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# Optimal bounds for arithmetic-geometric and Toader means in terms of generalized logarithmic mean

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**Abstract**
In this paper, we find the greatest values $\alpha_1, \alpha_2$ and the smallest values $\beta_1, \beta_2$ such that the double inequalities $L_{\alpha_1}(a,b) < \mathrm{AG}(a,b) < L_{\beta_1}(a,b)$ and $L_{\alpha_2}(a,b) < T(a,b) < L_{\beta_2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$, where $\mathrm{AG}(a,b)$, $T(a,b)$ and $L_p(a,b)$ are the arithmetic-geometric, Toader and generalized logarithmic means of two positive numbers $a$ and $b$, respectively.

**MSC:** 26E60

**Keywords:** arithmetic-geometric mean; Toader mean; logarithmic mean

## 1 Introduction

For $p \in \mathbb{R}$, the $p$th generalized logarithmic mean $L_p(a,b)$ [1] of two positive numbers $a$ and $b$ is defined by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & a \neq b, p \neq -1, p \neq 0, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/b-a}, & a \neq b, p = 0, \\ \frac{b-a}{\log b - \log a}, & a \neq b, p = -1, \\ a, & a = b. \end{cases} \tag{1.1}$$

It is well known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many remarkable inequalities for the generalized logarithmic mean can be found in the literature [2–17].

The classical arithmetic-geometric mean $\mathrm{AG}(a,b)$ of two positive numbers $a$ and $b$ is defined by starting with $a_0 = a, b_0 = b$ and then iterating

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n} \tag{1.2}$$

for $n \in \mathbb{N}$ until two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same number.

The well-known Gauss identity [18] shows that

$$\mathrm{AG}(1,r)\mathcal{K}\left(\sqrt{1-r^2}\right) = \frac{\pi}{2} \tag{1.3}$$

for $r \in (0,1)$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}\,dt$, $r \in [0,1)$, is the complete elliptic integral of the first kind.

In [19], the Toader mean $T(a,b)$ of two positive numbers $a$ and $b$ was given by

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$
$$= \begin{cases} \frac{2a\mathcal{E}(\sqrt{1-(b/a)^2})}{\pi}, & a > b, \\ \frac{2b\mathcal{E}(\sqrt{1-(a/b)^2})}{\pi}, & a < b, \\ a, & a = b, \end{cases} \tag{1.4}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}\,d\theta$, $r \in [0,1]$ is the complete elliptic integral of the second kind.

Recently, the bounds for the arithmetic-geometric mean $\mathrm{AG}(a,b)$ and Toader mean $T(a,b)$ have attracted the attention of many mathematicians. The double inequality

$$L_{-1}(a,b) = L(a,b) < \mathrm{AG}(a,b) < L^{2/3}\left(a^{3/2}, b^{3/2}\right) \tag{1.5}$$

holds for all $a,b > 0$ with $a \neq b$. The left inequality of (1.5) was first proposed by Carlson and Vuorinen [20] and also was proved by different methods in [21–23]. Vamanamurthy and Vuorinen [24] proved that $\mathrm{AG}(a,b) < (\pi/2)L(a,b)$ for all $a,b > 0$ with $a \neq b$. The second inequality of (1.5) was proved by Borwein and Borwein [25] and Yang [23].

Vuorinen [26] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.6}$$

for all $a,b > 0$ with $a \neq b$, where $M_p(a,b) = [(a^p + b^p)/2]^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the power mean of order $p$. This conjecture was proved by Qiu and Shen [27] and Barnard et al. [28].

In [29], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.7}$$

for all $a,b > 0$ with $a \neq b$.

In [30–32], the authors proved that

$$\widehat{L}_0(a,b) < T(a,b) < \widehat{L}_{1/4}(a,b), \tag{1.8}$$
$$\widehat{S}_{\sqrt{3}/4}(a,b) < T(a,b) < \widehat{S}_{1/2}(a,b) \tag{1.9}$$

for all $a,b > 0$ with $a \neq b$, where $\widehat{L}_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ denotes the $p$th Lehmer mean and $\widehat{S}_p(a,b)$ is the generalized Seiffert mean given by $\widehat{S}_p(a,b) = p(a-b)/\arctan[2p(a-b)/(a+b)]$ $(0 < p \leq 1, a \neq b)$, $\widehat{S}_0(a,b) = (a+b)/2 (a \neq b)$ and $\widehat{S}_p(a,a) = a$.

