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Dynamic Opial diamond- α integral inequalities involving the power of a function

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Abstract

In this paper, we present some new dynamic Opial-type diamond alpha inequalities on time scales. The obtained results are related to the function f^k .

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1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, we say that t is right-scattered, whereas if $\rho(t) < t$, we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The mappings $\mu, \nu : \mathbb{T} \rightarrow [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, are called the forward and backward graininess function, respectively. If $\mathbb{T}$ has a left-scattered maximum t_1 , then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2 , then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$.

Theorem 1.1 Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then:

1. The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable at t with

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

2. For any constant α , $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable at t with

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

3. The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

Theorem 1.2 Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_k$. Then:

1. The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable at t with

$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$

2. For any constant α , $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable at t with

$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t).$$

3. The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable at t with

$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f^\rho(t)g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g^\rho(t).$$

The following formulas will be used in our paper:

$$(f^{l+1})^\Delta = \left\{ \sum_{k=0}^l f^k (f^\sigma)^{l-k} \right\} f^\Delta, \quad l \in \mathbb{N},$$

$$(f^{l+1})^\nabla = \left\{ \sum_{k=0}^l f^k (f^\rho)^{l-k} \right\} f^\nabla, \quad l \in \mathbb{N}.$$

Definition 1.3 Let $0 \leq \alpha \leq 1$ and let f be both delta and nabla differentiable at $t \in \mathbb{T}_k^k$. Then f is diamond- α differentiable at t and $f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t)$.

Definition 1.4 Let $a, b \in \mathbb{T}$, $a < b$, $f : \mathbb{T} \rightarrow \mathbb{R}$ and $\alpha \in [0, 1]$. The diamond- α integral of t on $[a, b]_{\mathbb{T}}$ is defined by

$$\int_a^b f(t) \diamond_\alpha t = \alpha \int_a^b f(t) \Delta t + (1 - \alpha) \int_a^b f(t) \nabla t.$$

Theorem 1.5 Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be $\diamond_\alpha$ -differentiable at $t \in \mathbb{T}$. Then

1. $f + g$ is $\diamond_\alpha$ -differentiable at $t \in \mathbb{T}$ with $(f + g)^{\diamond_\alpha} = f^{\diamond_\alpha} + g^{\diamond_\alpha}$,
2. fg is $\diamond_\alpha$ -differentiable at $t \in \mathbb{T}$ with $(fg)^{\diamond_\alpha} = f^{\diamond_\alpha}g + \alpha f^\sigma g^\Delta + (1 - \alpha)f^\rho g^\nabla$.

Many authors have studied the theory of integral inequalities on time scales (see, for example, [1–10]). In [3], the following Opial inequality on time scales was established.

Theorem 1.6 ([3]) For a delta differentiable $f : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $f(0) = 0$, we have

$$\int_0^h |(f + f^\sigma)f^\Delta| \Delta t \leq h \int_0^h |f^\Delta|^2 \Delta t, \quad (1)$$

with equality when $f(t) = ct$.

In [1], the authors established the following theorem.

Theorem 1.7 ([1]) Let $\omega(t)$ be positive and continuous on $(0, h)$ with $\int_0^h \omega^{1-q} \Delta t < \infty$, $q > 1$. For a differentiable $f : [0, h] \rightarrow \mathbb{R}$ with $f(0) = 0$, we have

$$\int_0^h |(f + f^\sigma)f^\Delta| \Delta t \leq \left(\int_0^h \omega^{1-q} \Delta t \right)^{\frac{2}{q}} \left(\int_0^h \omega |f^\Delta|^p \Delta t \right)^{\frac{2}{p}},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and with equality when $f(t) = c \int_0^t \omega^{1-q} \Delta \tau$ for a constant c .

2 Main results

In this section, we present our results.

