subsequence $\{x_{n_k}\} = \bigcup_{s\in\mathbb{N}} A^{(s)}$ which satisfies (a) and (b).

Set $\{x_{n'_k}\} := \{x_n\} \setminus \{x_{n_k}\}$, it is obvious that $\{n_k, k \geq 0\} \cup \{n'_k, k \geq 0\} = \mathbb{N}$. Then from ($\mathcal{A}$) and (a), we have seen that $[n]_{p_{n_k},q_{n_k}} \to \infty \Leftrightarrow e^{n_k\beta_{n_k}}/n_k \to 0$ as $k \to \infty$. And since

$\overline{\lim}_{n\to\infty} x_{n'_k} < 1$ from (b), we also have seen from $(\mathcal{B})$ that $[n]_{p_{n'_k},q_{n'_k}} \to \infty \Leftrightarrow e^{n'_k\beta_{n'_k}}(\alpha_{n'_k} - \beta_{n'_k}) \to 0$ as $k \to \infty$. In summary,

(i) $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty \Leftrightarrow$ for any subsequence $\{N_k\}$ of a natural number set such that $\lim_{k\to\infty} N_k = \infty$, we have $\lim_{k\to\infty}[n]_{p_{N_k},q_{N_k}} = \infty$.

(ii) Since $\{n_k\}_{k\geq 0}$ and $\{n'_k\}_{k\geq 0}$ are two subsequences of a natural number set such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} n'_k = \infty$, thus as $k \to \infty$

$$\lim_{n\to\infty}[n]_{p_n,q_n} = \infty \quad \Rightarrow \quad \begin{cases} [n]_{p_{n_k},q_{n_k}} \to \infty & \Leftrightarrow \quad e^{n_k\beta_{n_k}}/n_k \to 0, \\ [n]_{p_{n'_k},q_{n'_k}} \to \infty & \Leftrightarrow \quad e^{n'_k\beta_{n'_k}}(\alpha_{n'_k} - \beta_{n'_k}) \to 0. \end{cases}$$

We assert that the inverse proposition of (ii) also holds. Indeed, for each sufficiently large $N > 0$, there exists a positive integer K_1 such that for every natural number $k > K_1$, we have $[n]_{p_{n_k},q_{n_k}} > N$; meanwhile, for the previous $N > 0$, there exists a positive integer K_2 such that for every natural number $k > K_2$, we have $[n]_{p_{n'_k},q_{n'_k}} > N$; take $n_0 = \max\{n_{K_1}, n'_{K_2}\}$, for every natural number $n > n_0$, we have $[n]_{p_n,q_n} > N$, i.e., $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$. Now we have proved that as $k \to \infty$

$$\lim_{n\to\infty}[n]_{p_n,q_n} = \infty \quad \Leftrightarrow \quad \Delta := \begin{cases} e^{n_k\beta_{n_k}}/n_k \to 0, & \text{(c)} \\ e^{n'_k\beta_{n'_k}}(\alpha_{n'_k} - \beta_{n'_k}) \to 0. & \text{(d)} \end{cases} \tag{2.11}$$

Next, we infer that as $k \to \infty$

$$\Delta \quad \Leftrightarrow \quad \lim_{n\to\infty} \max\left\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n - \beta_n)\right\} = 0. \tag{2.12}$$

'$\Leftarrow$' of (2.12) is straightforward. Now we show '$\Rightarrow$'. On the one hand, by Remark (2.2), we know from (d) of Δ that $e^{n'_k\beta_{n'_k}}/n'_k \to 0$, and combined with (c) $e^{n_k\beta_{n_k}}/n_k \to 0$, we can show $e^{n\beta_n}/n \to 0$ by using a similar method as in the previous paragraph. On the other hand, $e^{n_k\beta_{n_k}}(\alpha_{n_k} - \beta_{n_k}) \to 0$ is straightforward (since $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} e^{n_k(\beta_{n_k}-\alpha_{n_k})} = 1$, and note (c) in Δ), and combined with $e^{n'_k\beta_{n'_k}}(\alpha_{n'_k} - \beta_{n'_k}) \to 0$, we can also deduce that $e^{n\beta_n}(\alpha_n - \beta_n) \to 0$. This yields the proof of '$\Rightarrow$' in (2.12).