Very recently, Chu and Wang [33] proved that

$$S_{p_1}(a,b) < \mathrm{AG}(a,b) < S_{q_1}(a,b), \tag{1.10}$$

$$S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b) \tag{1.11}$$

for all $a,b > 0$ with $a \neq b$ if and only if $p_1 \leq 1/2, q_1 \geq 1$ and $p_2 \leq 1, q_2 \geq 3/2$. Here the $p$th Gini mean of two positive numbers $a$ and $b$ is defined by

$$S_p(a,b) = \begin{cases} (\frac{a^{p-1}+b^{p-1}}{a+b})^{1/(p-2)}, & p \neq 2, \\ (a^a b^b)^{1/(a+b)}, & p = 2. \end{cases} \tag{1.12}$$

The main purpose of this paper is to find the greatest values $\alpha_1, \alpha_2$ and the smallest values $\beta_1, \beta_2$ such that the double inequalities $L_{\alpha_1}(a,b) < \mathrm{AG}(a,b) < L_{\beta_1}(a,b)$ and $L_{\alpha_2}(a,b) < T(a,b) < L_{\beta_2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$ and give some new bounds for the complete elliptic integrals.

## 2 Basic knowledge and lemmas

In order to prove our main results, we need several formulas and lemmas, which we present in this section.

For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, the well-known complete elliptic integrals of the first and second kinds are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2}\, d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = +\infty, \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{1/2}\, d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r') \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1, \end{cases}$$

respectively, and the following formulas were presented in [18], Appendix E, pp.474-475:

$$\begin{gathered} \frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}, \\ \frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}, \end{gathered} \tag{2.1}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r}. \tag{2.2}$$

In what follows, four special values $\mathcal{E}(\sqrt{2}/2), \mathcal{K}(\sqrt{2}/2)$ and $\mathcal{E}(0.9), \mathcal{K}(0.9)$ will be used. By numerical computations, these are given by

$$\mathcal{E}(\sqrt{2}/2) = 1.35064\cdots, \qquad \mathcal{K}(\sqrt{2}/2) = 1.85407\cdots, \tag{2.3}$$

$$\mathcal{E}(0.9) = 1.1717\cdots, \qquad \mathcal{K}(0.9) = 2.28055\cdots. \tag{2.4}$$

**Lemma 2.1** (See [18], Theorem 1.25) *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and be differentiable on $(a,b)$, let $g'(x) \neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a,b)$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.2** (1) *The function $r \to (\mathcal{E}-r'^2\mathcal{K})/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;*
(2) *The function $r \to 2\mathcal{E}-r'^2\mathcal{K}$ is increasing and log-convex from $(0,1)$ onto $(\pi/2,2)$;*
(3) *The function $\mathcal{K}/\log(e^2/r')$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;*
(4) *The function $(\mathcal{K}-\mathcal{E})/r^2$ is strictly increasing on $(0,1)$; in particular, $\mathcal{K}-\mathcal{E} > (\pi/4)r^2$ for all $r \in (0,1)$.*

*Proof* Parts (1) and (2) follow from [18], Theorem 3.21(1) and Exercise 3.43(13). □

**Lemma 2.3** *The equation*

$$(1+p)^{1/p} = \frac{\pi}{2}$$

*has a unique solution $p = p_0 = 3.15295\cdots$.*

*Proof* Let

$$\varphi(p) = \begin{cases} (1+p)^{1/p} - \pi/2, & p \in (-1,0) \cup (0,+\infty), \\ e - \pi/2, & p = 0. \end{cases}$$