Theorem 2.1 *Let T be a time scale. For $\diamond_\alpha$ differentiable $f : [0, h] \cap T \rightarrow \mathbb{R}$, with $f(0) = 0$ we have*

$$\int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha t \leq h^{k-1} \int_0^h |f^{\diamond_\alpha}|^k(t) \diamond_\alpha t. \quad (2)$$

Proof Starting with the left side of (2), we obtain

$$\begin{aligned} \int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha t &= \int_0^h |f \cdot f^{k-1}|^{\diamond_\alpha}(t) \diamond_\alpha t \\ &= \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \diamond_\alpha t \\ &= \alpha \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + (1-\alpha) \int_0^h |f^{k-1} f^{\diamond_\alpha} + \alpha f^\sigma (f^{k-1})^\Delta + (1-\alpha) f^\rho (f^{k-1})^\nabla|(t) \nabla t \\ &\leq \alpha \int_0^h |f^{k-1} f^{\diamond_\alpha}|(t) \Delta t + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t + (1-\alpha) \int_0^h |f^{k-1} f^{\diamond_\alpha}|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t. \end{aligned}$$

Using Definition 1.3, we get

$$\begin{aligned} \int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha t &\leq \alpha \int_0^h |\alpha f^{k-1} f^\Delta + (1-\alpha) f^{k-1} f^\nabla|(t) \Delta t \\ &\quad + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + (1-\alpha) \int_0^h |\alpha f^{k-1} f^\Delta + (1-\alpha) f^{k-1} f^\nabla|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t \\ &\leq \alpha^2 \int_0^h |f^{k-1} f^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\nabla|(t) \Delta t \\ &\quad + \alpha^2 \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \Delta t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^{k-1} f^\nabla|(t) \nabla t \\ &\quad + \alpha(1-\alpha) \int_0^h |f^\sigma (f^{k-1})^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^\rho (f^{k-1})^\nabla|(t) \nabla t. \end{aligned}$$

We find that

$$\begin{aligned}
 \int_0^h |f^\sigma(f^{k-1})^\Delta|(t) \Delta t &= \int_0^h |f^\sigma(f \cdot f^{k-2})^\Delta|(t) \Delta t \\
 &= \int_0^h |f^\sigma(f^\Delta f^{k-2} + f^\sigma(f \cdot f^{k-3})^\Delta)|(t) \Delta t \\
 &\quad \vdots \\
 &= \int_0^h |f^\sigma(f^\Delta f^{k-2} + f^\sigma f^\Delta f^{k-3} + \dots + (f^\sigma)^2 f^\Delta)|(t) \Delta t \\
 &= \int_0^h |f^\sigma f^{k-2} + (f^\sigma)^2 f^{k-3} + \dots + (f^\sigma)|f^\Delta|(t) \Delta t \\
 &= \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta|(t) \Delta t.
 \end{aligned}$$

Similarly,

$$\int_0^h |f^\rho(f^{k-1})^\nabla|(t) \Delta t = \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla|(t) \nabla t.$$

Therefore,

$$\begin{aligned}
 \int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha t &\leq \alpha^2 \int_0^h |f^{k-1} f^\Delta|(t) \Delta t + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\nabla|(t) \Delta t \\
 &\quad + \alpha^2 \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta|(t) \Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla|(t) \Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h |f^{k-1} f^\Delta|(t) \nabla t + (1-\alpha)^2 \int_0^h |f^{k-1} f^\nabla|(t) \nabla t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| |f^\Delta| \nabla t \\
 &\quad + (1-\alpha)^2 \int_0^h \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| |f^\nabla| \nabla t \\
 &= \alpha^2 \int_0^h \left(|f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| \right) |f^\Delta|(t) \Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left(|f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\rho)^{k-1-n} \right| \right) |f^\nabla|(t) \Delta t \\
 &\quad + \alpha(1-\alpha) \int_0^h \left(|f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n (f^\sigma)^{k-1-n} \right| \right) |f^\Delta|(t) \nabla t
 \end{aligned}$$

