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Two generalized Lyapunov-type inequalities for a fractional *p*-Laplacian equation with fractional boundary conditions

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Abstract
In this paper, we investigate the existence of positive solutions for the boundary value problem of nonlinear fractional differential equation with mixed fractional derivatives and *p*-Laplacian operator. Then we establish two smart generalizations of Lyapunov-type inequalities. Some applications are given to demonstrate the effectiveness of the new results.

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1 Introduction

Lyapunov's inequality [1] has proved to be very useful in various problems related with differential equations; for examples, see [2, 3] and the references therein. Recently, many researchers have given some Lyapunov-type inequalities for different classes of fractional boundary value problems (see [4–10]). In [7], Ferreira investigated a Lyapunov-type inequality for the fractional boundary value problem

$$\begin{cases} D_{a^+}^{\alpha} y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.1}$$

where $D_{a^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $1 < \alpha \leq 2$, a and b are consecutive zeros, and q is a real and continuous function. It was proved that if (1.1) has a nontrivial solution, then

$$\int_a^b \left|q(t)\right| \mathrm{d}s > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.2}$$

Obviously, if we set $\alpha = 2$ in (1.2), one can obtain the classical Lyapunov inequality [1].

In [8], Jleli and Samet considered the fractional differential equation

$${}^{C}D_{a^+}^{\alpha} y(t) + q(t)y(t) = 0, \quad a < t < b, 1 < \alpha \leq 2, \tag{1.3}$$

with the mixed boundary conditions

$$y(a) = y'(b) = 0 \tag{1.4}$$

or

$$y'(a) = y(b) = 0, \tag{1.5}$$

where ${}^C D^{\alpha}_{a^+}$ is the Caputo fractional derivative of order $1 < \alpha \le 2$. For boundary conditions (1.4) and (1.5), two Lyapunov-type inequalities were established, respectively, as follows:

$$\int_a^b (b-s)^{\alpha-2} \left|q(s)\right| \,\mathrm{d}s \ge \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)} \tag{1.6}$$

and

$$\int_a^b (b-s)^{\alpha-1} \left|q(s)\right| \,\mathrm{d}s \ge \Gamma(\alpha). \tag{1.7}$$

Recently, we considered in [11] the same equation (1.3) with the fractional boundary condition

$$y(a) = {}^C D^{\beta}_{a^+} y(b) = 0,$$

where $0 < \beta \le 1$.

In [12], Arifi *et al.* considered the following nonlinear fractional boundary value problem with p-Laplacian operator:

$$\begin{cases} D^{\beta}_{a^+}(\Phi_p(D^{\alpha}_{a^+} u(t))) + \chi(t)\Phi_p(u(t)) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, & D^{\alpha}_{a^+} u(a) = D^{\alpha}_{a^+} u(b) = 0, \end{cases} \tag{1.8}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, $D^{\alpha}_{a^+}$, $D^{\beta}_{a^+}$ are the Riemann-Liouville fractional derivative of orders α, β, $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, and $\chi : [a,b] \to \mathbb{R}$ is a continuous function. It was proved that if (1.8) has a nontrivial continuous solution, then

$$\begin{aligned} &\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}\left|\chi(s)\right| \,\mathrm{d}s \\ &\quad \ge \bigl(\Gamma(\alpha)\bigr)^{p-1}\Gamma(\beta)(b-a)^{\beta-1}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,\mathrm{d}s\right)^{1-p}. \end{aligned} \tag{1.9}$$

More recently, Chidouh and Torres in [13] considered the following boundary value problem:

$$\begin{cases} D^{\alpha}_{a^+} y(t) + q(t)f(y(t)) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1.10}$$

where $D^{\alpha}_{a^+}$ is the Riemann-Liouville fractional derivative with $1 < \alpha \le 2$, and $q : [a,b] \to \mathbb{R}_+$ is a nontrivial Lebesgue integrable function. Under the assumption that the nonlinear term $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a concave and decreasing function, it was proved that if (1.10) has a nontrivial solution, then

$$\int_a^b |q(t)|\,\mathrm{d}s > \frac{4^{\alpha-1}\Gamma(\alpha)\eta}{(b-a)^{\alpha-1}f(\eta)}, \tag{1.11}$$

where $\eta = \max_{t\in[a,b]} y(t)$. Obviously, if we set $f(y) = y$ in (1.11), one can obtain a Lyapunov inequality (1.2).