It is easy to verify that the function $\varphi$ is continuous and strictly decreasing from $(-1,+\infty)$ onto $(1,+\infty)$. Therefore, Lemma 2.3 easily follows from the continuity and monotonicity of $\varphi$ together with the facts that $\varphi(3.15295) = 6.14999 \times 10^{-7}$ and $\varphi(3.15296) = -4.35155 \times 10^{-7}$. □

**Lemma 2.4** *The function*

$$f(r) = \frac{2(2\mathcal{E}-r'^2\mathcal{K})/\pi - 1 - r^2/4}{r^4}$$

*is strictly increasing from $(0,1)$ onto $(1/64, 4/\pi - 5/4)$.*

*Proof* Let $f_1(r) = 2(2\mathcal{E}-r'^2\mathcal{K})/\pi - 1 - r^2/4$ and $\widehat{f_1}(r) = r^4$, then $f_1(0) = \widehat{f_1}(0) = 0$ and $f(r) = f_1(r)/\widehat{f_1}(r)$.

A simple calculation yields

$$\frac{f_1'(r)}{\widehat{f_1}'(r)} = \frac{4(\mathcal{E}-r'^2\mathcal{K}) - \pi r^2}{8\pi r^4} \triangleq \frac{f_2(r)}{\widehat{f_2}(r)}, \tag{2.5}$$

$$f_2(0) = \widehat{f_2}(0), \tag{2.6}$$

$$\frac{f_2'(r)}{\widehat{f}_2'(r)} = \frac{2\mathcal{K} - \pi}{16\pi r^2} \triangleq \frac{f_3(r)}{\widehat{f}_3(r)}, \tag{2.7}$$

$$f_3(0) = \widehat{f}_3(0), \tag{2.8}$$

$$\frac{f_3'(r)}{\widehat{f}_3'(r)} = \frac{1}{16\pi} \cdot \frac{\mathcal{E} - r'^2\mathcal{K}}{r^2} \cdot \frac{1}{r'^2}. \tag{2.9}$$

Following from Lemma 2.2(1) and (2.9) together with the monotonicity of $1/r'^2$, we clearly see that $f_3'(r)/\widehat{f}_3'(r)$ is strictly increasing on $(0,1)$. Equations (2.5)-(2.8) and Lemma 2.1 lead to the conclusion that $f(r)$ is strictly increasing on $(0,1)$.

Therefore, Lemma 2.4 follows from the monotonicity of $f(r)$ together with the facts that $f(0^+) = 1/64$ and $f(1^-) = 4/\pi - 5/4$. □

The following double inequalities can be obtained from Lemma 2.4 immediately.

**Corollary 2.5** *Inequalities*

$$1 + \frac{r^2}{4} + \frac{r^4}{64} < \frac{2}{\pi}\left(2\mathcal{E} - r'^2\mathcal{K}\right) < 1 + \frac{r^2}{4} + \left(\frac{4}{\pi} - \frac{5}{4}\right)r^4$$

*hold for* $0 < r < 1$.

**Lemma 2.6** *The inequality*

$$\left[\frac{(1+r)^{7/2} - (1-r)^{7/2}}{7r}\right]^{2/5} < 1 + \frac{r^2}{4} \tag{2.10}$$

*holds for* $0 < r < 1$.

*Proof* In order to prove inequality (2.10), it suffices to prove that

$$\begin{aligned} g(r) &= \left[(1+r)^{7/2} - (1-r)^{7/2}\right]^2 - 49r^2\left(1 + \frac{r^2}{4}\right)^5 \\ &= (1+r)^7 + (1-r)^7 - 49r^2\left(1 + \frac{r^2}{4}\right)^5 - 2\left(1 - r^2\right)^{7/2} \\ &= g_1(r) - g_2(r) < 0 \end{aligned} \tag{2.11}$$

for $0 < r < 1$, where

$$g_1(r) = (1+r)^7 + (1-r)^7 - 49r^2\left(1 + \frac{r^2}{4}\right)^5,$$

$$g_2(r) = 2\left(1 - r^2\right)^{7/2}.$$

Observe that

$$g_1'(r) = -\frac{7r}{256}\left[32\left(4 - 5r^2\right)^2 + 2{,}848r^4 + 2{,}240r^6 + 350r^8 + 21r^{10}\right] < 0, \tag{2.12}$$

$$g_1(0.56) = 0.0755\cdots > 0, \qquad g_1(0.57) = -0.00966\cdots < 0, \tag{2.13}$$

we conclude, from (2.12) and (2.13), that there exists $r_0 \in (0.56, 0.57)$ such that $g_1(r) > 0$ for $r \in (0, r_0)$ and $g_1(r) < 0$ for $r \in (r_0, 1)$.