Therefore, $(\mathcal{C})$ and $(\mathcal{C}')$ follow from (2.11) and (2.12). □

Remark 2.1 In general, from (2.1) we have only $[n]_{p_n,q_n} \to \infty \Rightarrow e^{n\beta_n}/n \to 0$. However, if $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ holds, then we have the equivalent relation $e^{n\beta_n}/n \to 0 \Leftrightarrow [n]_{p_n,q_n} \to \infty$ from $(\mathcal{A})$. Thus, we have:

$(\mathcal{A}_0)$ If $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$, then $e^{n\beta_n}/n \to 0 \Leftrightarrow [n]_{p_n,q_n} \to \infty$.

Similarly, from $(\mathcal{B})$ and $(\mathcal{B}')$, $(\mathcal{C})$ and $(\mathcal{C}')$ we have

$(\mathcal{B}_0)$ If $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, then $e^{n\beta_n}(\alpha_n - \beta_n) \to 0 \Leftrightarrow [n]_{p_n,q_n} \to \infty$.

$(\mathcal{C}_0)$ If $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$, then $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n - \beta_n)\} \to 0 \Leftrightarrow [n]_{p_n,q_n} \to \infty$.

Remark 2.2 In Case $(\mathcal{B})$, we can also deduce directly from $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ and $e^{n\beta_n}(\alpha_n - \beta_n) \to 0$ that $e^{n\beta_n}/n \to 0$ as $n \to \infty$. Indeed, since $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, and combined with the classical inequality on upper (lower) limit

$$\underline{\lim}_{n\to\infty} e^{n(\alpha_n-\beta_n)} \cdot \overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} \geq \underline{\lim}_{n\to\infty} \left(e^{n(\alpha_n-\beta_n)} \cdot e^{n(\beta_n-\alpha_n)}\right) = 1,$$

we have $\underline{\lim}_{n\to\infty} e^{n(\alpha_n-\beta_n)} > 1$. Thus, for sufficiently large n, there exists $c_0 > 1$ such that $e^{n(\alpha_n-\beta_n)} > c_0$. This means that $0 < (\log c_0)e^{n\beta_n}/n < e^{n\beta_n}/n \cdot (n(\alpha_n - \beta_n)) \to 0$, and we have seen that $e^{n\beta_n}/n \to 0$.

Remark 2.3 Now we utilize Theorem 2.1 (Remark 2.1) to elaborate Example 1.2 again. In the example, $\beta_n = 1/\sqrt{n}$, while $e^{n\beta_n}/n = e^{\sqrt{n}}/n \nrightarrow 0$ as $n \to \infty$, thus $[n]_{p_n,q_n} \nrightarrow \infty$.

For $p_n = 1$, $\beta_n = 0$, $q_n = 1 - \alpha_n$, then $e^{n\beta_n}/n = 1/n \to 0$ and $e^{n\beta_n}(\alpha_n - \beta_n) = \alpha_n \to 0$. Thus any case of $(\mathcal{A}_0)$-$(\mathcal{C}_0)$ is straightforward. In this case, (p_n, q_n)-integer reduces to q_n-integer, and it is known that $[n]_{q_n} \to \infty \Leftrightarrow q_n \to 1$ as $n \to \infty$. See [19], Theorem 2, and [22], formula (2.7).

3 Conclusion

In this note, we mainly obtain the sufficient and necessary conditions for (p,q)-integer $[n]_{p,q}$ tending to infinity as $n \to \infty$. The conclusion guarantees the (p,q)-analogue of Bernstein-type operators to be approximation processes as $n \to \infty$.

Acknowledgements
The paper has been greatly improved by the efforts of the anonymous referees through their diligent reading of and perceptive comments on an initial draft.
This work is supported by the National Natural Science Foundation of China (Grant No. 61572020,11601266), the China Postdoctoral Science Foundation funded project (Grant No. 2015M582036) and the Natural Science Foundation of Fujian Province of China (Grant No. 2016J05017). We also thank Fujian Provincial Key Laboratory of Data Intensive Computing and Key Laboratory of Intelligent Computing and Information Processing of Fujian Province University.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XWX carried out the proof of the main results. QBC read and approved the final manuscript.