Motivated by the above work, in this paper, we consider the fractional boundary value problem

$$\begin{cases} D_{a^+}^{\beta}(\Phi_p({}^C D_{a^+}^{\alpha} u(t))) - k(t)f(u(t)) = 0, & a < t < b, \\ u'(a) = {}^C D_{a^+}^{\alpha} u(a) = 0, & u(b) = {}^C D_{a^+}^{\alpha} u(b) = 0, \end{cases} \tag{1.12}$$

where $1 < \alpha, \beta \leq 2$, and $k : [a, b] \to R$ is a continuous function. We write (1.12) as an equivalent integral equation and then, by using some properties of its Green function and the Guo-Krasnoselskii fixed point theorem, we can obtain our first result asserting existence of nontrivial positive solutions to problem (1.12). Then, under some assumptions on the nonlinear term f, we are able to get two corresponding Lyapunov-type inequalities. Finally in this paper, two corollaries and an example are given to demonstrate the effectiveness of the obtained results.

2 Preliminaries

In this section, we recall the definitions of the Riemann-Liouville fractional integral, fractional derivative, and the Caputo fractional derivative and give some lemmas which are useful in this article. For more details, we refer to [14, 15].

Definition 2.1 Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by ${}_aI^0 f \equiv f$ and

$$\left(I_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad \alpha > 0, t \in [a, b].$$

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : [a, b] \to \mathbb{R}$ is given by

$$\left(D_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where n is the smallest integer greater or equal to α and Γ denotes the Gamma function.

Definition 2.3 The Caputo derivative of fractional order $\alpha \geq 0$ is defined by ${}^C D_{a^+}^0 f \equiv f$ and

$$\left({}^C D_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \quad \alpha > 0, t \in [a, b],$$

where n is the smallest integer greater or equal to α.

Lemma 2.1 (Guo-Krasnoselskii fixed point theorem [16]) *Let X be a Banach space and let $P \subset X$ be a cone. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.2 (Jensen's inequality [17]) *Let ν be a positive measure and let Ω be a measurable set with $\nu(\Omega) = 1$. Let I be an interval and suppose that u is a real function in $L(d\nu)$ with $u(t) \in I$ for all $t \in \Omega$. If f is convex on I, then*

$$f\left(\int_\Omega u(t)\,\mathrm{d}\nu(t)\right) \leq \int_\Omega (f \circ u)(t)\,\mathrm{d}\nu(t). \tag{2.1}$$

If f is concave on I, then the inequality (2.1) *holds with $\leq$ substituted by $\geq$.*

3 Main results

We begin to write problem (1.12) in its equivalent integral form.

Lemma 3.1 *If $u \in C[a,b]$, $1 < \alpha, \beta \leq 2$, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then BVP* (1.12) *has a unique solution*

$$u(t) = \int_a^b G(t,s)\Phi_q\left(\int_a^b H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \tag{3.1}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b,\\ (b-s)^{\alpha-1}, & a \leq t \leq s \leq b,\end{cases} \tag{3.2}$$

and

$$H(s,\tau) = \frac{1}{\Gamma(\beta)}\begin{cases}(\frac{s-a}{b-a})^{\beta-1}(b-\tau)^{\beta-1} - (s-\tau)^{\beta-1}, & a \leq \tau \leq s \leq b,\\ (\frac{s-a}{b-a})^{\beta-1}(b-\tau)^{\beta-1}, & a \leq s \leq \tau \leq b.\end{cases} \tag{3.3}$$