In order to prove (2.11), we divide it into two cases.

*Case* A $r \in [r_0, 1)$. In this case, we clearly see that $g_1(r) \leq 0$ and $g_2(r) > 0$. This implies that $g(r) = g_1(r) - g_2(r) < 0$.

*Case* B $r \in (0, r_0)$. In this case, $g_1(r) > 0$. Let $g_3(r) = 2 - 7r^2 + \frac{35}{4}r^4 - 6r^6$, the difference between $g_1(r)$ and $g_3(r)$ yields

$$g_1(r) - g_3(r) = -\frac{r^6(10{,}880 + 7{,}840r^2 + 980r^4 + 49r^6)}{1{,}024} < 0. \tag{2.14}$$

We know from (2.14) that $g_3(r) > g_1(r) > 0$. Moreover,

$$g_3^2(r) - g_2^2(r) = -\frac{r^6}{16}\left[\left(2 - 4r^2\right)\left(4 + r^2\right) + r^2\right]\left[16\left(r^2 - \frac{5}{8}\right) + \frac{27}{4}\right] < 0,$$

this in conjunction with $g_3(r) > 0$ implies that

$$g_3(r) - g_2(r) < 0. \tag{2.15}$$

Therefore, we clearly see that $g(r) = [g_1(r) - g_3(r)] + [g_3(r) - g_2(r)] < 0$ from (2.14) and (2.15). □

**Lemma 2.7** *Let* $\eta(r) = [(1+r)^{p_0+1} - (1-r)^{p_0+1}]/r$ *and* $\omega(r) = [(1-r)^{p_0}(1+p_0r) - (1+r)^{p_0}(1-p_0r)]/r^2$, *then the functions* $\eta(r)$ *and* $\omega(r)$ *both are strictly increasing on* $(0,1)$.

*Proof* We assume that

$$\eta_1(r) = (1+r)^{p_0+1} - (1-r)^{p_0+1}, \qquad \eta_2(r) = r,$$
$$\omega_1(r) = (1-r)^{p_0}(1+p_0r) - (1+r)^{p_0}(1-p_0r), \qquad \omega_2(r) = r^2,$$

then $\eta(r) = \eta_1(r)/\eta_2(r)$ and $\omega(r) = \omega_1(r)/\omega_2(r)$.

A simple calculation yields

$$\eta_1(0) = \eta_2(0) = \omega_1(0) = \omega_2(0) = 0, \tag{2.16}$$

$$\frac{\eta_1'(r)}{\eta_2'(r)} = (1+p_0)\left[(1+r)^{p_0} - (1-r)^{p_0}\right], \tag{2.17}$$

$$\frac{\omega_1'(r)}{\omega_2'(r)} = \frac{p_0(p_0+1)[(1+r)^{p_0-1} - (1-r)^{p_0-1}]}{2}. \tag{2.18}$$

Lemma 2.1 and (2.16)-(2.18) lead to the conclusion that $\eta(r)$ and $\omega(r)$ are strictly increasing on $(0,1)$. □

**Lemma 2.8** *Let*

$$\phi_p(r) = \frac{2}{\pi}\left(2\mathcal{E} - r'^2\mathcal{K}\right) - \left[\frac{(1+r)^{p+1} - (1-r)^{p+1}}{2(p+1)r}\right]^{1/p},$$

*then* $\phi_p(r) > 0$ *for* $0 < r < 1$ *if and only if* $p \leq 5/2$; $\phi_p(r) < 0$ *for* $0 < r < 1$ *if and only if* $p \geq p_0$.