$$\begin{aligned}
& + (1-\alpha)^2 \int_0^h \left(|f^{k-1}| + \left| \sum_{n=0}^{k-2} f^n(f^\rho)^{k-1-n} \right| \right) |f^\nabla(t)| \nabla t \\
& \leq \alpha^2 \int_0^h \left(|f^{k-1}| + \sum_{n=0}^{k-2} |f^n(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(|f^{k-1}| + \sum_{n=0}^{k-2} |f^n(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(|f^{k-1}| + \sum_{n=0}^{k-2} |f^n(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \nabla t \\
& \quad + (1-\alpha)^2 \int_0^h \left(|f^{k-1}| + \sum_{n=0}^{k-2} |f^n(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \nabla t \\
& = \alpha^2 \int_0^h \left(\sum_{n=0}^{k-1} |f^n| |(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(\sum_{n=0}^{k-1} |f^n| |(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(\sum_{n=0}^{k-1} |f^n| |(f^\sigma)^{k-1-n}| \right) |f^\Delta(t)| \nabla t \\
& \quad + (1-\alpha)^2 \int_0^h \left(\sum_{n=0}^{k-1} |f^n| |(f^\rho)^{k-1-n}| \right) |f^\nabla(t)| \nabla t.
\end{aligned}$$

Consider $g(t) = \int_0^t |f^{\diamond_\alpha}(s)| \diamond_\alpha s$. Then we have $g^\Delta(t) = |f^\Delta(t)|$, $g^\nabla(t) = |f^\nabla(t)|$, and $|f| \leq g$, so that $g(t) = \int_0^t |f^{\diamond_\alpha}(s)| \diamond_\alpha s \geq |\int_0^t f^{\diamond_\alpha}(s) \diamond_\alpha s| = |f(t) - f(0)| = |f(t)|$.

The above inequality becomes

$$\begin{aligned}
\int_0^h |f^k|^{\diamond_\alpha}(t) \diamond_\alpha & \leq \alpha^2 \int_0^h \left(\sum_{n=0}^{k-1} g^n (g^\sigma)^{k-1-n} \right) (g^\Delta)(t) \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(\sum_{n=0}^{k-1} g^n (g^\rho)^{k-1-n} \right) (g^\nabla)(t) \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h \left(\sum_{n=0}^{k-1} g^n (g^\sigma)^{k-1-n} \right) (g^\Delta)(t) \nabla t \\
& \quad + (1-\alpha)^2 \int_0^h \left(\sum_{n=0}^{k-1} g^n (g^\rho)^{k-1-n} \right) (g^\nabla)(t) \nabla t \\
& = \alpha^2 \int_0^h (g^k)^\Delta(t) \Delta t + \alpha(1-\alpha) \int_0^h (g^k)^\nabla(t) \Delta t \\
& \quad + \alpha(1-\alpha) \int_0^h (g^k)^\Delta(t) \nabla t + (1-\alpha)^2 \int_0^h (g^k)^\nabla(t) \nabla t \\
& = \alpha \left[\alpha \int_0^h (g^k)^\Delta(t) \Delta t + (1-\alpha) \int_0^h (g^k)^\Delta(t) \nabla t \right]
\end{aligned}$$

$$\begin{aligned}
& + (1-\alpha) \left[\int_0^h \alpha (g^k)^\nabla \Delta t + (1-\alpha) \int_0^h (g^k)^\nabla \nabla t \right] \\
& = \alpha \int_0^h (g^k)^\Delta \diamond_\alpha + (1-\alpha) \int_0^h (g^k)^\nabla \diamond_\alpha = \int_0^h (g^k)(t)^{\diamond_\alpha} \diamond_\alpha \\
& = g^k(t)|_0^h = g^k(h) - g^k(0) = [g(h)]^k = \left[\int_0^h |f^{\diamond_\alpha}(s)| \diamond_\alpha s \right]^k.
\end{aligned}$$