Proof Set $\Phi_p({}^cD^\alpha_{a^+}u(t)) = v(t)$. Then BVP (1.12) can be turned into the following coupled boundary value problems:

$$\begin{cases}D^\beta_{a^+}v(t) = k(t)f(u(t)), & a < t < b,\\ v(a) = v(b) = 0,\end{cases} \tag{3.4}$$

and

$$\begin{cases}{}^cD^\alpha_{a^+}u(t) = \Phi_q(v(t)), & a < t < b,\\ u'(a) = u(b) = 0.\end{cases} \tag{3.5}$$

From Lemma 2 of [7], we see that BVP (3.4) has a unique solution, which is given by

$$v(t) = -\int_a^b H(t,s)k(s)f\big(u(s)\big)\,\mathrm{d}s, \tag{3.6}$$

where $H(t,s)$ is as in (3.3). Moreover, by Lemma 5 of [8], we see that BVP (3.5) has a unique solution, which is given by

$$u(t) = -\int_a^b G(t,s)\Phi_q\big(v(s)\big)\,\mathrm{d}s, \tag{3.7}$$

where $G(t,s)$ is as in (3.2). Substitute (3.6) into (3.7), we see that BVP (1.12) has a unique solution which is given by (3.1). □

Lemma 3.2 *The Green's function H defined by* (3.3) *satisfies the following properties*:

(1) $H(t,s) \geq 0$ *for all* $a \leq t,s \leq b$;
(2) $\max_{t\in[a,b]} H(t,s) = H(s,s), s \in [a,b]$;
(3) $H(s,s)$ *has a unique maximum given by*

$$\max_{s\in[a,b]} H(s,s) = \frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)};$$

(4) $\min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} H(t,s) \geq \sigma(s)H(s,s), a < s < b$,
where

$$\begin{aligned} &\sigma(s) = \begin{cases} \frac{(\frac{3(b-a)(b-s)}{4})^{\beta-1}-(b-a)^{\beta-1}(\frac{a+3b}{4}-s)^{\beta-1}}{(s-a)^{\beta-1}(b-s)^{\beta-1}} & \text{if } s \in (a, c_\beta], \\ (\frac{b-a}{4(s-a)})^{\beta-1} & \text{if } s \in [c_\beta, b), \end{cases} \\ &c_\beta := \frac{\frac{a+3b}{4} - bA_\beta}{1-A_\beta}, \qquad A_\beta = \left(\left(\frac{3}{4}\right)^{\beta-1} - \left(\frac{1}{4}\right)^{\beta-1}\right)^{\frac{1}{\beta-1}}. \end{aligned} \tag{3.8}$$

Proof The first three properties are proved in [7]. For convenience, we set

$$h_1(t,s) = \frac{1}{\Gamma(\beta)}\left(\left(\frac{t-a}{b-a}\right)^{\beta-1}(b-s)^{\beta-1} - (t-s)^{\beta-1}\right), \quad s \leq t$$

and

$$h_2(t,s) = \frac{1}{\Gamma(\beta)}\left(\frac{t-a}{b-a}\right)^{\beta-1}(b-s)^{\beta-1}, \quad t \leq s.$$

From [7], we know that $h_1(t,s)$ is decreasing with respect to t for $s \leq t$, and $h_2(t,s)$ is increasing with respect to t for $t \leq s$. Thus

$$\min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} H(t,s) = \begin{cases} h_1(\frac{a+3b}{4},s) & \text{if } s \in (a, \frac{3a+b}{4}], \\ \min\{h_1(\frac{a+3b}{4},s), h_2(\frac{3a+b}{4},s)\} & \text{if } s \in [\frac{3a+b}{4}, \frac{a+3b}{4}], \\ h_2(\frac{3a+b}{4},s) & \text{if } s \in [\frac{a+3b}{4}, b). \end{cases}$$