*Proof* It is well known that $L_p(a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, then $\phi_p(r)$ is strictly decreasing with respect to $p \in \mathbb{R}$. In order to prove Lemma 2.8, we divide it into three cases.

*Case* 1 $p = 5/2$.

From Corollary 2.5 and Lemma 2.6, we clearly see that

$$\begin{aligned}\phi_{5/2}(r) &= \frac{2}{\pi}\left(2\mathcal{E} - r'^2\mathcal{K}\right) - \left[\frac{(1+r)^{7/2} - (1-r)^{7/2}}{7r}\right]^{2/5} \\ &> 1 + \frac{r^2}{4} + \frac{r^4}{64} - \left[\frac{(1+r)^{7/2} - (1-r)^{7/2}}{7r}\right]^{2/5} \\ &> \frac{r^4}{64} > 0\end{aligned}$$

for $0 < r < 1$.

*Case* 2 $p = p_0$.

We divide it into two subcases.

*Subcase* A $\phi_{p_0}(r) < 0$ for $r \in (0, 0.9)$.

Since $\phi_p(r)$ is strictly decreasing with respect to $p \in \mathbb{R}$, we clearly see that $\phi_{p_0}(r) < \phi_3(r)$. It suffices to prove that $\phi_3(r) < 0$ for $r \in (0, 0.9)$.

For $r \in (0, \sqrt{2}/2]$, it follows from Corollary 2.5 that

$$\begin{aligned}\phi_3(r) &= \frac{2}{\pi}\left(2\mathcal{E} - r'^2\mathcal{K}\right) - \left[\frac{(1+r)^4 - (1-r)^4}{8r}\right]^{1/3} \\ &< 1 + \frac{r^2}{4} + \left(\frac{4}{\pi} - \frac{5}{4}\right)r^4 - \left(1 + \frac{r^2}{3} - \frac{r^4}{9}\right)^3 \\ &= -\frac{r^2}{12}\left[1 - \left(\frac{48}{\pi} - \frac{41}{3}\right)r^2\right] \\ &\leq -\frac{r^2}{12}\left[1 - \frac{1}{2}\left(\frac{48}{\pi} - \frac{41}{3}\right)\right] \\ &= -\frac{(47\pi - 144)r^2}{72\pi} < 0,\end{aligned}$$

where the first inequality easily follows from

$$\frac{(1+r)^4 - (1-r)^4}{8r} - \left(1 + \frac{r^2}{3} - \frac{r^4}{9}\right)^3 = \frac{r^6}{729}\left[126 + 9\left(1 - r^4\right) + r^6\right] > 0.$$

For $r \in (\sqrt{2}/2, 0.9)$, taking the derivative of $\phi_3(r)$ yields

$$\phi_3'(r) = \frac{2(\mathcal{E} - r'^2\mathcal{K})}{\pi r} - \frac{2r}{3(1+r^2)^{2/3}} = \mu_1(r) + \mu_2(r), \tag{2.19}$$

where

$$\mu_1(r) = \frac{2(\mathcal{E} - r'^2\mathcal{K})}{\pi r} - \frac{r}{2}, \qquad \mu_2(r) = \frac{r}{2} - \frac{2r}{3(1+r^2)^{2/3}}.$$

From Lemma 2.2(4), we clearly see that

$$\frac{d\mu_1(r)}{dr} = \frac{2}{\pi r^2}\left(\mathcal{K} - \mathcal{E} - \frac{\pi}{4}r^2\right) > 0 \tag{2.20}$$

for $r \in (0,1)$ and

$$\begin{aligned}\frac{d\mu_2(r)}{dr} &= \frac{9(1+r^2)^{5/3} - 12 + 4r^2}{18(1+r^2)^{5/3}} > \frac{9[1+(5/3)r^2] - 12 + 4r^2}{18(1+r^2)^{5/3}} \\ &= \frac{19r^2 - 3}{18(1+r^2)^{5/3}} > 0 \end{aligned}\tag{2.21}$$

for $r \in (\sqrt{2}/2, 0.9)$. Equations (2.19)-(2.21) lead to the conclusion that $\phi_3'(r)$ is strictly increasing on $(\sqrt{2}/2, 0.9)$. This in conjunction with (2.3) implies that

$$\phi_3'(r) > \phi_3'(\sqrt{2}/2) = 0.02163\cdots > 0 \tag{2.22}$$

for $r \in (\sqrt{2}/2, 9/10)$. Therefore, from (2.22) we clearly see that $\phi_3(r)$ is strictly increasing on $(\sqrt{2}/2, 0.9)$. This in conjunction with (2.4) yields $\phi_3(r) < \phi_3(0.9) = -0.002687\cdots < 0$ for $r \in (\sqrt{2}/2, 0.9)$.