By using Hölder's inequality with indices $p = \frac{k}{k-1}$ and $q = k$, we obtain

$$\begin{aligned}
\left[\int_0^h 1 \cdot |f^{\diamond_\alpha}(s)| \diamond_\alpha s \right]^k & \leq \left[\left(\int_0^h 1^{\frac{k}{k-1}} \diamond_\alpha s \right)^{\frac{k-1}{k}} \left(\int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \right)^{\frac{1}{k}} \right]^k \\
& = \left(\int_0^h \diamond_\alpha s \right)^{k-1} \left(\int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \right) \\
& = (s|_0^h)^{k-1} \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s \\
& = h^{k-1} \int_0^h |f^{\diamond_\alpha}(s)|^k \diamond_\alpha s,
\end{aligned}$$

hence the proof is complete. $\square$

Theorem 2.2 Let $\omega(t)$ be positive and continuous on $(0, h)$, with $\int_0^h \omega^{1-q}(t) \Delta t < \infty$, $q > 1$. For differentiable $f : [0, h] \rightarrow \mathbb{R}$ with $f(0) = 0$ we have

$$\int_0^h |f^k|^\Delta \Delta t \leq \left(\int_0^h \omega^{1-q} \Delta t \right)^{\frac{k}{q}} \left(\int_0^h \omega |f^\Delta|^p \Delta t \right)^{\frac{k}{p}}, \quad (3)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof We take $g(t) = \int_0^t |f^\Delta(s)| \Delta s$. Then $|f(t)| \leq g(t)$, $g^\Delta(t) = |f^\Delta(t)|$, so we have

$$\begin{aligned}
\int_0^h |f^k|^\Delta \Delta t & = \int_0^h \left| \sum_{k=0}^{n-1} f^k(f^\sigma)^{n-1-k} \right| |f^\Delta|(t) \Delta t \\
& \leq \int_0^h \left(\sum_{k=0}^{n-1} g^k(g^\sigma)^{n-1-k} \right) (g^\Delta)(t) \Delta t = \int_0^h (g^k)^\Delta \Delta t \\
& = g^k(h) - g^k(0) = g^k(h) = \left(\int_0^h |f^\Delta|(t) \Delta t \right)^k \\
& = \left(\int_0^h \omega^{-\frac{1}{p}} \omega^{\frac{1}{p}} |f^\Delta|(t) \Delta t \right)^k \\
& \leq \left[\left(\int_0^h (\omega^{-\frac{1}{p}})^q \Delta t \right)^{\frac{1}{q}} \left(\int_0^h (\omega^{\frac{1}{p}} |f^\Delta|)^p(t) \Delta t \right)^{\frac{1}{p}} \right]^k \\
& = \left(\int_0^h \omega^{1-q} \Delta t \right)^{\frac{k}{q}} \left(\int_0^h (\omega |f^\Delta|)^p(t) \Delta t \right)^{\frac{k}{p}}.
\end{aligned}$$

$\square$

Theorem 2.3 Let $\omega(t)$ be positive and continuous on $(0, h)$, with $\int_0^h \omega^{1-q}(t) \nabla t < \infty$, $q > 1$. For differentiable $f : [0, h] \rightarrow \mathbb{R}$ with $f(0) = 0$ we have

$$\int_0^h |f^k|^\nabla \nabla t \leq \left(\int_0^h \omega^{1-q} \nabla t \right)^{\frac{k}{q}} \left(\int_0^h \omega |f^\nabla|^p \nabla t \right)^{\frac{k}{p}}, \quad (4)$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.4 Assume that $p > 1$, $q = \frac{p}{p-1}$, $\alpha \in [0, 1]$, $h \in (0, \infty)_{\mathbb{T}}$, $\omega \in \mathcal{C}([0, h]_{\mathbb{T}}, (0, \infty))$ and $f \in \mathcal{C}_{\diamond_\alpha}^1([0, h]_{\mathbb{T}}, \mathbb{R})$. If $\alpha f^\Delta \geq 0$, $(1 - \alpha)f^\nabla \geq 0$ and $f(0) = 0$ then

$$\begin{aligned} & \alpha^k \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^k \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & \leq \left(\int_0^h \omega^{1-q}(t) \diamond_\alpha t \right)^{\frac{k}{q}} \left(\int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \diamond_\alpha t \right)^{\frac{k}{p}}. \end{aligned} \quad (5)$$