From

$$h_1\left(\frac{a+3b}{4},s\right)=h_2\left(\frac{3a+b}{4},s\right)$$

we have

$$\left(\frac{\frac{a+3b}{4}-s}{b-s}\right)^{\beta-1}=\left(\frac{3}{4}\right)^{\beta-1}-\left(\frac{1}{4}\right)^{\beta-1},$$

which implies that

$$s=\frac{\frac{a+3b}{4}-bA_\beta}{1-A_\beta}=c_\beta,$$

where c_β and A_β are as in (3.8). It is easy to check that $A_\beta<\frac{3}{4}$ and $c_\beta<\frac{a+3b}{4}$. On the other hand, since

$$3^{\beta-1}+8^{\beta-1}\geq 2\sqrt{3^{\beta-1}8^{\beta-1}}\geq 4^{\frac{\beta-1}{2}}3^{\frac{\beta-1}{2}}8^{\frac{\beta-1}{2}}=96^{\frac{\beta-1}{2}}>9^{\beta-1},$$

we have

$$\left(\frac{3}{4}\right)^{\beta-1}<\left(\frac{2}{3}\right)^{\beta-1}+\left(\frac{1}{4}\right)^{\beta-1},$$

from which we deduce that $A_\beta<\frac{2}{3}$ and $c_\beta>\frac{3a+b}{4}$. So $c_\beta\in(\frac{3a+b}{4},\frac{a+3b}{4})$ is the unique solution of the equation $h_1(\frac{a+3b}{4},s)=h_2(\frac{3a+b}{4},s)$. Hence

$$\begin{aligned}\min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]}H(t,s)&=\begin{cases}h_1(\frac{a+3b}{4},s) & \text{if } s\in(a,c_\beta],\\ h_2(\frac{3a+b}{4},s) & \text{if } s\in[c_\beta,b)\end{cases}\\ &=\frac{1}{\Gamma(\beta)}\begin{cases}(\frac{3(b-s)}{4})^{\beta-1}-(\frac{a+3b}{4}-s)^{\beta-1} & \text{if } s\in(a,c_\beta],\\ (\frac{b-s}{4})^{\beta-1} & \text{if } s\in[c_\beta,b)\end{cases}\\ &\geq\sigma(s)H(s,s).\end{aligned}$$

□

Remark 3.1 Since $\frac{2a+b}{3}<\frac{2b-a}{3}$ implies $3a<b$, we see that the conclusion of Lemma 7(4) in [13] only holds for $a<\frac{b}{3}$.

Lemma 3.3 ([8]) *The Green's function G defined by* (3.2) *satisfies the following properties*:

(i) $0\leq G(t,s)\leq G(s,s)=\frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-1}$ *for all* $a\leq t,s\leq b$;

(ii) $G(s,s)$ *has a unique maximum given by*

$$\max_{s\in[a,b]}G(s,s)=\frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1};$$

(iii) $\min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]}G(t,s)\geq\mu(s)G(s,s), a<s<b$, *where*

$$\mu(s)=\begin{cases}1-(\frac{\frac{a+3b}{4}-s}{b-s})^{\alpha-1} & \textit{if } s\in(a,\frac{a+3b}{4}],\\ 1 & \textit{if } s\in[\frac{a+3b}{4},b).\end{cases}$$

Let $E = C[a,b]$ be endowed with the norm $\|x\| = \max_{t\in[a,b]} |x(t)|$. Define the cone $P \subset E$ by

$$P = \big\{x \in E | x(t) \geq 0 \ \forall t \in [a,b] \text{ and } \|x\| \neq 0\big\}.$$

Theorem 3.4 *Let $k : [a,b] \to \mathbb{R}_+ = [0,+\infty)$ be a nontrivial Lebesgue integrable function. Suppose that there exist two positive constants $r_2 > r_1 > 0$ such that the following assumptions:*