*Subcase* B $\phi_{p_0}(r) < 0$ for $r \in [0.9, 1)$.

For $0.9 \leq r < 1$, taking the derivation of $\phi_{p_0}(r)$ yields

$$\phi_{p_0}'(r) = \frac{2(\mathcal{E} - r'^2\mathcal{K})}{\pi r} - \frac{\omega(r)}{p_0(p_0+1)2^{1/p_0}\eta(r)^{1-1/p_0}}, \tag{2.23}$$

where $\omega(r)$ and $\eta(r)$ are defined as in Lemma 2.7. From Lemma 2.2(1), we clearly see that $(\mathcal{E} - r'^2\mathcal{K})/r$ is strictly increasing on $(0,1)$. Lemma 2.7 and (2.4), (2.20) lead to the conclusion that

$$\begin{aligned}\phi_{p_0}'(r) &\geq \frac{2[\mathcal{E}(0.9) - (1-0.9^2)\mathcal{K}(0.9)]}{0.9\pi} - \frac{\omega(1)}{p_0(p_0+1)2^{1/p_0}\eta(0.9)^{1-1/p_0}} \\ &= 0.522306\cdots - 0.46787\cdots = 0.054436\cdots > 0\end{aligned}$$

for $0.9 \leq r < 1$.

Therefore, it follows from the monotonicity of $\phi_{p_0}'(r)$ on $(9/10, 1)$ that $\phi_{p_0}(r) < \phi_{p_0}(1) = 2[2/\pi - 1/(1+p_0)^{1/p_0}] = 0$ for $0 < r < 1$.

*Case* 3 $5/2 < p < p_0$.

Taking the Taylor series of $\phi_p(r)$ at $r = 0$ yields

$$\phi_p(r) = \left(\frac{5}{12} - \frac{p}{6}\right)r^2 + \frac{(149 - 144p + 24p^2 + 16p^3)r^4}{2{,}880} + o(r^4). \tag{2.24}$$

From (2.24) we clearly see that there exists a sufficiently small $\delta_1 > 0$ such that $\phi_p(r) < 0$ for $r \in (0, \delta_1)$ if $p > 5/2$. If $p < p_0$, then $\phi_p(1) = 2[2/\pi - 1/(1+p)^{1/p}] > 0$. By the continuity of $\phi_p(r)$ with respect to $r$, there exists a sufficiently small $\delta_2 > 0$ such that $\phi_p(r) > 0$ for $r \in (\delta_2, 1)$. □

## 3 Main results

**Theorem 3.1** *Inequality $L_{-1}(a,b) < \mathrm{AG}(a,b) < L_{-1/2}(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $L_{-1}(a,b)$ and $L_{-1/2}(a,b)$ are the best possible lower and upper generalized logarithmic mean bounds for the arithmetic-geometric mean $\mathrm{AG}(a,b)$, respectively.*

*Proof* Firstly, from (1.5) we clearly see that $L_{-1}(a,b) < \mathrm{AG}(a,b)$ for all $a,b>0$ with $a\neq b$.