Proof By Theorems 2.2, 2.3, Hölder's inequality and $k = \frac{k}{q} + (1 + p)\frac{k}{p}$, we get

$$\begin{aligned} & \alpha^k \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^k \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & = \alpha^{\frac{k}{q} + (1+p)\frac{k}{p}} \int_0^h |(f^k)^\Delta(t)| \Delta t + (1 - \alpha)^{\frac{k}{q} + (1+p)\frac{k}{p}} \int_0^h |(f^k)^\nabla(t)| \nabla t \\ & \leq \alpha^{\frac{k}{q} + (1+p)\frac{k}{p}} \left(\int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left(\int_0^h \omega(t) |f^\Delta(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + (1 - \alpha)^{\frac{k}{q} + (1+p)\frac{k}{p}} \left(\int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \left(\int_0^h \omega(t) |f^\nabla(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & \leq \left(\alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left(\alpha \int_0^h \omega(t) |\alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + \left((1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \\ & \quad \cdot \left((1 - \alpha) \int_0^h \omega(t) |\alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & = \left(\alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^{\frac{k}{q}} \left(\alpha \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \Delta t \right)^{\frac{k}{p}} \\ & \quad + \left((1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \left((1 - \alpha) \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \nabla t \right)^{\frac{k}{p}} \\ & \leq \left[\left(\alpha \int_0^h \omega^{1-q}(t) \Delta t \right)^k + \left((1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^k \right]^{\frac{1}{q}} \\ & \quad \cdot \left[\left(\alpha \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \Delta t \right)^k + \left((1 - \alpha) \int_0^h \omega(t) |f^{\diamond_\alpha}(t)|^p \nabla t \right)^k \right]^{\frac{1}{p}} \\ & \leq \left(\alpha \int_0^h \omega^{1-q}(t) \Delta t + (1 - \alpha) \int_0^h \omega^{1-q}(t) \nabla t \right)^{\frac{k}{q}} \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\alpha \int_0^h \omega(t) |f^{\diamond \alpha}(t)|^p \Delta t + (1-\alpha) \int_0^h \omega(t) |f^{\nabla \alpha}(t)|^p \nabla t \right)^{\frac{k}{p}} \\
& = \left(\int_0^h \omega^{1-q}(t) \diamond_{\alpha} t \right)^{\frac{k}{q}} \left(\int_0^h \omega(t) |f^{\diamond \alpha}(t)|^p \diamond_{\alpha} t \right)^{\frac{k}{p}}.
\end{aligned}$$

□

Theorem 2.5 Assume that $1 < p \leq 2$, $q = \frac{p}{p-1}$, $\alpha \in [0, 1]$, $h \in (0, \infty)_{\mathbb{T}}$, $\omega \in \mathbb{C}([0, h]_{\mathbb{T}}, (0, \infty))$ and $f \in \mathbb{C}_{\diamond_{\alpha}}^1([0, h]_{\mathbb{T}}, \mathbb{R})$. If $\alpha f^{\Delta} \geq 0$, $(1-\alpha)f^{\nabla} \geq 0$ and $f(0) = 0$, then

$$\begin{aligned}
& \alpha^k \int_0^u |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_0^u |(f^k)^{\nabla}(t)| \nabla t \\
& \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} \gamma^j \beta^{\frac{k-j}{q}} \left[\int_0^h \omega(t) |g^{\diamond \alpha}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-1}{p}} \\
& \quad + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(h) - f(0)),
\end{aligned} \tag{6}$$

where $\beta := \min_{u \in [0, h]_{\mathbb{T}}} v(u)$, $v(u) = \max\{\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t, \int_u^h \omega^{1-q}(t) \diamond_{\alpha} t\}$, $\gamma := \max\{|f(0)|, |f(h)|\}$.