(H1) *$f(x) \geq \rho\Phi_p(r_1)$ for $x \in [0,r_1]$,*

(H2) *$f(x) \leq \omega\Phi_p(r_2)$ for $x \in [0,r_2]$,*

are satisfied, where

$$\rho = \left[\int_a^b \sigma(\tau)H(\tau,\tau)k(\tau)\,\mathrm{d}\tau \times \Phi_p\left(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \mu(s)G(s,s)\,\mathrm{d}s\right)\right]^{-1}$$

and

$$\omega = \left[\int_a^b H(\tau,\tau)k(\tau)\,\mathrm{d}\tau \times \Phi_p\left(\int_a^b G(s,s)\,\mathrm{d}s\right)\right]^{-1}.$$

Then FBVP (1.12) *has at least one nontrivial positive solution u belonging to E such that $r_1 \leq \|u\| \leq r_2$.*

Proof Let $T : P \to E$ be the operator defined by

$$Tu(t) = \int_a^b G(t,s)\Phi_q\left(\int_a^b H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

By using the Arzela-Ascoli theorem, we can prove that $T : P \to P$ is completely continuous. Let $\Omega_i = \{u \in P : \|u\| \leq r_i\}$, $i = 1,2$. From (H1), and Lemmas 3.2 and 3.3, we obtain for $t \in [\frac{3a+b}{4}, \frac{a+3b}{4}]$ and $u \in P \cap \partial\Omega_1$

$$\begin{aligned}
(Tu)(t) &\geq \int_a^b \min_{t\in[\frac{3a+b}{4},\frac{a+3b}{4}]} G(t,s)\Phi_q\left(\int_a^b H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \int_a^b \mu(s)G(s,s)\Phi_q\left(\int_a^b H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \mu(s)G(s,s)\,\mathrm{d}s \cdot \Phi_q\left(\int_a^b \min_{s\in[\frac{3a+b}{4},\frac{a+3b}{4}]} H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right) \\
&\geq \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \mu(s)G(s,s)\,\mathrm{d}s \cdot \Phi_q\left(\int_a^b \sigma(\tau)H(\tau,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right) \\
&\geq \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \mu(s)G(s,s)\,\mathrm{d}s \cdot \Phi_q\left(\int_a^b \sigma(\tau)H(\tau,\tau)k(\tau)\,\mathrm{d}\tau\right)\Phi_q(\rho)\cdot r_1 \\
&= \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$. On the other hand, from (H2), Lemmas 3.2 and 3.3, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[a,b]} \int_a^b G(t,s)\Phi_q\left(\int_a^b H(s,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leq \int_a^b G(s,s)\,\mathrm{d}s\cdot\Phi_q\left(\int_a^b H(\tau,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right) \\ &\leq \int_a^b G(s,s)\,\mathrm{d}s\cdot\Phi_q\left(\int_a^b H(\tau,\tau)k(\tau)\,\mathrm{d}\tau\right)\Phi_q(\omega)r_2 = \|u\|\end{aligned}$$

for $u \in P \cap \partial\Omega_2$. Thus, by Lemma 2.1, we see that the operator T has a fixed point in $u \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution for FBVP (1.12).

Next, we will give two Lyapunov inequalities for FBVP (1.12). □

Theorem 3.5 *Let $k: [a,b] \to \mathbb{R}_+$ be a real nontrivial Lebesgue function. Suppose that there exists a positive constant M satisfying $0 \leq f(x) \leq M\Phi_p(x)$ for any $x \in \mathbb{R}_+$. If* (1.12) *has a nontrivial solution in P, then the following Lyapunov inequality holds*:

$$\int_a^b k(\tau)\,\mathrm{d}\tau > \frac{4^{\beta-1}\Gamma(\beta)}{M(b-a)^{\beta-1}}\Phi_p\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\right).$$