Next, we prove that $\mathrm{AG}(a,b) < L_{-1/2}(a,b)$ for all $a,b>0$ with $a\neq b$. Since $\mathrm{AG}(a,b)$ and $L_p(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, it suffices to give an assumption that $a=1>b$. Let $t=b\in(0,1)$, $r=(1-t)/(1+t)$, then (1.1) and (1.3) lead to

$$\begin{aligned}\mathrm{AG}(a,b)-L_{-1/2}(a,b) &= \frac{\pi}{2\mathcal{K}(\sqrt{1-t^2})}-\left[\frac{1-t}{2(1-\sqrt{t})}\right]^2\\ &= \frac{1}{1+r}\left[\frac{\pi}{2\mathcal{K}}-\left(\frac{r}{\sqrt{1+r}-\sqrt{1-r}}\right)^2\right]\\ &= \frac{h(r)}{2(1+r)\mathcal{K}(r)},\end{aligned}\tag{3.1}$$

where

$$h(r)=\pi-\left(1+r'\right)\mathcal{K}(r).$$

We can rewrite $h(r)$ as

$$h(r)=\pi-\lambda\left(r'\right)\cdot\frac{\mathcal{K}}{\log(e^2/r'2)},\tag{3.2}$$

where $\lambda(r')=(1+r')\log(e^2/r')$.

A simple calculation yields

$$\lambda'\left(r'\right)=1-\frac{1}{r'}-\log r',\tag{3.3}$$

$$\lambda'(1)=0,\tag{3.4}$$

$$\lambda''\left(r'\right)=\frac{1-r'}{r'}>0.\tag{3.5}$$

Equations (3.3)-(3.5) lead to the conclusion that $\lambda(r')$ is strictly decreasing on $(0,1)$ with respect to $r'$. Moreover, the function $r'=\sqrt{1-r^2}$ is strictly decreasing on $(0,1)$. Hence the function $\lambda(r')$ is strictly increasing on $(0,1)$ with respect to $r$. It follows from (3.2) and Lemma 2.2(3) that $h(r)$ is strictly decreasing on $(0,1)$. This implies that $h(r)<0$ for $0<r<1$ together with $h(0)=0$.

Therefore, $\mathrm{AG}(a,b) < L_{-1/2}(a,b)$ for all $a,b>0$ with $a\neq b$ follows from (3.1) and $h(r)<0$.

Finally, we prove that $L_{-1}(a,b)$ and $L_{-1/2}(a,b)$ are the best possible lower and upper generalized logarithmic mean bounds for the arithmetic-geometric mean $\mathrm{AG}(a,b)$.

For any $0<\varepsilon<1/2$ and $0<x<1$, it follows from (1.1) and (1.3) that

$$\begin{aligned}\lim_{x\to0}\left[\mathrm{AG}(1,x)-L_{-1+\varepsilon}(1,x)\right] &= \lim_{x\to0}\left\{\frac{\pi}{2\mathcal{K}(\sqrt{1-x^2})}-\left[\frac{1-x^{\varepsilon}}{\varepsilon(1-x)}\right]^{\frac{1}{\varepsilon-1}}\right\}\\ &= -\varepsilon^{\frac{1}{1-\varepsilon}}\end{aligned}\tag{3.6}$$

and making use of the Taylor expansion as $x \to 0$, one has

$$\begin{aligned}
&\mathrm{AG}(1,1-x) - L_{-1/2-\varepsilon}(1,1-x) \\
&= \frac{\pi}{2\mathcal{K}(\sqrt{2x-x^2})} - \left[\frac{1-(1-x)^{1/2-\varepsilon}}{(1/2-\varepsilon)x}\right]^{-\frac{2}{1+2\varepsilon}} \\
&= \left[1 - \frac{x}{2} - \frac{x^2}{16} + o(x^2)\right] - \left[1 - \frac{x}{2} - \frac{3+2\varepsilon}{48}x^2 + o(x^2)\right] \\
&= \frac{\varepsilon}{24}x^2 + o(x^2).
\end{aligned} \tag{3.7}$$

Equations (3.6) and (3.7) imply that for any $0 < \varepsilon < 1/2$ there exist $\delta_1 = \delta_1(\varepsilon) \in (0,1)$ and $\delta_2 = \delta_2(\varepsilon) \in (0,1)$ such that $\mathrm{AG}(1,x) < L_{-1+\varepsilon}(1,x)$ for $x \in (0,\delta_1)$ and $\mathrm{AG}(1,1-x) > L_{-1/2-\varepsilon}(1,1-x)$ for $x \in (0,\delta_2)$. □