Proof We let $u \in [0, h]_{\mathbb{T}}$ be arbitrary. By applying Theorem 2.4 to the function $g(t) = f(t) - f(0)$, we obtain

$$\begin{aligned}
& \alpha^k \int_0^u |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_0^u |(f^k)^{\nabla}(t)| \nabla t \\
& = \alpha^k \int_0^u \left| \sum_{j=0}^{k-1} \binom{k}{j} (g^{k-j})^{\Delta} f^j(0) \right| \Delta t \\
& \quad + (1-\alpha)^k \int_0^u \left| \sum_{j=0}^{k-1} \binom{k}{j} (g^{k-j})^{\nabla} f^j(0) \right| \nabla t \\
& \leq \binom{k}{0} \left[\alpha^k \int_0^u |g^k|^{\Delta} \Delta t + (1-\alpha)^k \int_0^u |g^k|^{\nabla} \nabla t \right] \\
& \quad + \alpha \binom{k}{1} |f(0)| \left[\alpha^{k-1} \int_0^u |g^{k-1}|^{\Delta} \Delta t + (1-\alpha)^{k-1} \int_0^u |g^{k-1}|^{\nabla} \nabla t \right] \\
& \quad + \alpha^2 \binom{k}{2} |f^2(0)| \left[\alpha^{k-2} \int_0^u |g^{k-2}|^{\Delta} \Delta t + (1-\alpha)^{k-2} \int_0^u |g^{k-2}|^{\nabla} \nabla t \right] \\
& \quad \vdots \\
& \quad + \alpha^{k-2} \binom{k}{k-2} |f^{k-2}(0)| \left[\alpha^2 \int_0^u |g^2|^{\Delta} \Delta t + (1-\alpha)^2 \int_0^u |g^2|^{\nabla} \nabla t \right] \\
& \quad + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| \left[\alpha \int_0^u |f|^{\Delta} \Delta t + (1-\alpha) \int_0^u |f|^{\nabla} \nabla t \right] \\
& \leq \binom{k}{0} \left[\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k}{q}} \left[\int_0^u \omega(t) |g^{\diamond \alpha}(t)|^p \diamond_{\alpha} t \right]^{\frac{k}{p}}
\end{aligned}$$

$$\begin{aligned}
& + \alpha \binom{k}{1} |f(0)| \left[\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-1}{q}} \left[\int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-1}{p}} \\
& + \alpha^2 \binom{k}{2} |f^2(0)| \left[\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-2}{q}} \left[\int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-2}{p}} \\
& \vdots \\
& + \alpha^{k-2} \binom{k}{k-2} |f^{k-2}(0)| \left[\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{2}{q}} \\
& \cdot \left[\int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{2}{p}} \\
& + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| \alpha \int_0^u |f^{\Delta}(t)| \Delta t \\
& + \alpha^{k-1} \binom{k}{k-1} |f^{k-1}(0)| (1-\alpha) \int_0^u |f^{\nabla}(t)| \nabla t \\
& \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} |f^j(0)| \left[\int_0^u \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-j}{q}} \left[\int_0^u \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-j}{p}} \\
& + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(u) - f(0)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \alpha^k \int_u^h |(f^k)^{\Delta}(t)| \Delta t + (1-\alpha)^k \int_u^h |(f^k)^{\nabla}(t)| \nabla t \\
& \leq \sum_{j=0}^{k-2} \alpha^j \binom{k}{j} |f^j(0)| \left[\int_u^h \omega^{1-q}(t) \diamond_{\alpha} t \right]^{\frac{k-j}{q}} \left[\int_u^h \omega(t) |g^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right]^{\frac{k-j}{p}} \\
& + \alpha^{k-1} \binom{k}{k-1} \gamma^{k-1} (f(h) - f(u)).
\end{aligned}$$

Adding these two inequalities and taking into account that $a^r + b^r \leq (a+b)^r$ holds, for $a, b \geq 0$ and $r \geq 1$, yield the desired inequality. $\square$

3 Conclusion

In this paper, we have obtained several Opial-type integral inequalities on time scales via the notion of the diamond-alpha derivative. These inequalities are related to the function f^k .

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Author's contributions

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