Proof Assume $u \in P$ is a nontrivial solution for (1.12), then $\|u\| \neq 0$. From (3.1), and Lemmas 3.2 and 3.3, $\forall t \in [a,b]$, we have

$$\begin{aligned}0 \leq u(t) &\leq \int_a^b G(s,s)\Phi_q\left(\int_a^b H(\tau,\tau)k(\tau)f\big(u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &< \int_a^b G(s,s)\,\mathrm{d}s\cdot\Phi_q\left(\int_a^b H(\tau,\tau)k(\tau)\,\mathrm{d}\tau\right)\Phi_q(M)\|u\| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\,\mathrm{d}s\cdot\Phi_q\left(\int_a^b \frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}k(\tau)\,\mathrm{d}\tau\right)\Phi_q(M)\|u\| \\ &= \frac{1}{\Gamma(\alpha+1)}(b-a)^\alpha\cdot\Phi_q\left(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\right)\Phi_q\left(\int_a^b k(\tau)\,\mathrm{d}\tau\right)\Phi_q(M)\|u\|,\end{aligned}$$

which implies that

$$\int_a^b k(\tau)\,\mathrm{d}\tau > \frac{4^{\beta-1}\Gamma(\beta)}{M(b-a)^{\beta-1}}\Phi_p\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\right).$$ □

Theorem 3.6 *Let $k: [a,b] \to \mathbb{R}_+$ be a real nontrivial Lebesgue function. Assume that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a concave and nondecreasing function. If* (1.12) *has a nontrivial solution $u \in P$, then*

$$\int_a^b k(\tau)\,\mathrm{d}\tau > \frac{4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))\Phi_p(\eta)}{(b-a)^{\alpha p+\beta-\alpha-1}f(\eta)},$$

where $\eta = \max_{t\in[a,b]} u(t)$.

Proof By (3.1), Lemmas 3.2 and 3.3, we get

$$\begin{aligned} u(t) &\leq \int_a^b G(s,s)\Phi_q\biggl(\int_a^b H(\tau,\tau)k(\tau)f\bigl(u(\tau)\bigr)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \\ \|u\| &< \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\,\mathrm{d}s\cdot\Phi_q\biggl(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\biggr)\Phi_q\biggl(\int_a^b k(\tau)f\bigl(u(\tau)\bigr)\,\mathrm{d}\tau\biggr) \\ &= \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\cdot\Phi_q\biggl(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\biggr)\Phi_q\biggl(\int_a^b k(\tau)f\bigl(u(\tau)\bigr)\,\mathrm{d}\tau\biggr). \end{aligned}$$

Using Lemma 2.2, and taking into account that f is concave and nondecreasing, we see that

$$\begin{aligned} \|u\| &< \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\cdot\Phi_q\biggl(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\biggr)\Phi_q\biggl(\int_a^b k(s)\,\mathrm{d}s\biggr)\Phi_q\biggl(\int_a^b \frac{k(\tau)f(u(\tau))\,\mathrm{d}\tau}{\int_a^b k(s)\,\mathrm{d}s}\biggr) \\ &< \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\cdot\Phi_q\biggl(\frac{(b-a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)}\biggr)\Phi_q\biggl(\int_a^b k(s)\,\mathrm{d}s\biggr)\Phi_q\bigl(f(\eta)\bigr), \end{aligned}$$

where $\eta = \max_{t\in[a,b]} u(t)$. Hence,

$$\int_a^b k(s)\,\mathrm{d}s > \frac{4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))\Phi_p(\eta)}{(b-a)^{\alpha p+\beta-\alpha-1}f(\eta)}.$$

The proof is completed. □

4 Applications

In the following, some applications of the obtained results are presented.

Corollary 4.1 *If* $\lambda \in [0, 4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))]$, *then the following eigenvalue problem*:

$$\begin{cases} D_{0^+}^{\beta}(\Phi_p({}^{C}D_{0^+}^{\alpha}y(t))) - \lambda\Phi_p(y(t)) = 0, & 0 < t < 1, \\ y'(0) = {}^{C}D_{0^+}^{\alpha}y(0) = 0, \qquad y(1) = {}^{C}D_{0^+}^{\alpha}y(1) = 0, \end{cases} \tag{4.1}$$

has no corresponding eigenfunction $y \in P$, *where* $1 < \alpha, \beta \leq 2$, *and* $p > 1$.