**Theorem 3.2** *Inequality $L_{5/2}(a,b) < T(a,b) < L_{p_0}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $p_0$ is defined as in Lemma 2.3 and $L_{5/2}(a,b), L_{p_0}(a,b)$ are the best possible lower and upper generalized logarithmic mean bounds for the Toader mean $T(a,b)$, respectively.*

*Proof* From (1.1) and (1.4) we clearly see that both $T(a,b)$ and $L_p(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b \in (0,1)$, $r = (1-t)/(1+t)$, then from (1.1) and (1.4) together with (2.2) we have

$$\begin{aligned}
T(a,b) - L_p(a,b) &= \frac{2}{\pi}\mathcal{E}\left(\sqrt{1-t^2}\right) - \left[\frac{1-t^{p+1}}{(p+1)(1-t)}\right]^{1/p} \\
&= \frac{2}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - \frac{1}{1+r}\left[\frac{(1+r)^{p+1}-(1-r)^{p+1}}{2(p+1)r}\right]^{1/p} \\
&= \frac{1}{1+r}\left[\frac{2}{\pi}\left(2\mathcal{E} - r'^2\mathcal{K}\right) - \left(\frac{(1+r)^{p+1}-(1-r)^{p+1}}{2(p+1)r}\right)^{1/p}\right] \\
&= \frac{\phi_p(r)}{(1+r)},
\end{aligned} \tag{3.8}$$

where $\phi_p(r)$ is defined as in Lemma 2.8.

Therefore, Theorem 3.2 follows from (3.8) and Lemma 2.8. □

## 4 Corollaries and remarks

From Theorem 3.1 we get a lower bound for the complete elliptic integral of the first kind $\mathcal{K}(r)$ as follows.

**Corollary 4.1** *Inequality*

$$\mathcal{K}(r) > \frac{2\pi[1+\sqrt{1-r^2}-2(1-r^2)^{1/4}]}{(1-\sqrt{1-r^2})^2} \tag{4.1}$$

*holds for all $r \in (0,1)$.*

**Remark 4.1** We define $H(r) = 2\pi[1+\sqrt{1-r^2}-2(1-r^2)^{1/4}]/(1-\sqrt{1-r^2})^2$. Computational and numerical experiments show that the lower bound in (4.1) can be regarded as an approximation of $\mathcal{K}(r)$ for some $r \in (0,1)$, refer to Table 1 for numerical values.

**Table 1 Comparison of $\mathcal{K}(r)$ with $H(r)$ for some $r \in (0,1)$**

| $r$ | $\mathcal{K}(r)$ | $H(r)$ |
|---|---|---|
| 0.1 | $1.5747455615\cdots$ | $1.5747455614\cdots$ |
| 0.2 | $1.5868678474\cdots$ | $1.5868678471\cdots$ |
| 0.3 | $1.608048619\cdots$ | $1.608048612\cdots$ |
| 0.4 | $1.63999986\cdots$ | $1.63999977\cdots$ |
| 0.5 | $1.68575035\cdots$ | $1.68574965\cdots$ |
| 0.6 | $1.75075380\cdots$ | $1.75074958\cdots$ |
| 0.7 | $1.84569400\cdots$ | $1.84567106\cdots$ |
| 0.8 | $1.99530278\cdots$ | $1.99517293\cdots$ |

Theorem 3.2 enables us to give new bounds for the complete elliptic integrals of the second kind $\mathcal{E}(r)$.

**Corollary 4.2** *Inequality*

$$\frac{\pi}{2}\left[\frac{2(1-(1-r^2)^{7/4})}{7(1-\sqrt{1-r^2})}\right]^{2/5} < \mathcal{E}(r) < \frac{\pi}{2}\left[\frac{1-(1-r^2)^{(p_0+1)/2}}{(p_0+1)(1-\sqrt{1-r^2})}\right]^{1/p_0} \tag{4.2}$$

*holds for all* $r \in (0,1)$, *where* $p_0 = 3.15295\cdots$ *is defined as in Lemma* 2.3.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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