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Precise large deviations of aggregate claims in a size-dependent renewal risk model with stopping time claim-number process

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Abstract

In this paper, we consider a size-dependent renewal risk model with stopping time claim-number process. In this model, we do not make any assumption on the dependence structure of claim sizes and inter-arrival times. We study large deviations of the aggregate amount of claims. For the subexponential heavy-tailed case, we obtain a precise large-deviation formula; our method substantially relies on a martingale for the structure of our models.

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Keywords: aggregate claims; stopping time; dependent renewal risk model; precise large deviations

1 Introduction

Consider the following renewal risk model. Let $\{X_k, k \in \mathbb{N}\}$ and $\{\theta_k, k \in \mathbb{N}\}$ be claim sizes and inter-arrival times, respectively. Assume that (X_k, θ_k), $k \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) copies of a generic random pair (X, θ) with marginal distribution functions $F = 1 - \overline{F}$ on $[0, \infty)$ and G on $[0, \infty)$, with dependent components X and θ. The claim arrival times are $\tau_k = \sum_{i=1}^{k} \theta_i, k \in \mathbb{N}$, with $\tau_0 = 0$. The number of claims is defined by

$$N_t^* = \inf_k \{k \in \mathbb{N} : \tau_k \geq t\}, \tag{1.1}$$

then N_t^* is a stopping time. In this way, the aggregate amount of claims over the $[0, t]$ is of the form

$$S_t^* = \sum_{k=1}^{N_k^*} X_k, \qquad S_0^* = X_0 = 0, \quad t \geq 0. \tag{1.2}$$

Note that $N_t^* = \sup_k \{k \in \mathbb{N} : \tau_k \leq t\} \triangleq N_t$ if $\tau_k = t$, whereas $N_t^* = N_t + 1$ if $\tau_k \neq t$.

We study large deviations of S_t^* in (1.2). We only consider the case of heavy-tailed claim-size distributions. One of the most important classes of heavy-tailed distributions is the class $\mathcal{S}$ of subexponential distributions. By definition, a distribution F on $[0,\infty)$ is subexponential if $\overline{F}(x) = 1 - F(x)$ for all $x \geq 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \tag{1.3}$$

holds for all $n \geq 2$, where F^{*n} denotes the n-fold convolution of F. Clearly, (1.3) implies

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 + \cdots + X_n > x)}{P(\max\{X_1,\ldots,X_n\} > x)} = 1, \tag{1.4}$$

where $X_1, X_2, \ldots$ is a sequence of i.i.d. r.v.'s with the common distribution function (d.f.) F. See Embrechts *et al.* [1] for a nice review of subexponential distributions in the context of insurance and finance.

Our main result is given now.

Theorem 1.1 *Consider the aggregate amount of claims* S_t^* *in* (1.2), *assume that* $F \in \mathcal{S}$, $E[X] = \mu \in (0,\infty)$ *and* $E[\theta] = 1/\lambda \in (0,\infty)$. *Then, for arbitrarily given* $\gamma > 0$, *we have uniformly for all* $x \geq \gamma t$

$$P\bigl(S_t^* - \mu\lambda t > x\bigr) \sim \lambda t\overline{F}(x), \quad t \to \infty. \tag{1.5}$$

A non-standard renewal risk model with dependent components X and θ, which was firstly proposed by Albrecher and Teugels [2] and further studied by Boudreault *et al.* [3], Cossette *et al.* [4], Badescu *et al.* [5], and among many others. Recently, Asimit and Badescu [6] introduced a general dependence structure for (X,θ), via the conditional tail probability of X given θ; see also Li *et al.* [7]. In particular, Chen and Yuen [8] considered that there is a random variable $\tilde{\theta}$ such that

$$\Pr(\theta > t | X > x) \leq \Pr(\tilde{\theta} > t) \tag{1.6}$$

holds for $t \geq 0$ and large enough x, they studied the large deviations of the aggregate amount of $\mathcal{C}$ heavy-tailed claims, where $\mathcal{C} \subset \mathcal{S}$ (see Embrechts *et al.* [1]).

We now comment on the approaches used in this work. First, in Theorem 1.1, we do not make an assumption on the dependence structure of (X,θ). The existing results usually require a conditional tail probability of X given θ, *e.g.*, Chen and Yuen [8] made the assumption (1.6), to say the least. Second, we extend the asymptotic behavior of the large deviations of S_t^* to the case of $\mathcal{S}$ heavy-tailed claims. Finally, we construct a martingale to prove our result.

The rest of the paper is organized as follows. Section 2 recalls various preliminaries and prepares a few lemmas. Section 3 presents the proof of the main result. We end the paper with conclusions in Section 4.

2 Preliminaries

Throughout this paper, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \geq 1$, and write $a(x) \sim b(x)$ if both. Very often we equip limit relationships with certain uniformity, which is crucial for our purpose. For instance, for two positive bivariate functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we say that $a(\cdot,\cdot) \sim b(\cdot,\cdot)$ holds uniformly for $x \in \Delta \neq \emptyset$ if

$$\lim_{t\to\infty} \sup_{x\in\Delta} \left| \frac{a(x;t)}{b(x;t)} - 1 \right| = 0.$$

Clearly, the asymptotic relation $a(\cdot,\cdot) \sim b(\cdot,\cdot)$ holds uniformly for $x \in \Delta$ if and only if

$$\limsup_{t\to\infty} \sup_{x\in\Delta} \frac{a(x;t)}{b(x;t)} \le 1 \quad \text{and} \quad \liminf_{t\to\infty} \inf_{x\in\Delta} \frac{a(x;t)}{b(x;t)} \ge 1.$$

To obtain our desired results, we need to mention the following useful lemma.