Proof Assume that $y_0 \in P$ is an eigenfunction of (4.1) corresponding to an eigenvalue $\lambda_0 \in [0, 4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))]$. By using Theorem 3.5 with $a = 0$, $b = 1$, $k(s) = \lambda_0$ and $M = 1$ ($f(y) = \Phi_p(y)$), we get

$$\lambda_0 > 4^{\beta-1}\Gamma(\beta)\Phi_p\bigl(\Gamma(\alpha+1)\bigr),$$

which is a contradiction. □

From Theorems 3.4 and 3.6, we have the following.

Corollary 4.2 *For fractional boundary value problem* (1.12), *let* $k : [a,b] \to \mathbb{R}_+$ *be a nontrivial Lebesgue integrable function, and* $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be a concave and nondecreasing* *function. If there exist two positive constants* $r_2 > r_1 > 0$ *such that the assumptions* (H1) *and* (H2) *hold, then*

$$\int_a^b k(\tau)\,\mathrm{d}\tau > \frac{4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))\Phi_p(r_1)}{(b-a)^{\alpha p+\beta-\alpha-1}f(r_2)}.$$

Example 4.3 Consider the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{3/2}(\Phi_{1.8}({}^CD_{0^+}^{4/3}y)) - \sqrt{t}\ln(15+y) = 0, \quad 0 < t < 1, \\ y'(0) = {}^CD_{0^+}^{4/3}y(0) = 0, \qquad y(1) = {}^CD_{0^+}^{4/3}y(1) = 0. \end{cases}$$

Obviously, we have

(i) $f(y) = \ln(15+y) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, concave and nondecreasing;
(ii) $k(t) = \sqrt{t} : [0,1] \to \mathbb{R}_+$ is a Lebesgue integrable function with $\int_0^1 k(t)\,\mathrm{d}t = \frac{2}{3} > 0$.

We now compute the values of ρ and ω in (H1) and (H2), respectively.

Since $A_{3/2} = ((\frac{3}{4})^{1/2} - (\frac{1}{4})^{1/2})^2 = 1 - \frac{\sqrt{3}}{2}$, we have $c_{3/2} = \frac{\frac{3}{4}-A_{3/2}}{1-A_{3/2}} = 1 - \frac{\sqrt{3}}{6}$. where $A_{3/2}$ and $c_{3/2}$ ($\beta = 3/2$) are as in (3.8). Hence

$$\sigma(s) = \begin{cases} \frac{\sqrt{3}}{2}\frac{(1-s)^{1/2}-1}{(s(1-s))^{1/2}} & \text{if } s \in (0, 1-\frac{\sqrt{3}}{6}], \\ \frac{1}{2s^{1/2}} & \text{if } s \in [1-\frac{\sqrt{3}}{6}, 1). \end{cases}$$

Thus, by a simple computation, we obtain

$$\rho \approx 61.7797, \qquad \omega \approx 3.8213.$$

Choosing $r_1 = 1/50$ and $r_2 = 1$, we obtain

1. $f(y) = \ln(15+y) \ge \rho\Phi_{1.8}(r_1)$ for $y \in [0, 1/50]$;
2. $f(y) = \ln(15+y) \le \omega\Phi_{1.8}(r_2)$ for $y \in [0,1]$.

Hence, from Corollary 4.2, we obtain

$$\int_0^1 k(t)\,\mathrm{d}t > \frac{2\Gamma(3/2)\Phi_{1.8}(\frac{1}{50}\Gamma(7/3))}{\ln 16} \approx 0.0321.$$

5 Conclusions

In this paper, we prove existence of positive solutions to a nonlinear fractional boundary value problem involving a p-Laplacian operator. Then, under some mild assumptions on the nonlinear term, we present two new Lyapunov-type inequalities. A numerical example shows that the new results are efficient.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

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