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Publisher's Note

Received: 23 February 2017 Accepted: 1 June 2017 Published online: 17 June 2017

References

1. Ostrovska, S: On the q-Bernstein polynomials. Adv. Stud. Contemp. Math. **11**(2), 193-204 (2005)
2. Charalambides, CA: The q-Bernstein basis as a q-binomial distribution. J. Stat. Plan. Inference **140**, 2184-2190 (2010)
3. Jing, S: The q-deformed binomial distribution and its asymptotic behaviour. J. Phys. A, Math. Gen. **27**, 493-499 (1994)
4. Castellani, L, Wess, J (eds.): Quantum Groups and Their Applications in Physics. IOS Press, Amsterdam (1996)
5. Chakrabarti, R, Jagannathan, R: A (p,q)-oscillator realization of two parameter quantum algebras. J. Phys. A, Math. Gen. **24**, 711-718 (1991)
6. Mursaleen, M, Ansari, KJ, Khan, A: On (p,q)-analogue of Bernstein operators. Appl. Math. Comput. **266**, 874-882 (2015)
7. Mursaleen, M, Ansari, KJ, Khan, A: Erratum to "On (p,q)-analogue of Bernstein operators" [Appl. Math. Comput. 266 (2015) 874-882]. Appl. Math. Comput. **278**, 70-71 (2016)
8. Acar, T: (p,q)-Generalization of Szász-Mirakyan operators. Math. Methods Appl. Sci. **39**(10), 2685-2695 (2016)
9. Acar, T, Aral, A, Mohiuddine, SA: On Kantorovich modification of (p,q)-Baskakov operators. J. Inequal. Appl. (2016). doi:10.1186/s13660-016-1045-9
10. Mursaleen, M, Nasiruzzaman, MD, Khan, A, Ansari, KJ: Some approximation results on Bleimann-Butzer-Hahn operators defined by (p,q)-integers. Filomat **30**(3), 639-648 (2016)

11. Cai, QB, Zhou, GR: On (p,q)-analogue of Kantorovich type Bernstein-Stancu-Schurer operators. Appl. Math. Comput. **276**, 12-20 (2016)
12. Mishra, VN, Pandey, S: On (p,q)-Baskakov-Durrmeyer-Stancu operators (2016). arXiv:1602.06719
13. Mursaleen, M, Alotaibi, A, Ansari, KJ: On a Kantorovich variant of (p,q)-Szász-Mirakjan operators. J. Funct. Spaces **2016**, Article ID 1035253 (2016)
14. Mursaleen, M, Ansari, KJ, Khan, A: Some approximation results by (p,q)-analogue of Bernstein-Stancu operators. Appl. Math. Comput. **264**, 392-402 (2015)
15. Mursaleen, M, Nasiuzzaman, M, Nurgali, A: Some approximation results on Bernstein-Schurer operators defined by (p,q)-integers. J. Inequal. Appl. **2015**, Article ID 249 (2015)
16. Acar, T, Aral, A: On pointwise convergence of q-Bernstein operators and their q-derivatives. Numer. Funct. Anal. Optim. **36**(3), 287-304 (2015)
17. Acar, T, Agrawal, P, Kumar, A: On a modification of (p,q)-Szász-Mirakyan operators. Complex Anal. Oper. Theory (2016). doi:10.1007/s11785-016-0613-9
18. Ilarslan, H, Acar, T: Approximation by bivariate (p,q)-Baskakov-Kantorovich operators. Georgian Math. J. (2016). doi:10.1515/gmj-2016-0057
19. Phillips, GM: Bernstein polynomials based on the q-integers. Ann. Numer. Math. **4**, 511-518 (1997)
20. Ostrovska, S: The q-versions of the Bernstein operator: from mere analogies to further developments. Results Math. **69**(3-4), 275-295 (2016)
21. Khan, K, Lobiyal, DK: Bézier curves based on Lupas (p,q)-analogue of Bernstein polynomials in CAGD (2015). arXiv:1505.01810
22. Videnskii, VS: On q-Bernstein polynomials and related positive linear operators. In: Problems of Modern Mathematics and Mathematical Education, Hertzen Readings, pp. 118-126 (2004)