Lemma 2.1 *Consider the renewal counting process* N_t^* *in* (1.1). *Under the assumption* $E[\theta] = 1/\lambda < \infty$. *Then, for any* $p \ge 1$, *we have*

$$E\big[N_t^*\big]^p \sim (\lambda t)^p, \quad \text{as } t \to \infty. \tag{2.1}$$

Proof First, for arbitrarily $\varepsilon > 0$, by definition of N^* and $E[\theta] = 1/\lambda$, we have

$$\begin{aligned} \mathbb{P}\big\{N_t^* > (\lambda+\varepsilon)t\big\} &= \mathbb{P}\left\{\sum_{j=1}^{(\lambda+\varepsilon)t} \theta_j \le t\right\} \\ &= \mathbb{P}\left\{\frac{1}{(\lambda+\varepsilon)t} \sum_{j=1}^{(\lambda+\varepsilon)t} \theta_j \le \frac{1}{\lambda+\varepsilon}\right\} \\ &\to 0. \end{aligned} \tag{2.2}$$

Similarly,

$$\mathbb{P}\big\{N_t^* < (\lambda-\varepsilon)t\big\} \to 0. \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\frac{N_t^*}{t} \xrightarrow{P} \lambda. \tag{2.4}$$

It remains to show

$$\sup_{t\ge1} \mathbb{E}\left(\frac{N_t^*}{t}\right)^p < \infty, \quad \text{for } \forall p \ge 1.$$

Indeed, for any $\delta > 0$,

$$\begin{aligned} \mathbb{E}\left(\frac{N_t^*}{t}\right)^p &= p\int_0^\infty u^{p-1}\mathbb{P}\big\{N_t^* > ut\big\}\,du \\ &= p(\lambda+\delta)^p \int_0^\infty u^{p-1}\mathbb{P}\big\{N_t^* > (\lambda+\delta)ut\big\}\,du \\ &\le p(\lambda+\delta)^p \int_0^\infty u^{p-1}\mathbb{P}\left\{\sum_{j=1}^{(\lambda+\delta)ut} \theta_j \le t\right\} du. \end{aligned} \tag{2.5}$$

For the case when $u \geq 1$ is an integer

$$\mathbb{P}\left\{\sum_{j=1}^{(\lambda+\delta)ut} \theta_j \leq t\right\} \leq \mathbb{P}\left\{\bigcap_{k=1}^{u}\left\{\sum_{j=(k-1)(\lambda+\delta)t+1}^{k(\lambda+\delta)t} \theta_j \leq t\right\}\right\} = \left(\mathbb{P}\left\{\sum_{j=1}^{(\lambda+\delta)t} \theta_j \leq t\right\}\right)^{u}.$$

By the law of large numbers

$$\mathbb{P}\left\{\sum_{j=1}^{(\lambda+\delta)t} \theta_j \leq t\right\} \to 0 \quad (t \to \infty).$$

So we have the bound, for any constant $c > 0$,

$$\mathbb{P}\left\{\sum_{j=1}^{(\lambda+\delta)ut} \theta_j \leq t\right\} \leq e^{-cu}. \tag{2.6}$$

Combining (2.5) and (2.6), uniformly for large u and t,

$$\mathbb{E}\left(\frac{N_t^*}{t}\right)^p \leq p(\lambda+\delta)^p \int_0^\infty u^{p-1} e^{-cu}\,du < \infty. \tag{2.7}$$

By (2.4) and (2.7), we obtain Lemma 2.1. □

3 Proof of Theorem 1.1

By $F \in \mathcal{S}$ and (1.4), we need only to prove

$$\mathbb{P}\left\{\max_{k \leq N_t^*}(X_k - \mu) > x\right\} \sim \lambda t \overline{F}(x) \quad \text{for } t, x \to \infty.$$

Write $\xi_k = I_{\{X_k - \mu > x\}}$. First,

$$I_{\{\max_{k \leq N_t^*}(X_k-\mu)>x\}} \leq \sum_{k=1}^{N_t^*} \xi_k.$$

By Wald's equation

$$\mathbb{P}\left\{\max_{k \leq N_t^*}(X_k - \mu) > x\right\} \leq \left(\mathbb{E}N_t^*\right)\mathbb{P}\{X > x + \mu\}.$$

From Lemma 2.1 and $\overline{F}(x+\mu) \sim \overline{F}(x)$ (see class $\mathcal{S} \subset \mathcal{L}$ and definition of $\mathcal{L}$, Embrechts *et al.* [1]), we have

$$\mathbb{P}\left\{\max_{k \leq N_t^*}(X_k - \mu) > x\right\} \lesssim \lambda t \overline{F}(x) \quad \text{for } t, x \to \infty. \tag{3.1}$$

On the other hand,

$$I_{\{\max_{k \leq N_t^*}(X_k-\mu)>x\}} \geq \sum_{k=1}^{N_t^*} \xi_k - \sum_{1 \leq j < k \leq N_t^*} \xi_j \xi_k. \tag{3.2}$$

All we need is to bound the second term. Notice that

$$M_n \equiv \sum_{1 \le j < k \le n} \xi_j(\xi_k - \mathbb{E}\xi_k), \quad n = 1, 2, \ldots$$

is a martingale. By Doob's stopping rule,

$$\mathbb{E}M_{N_t^*} = \mathbb{E}M_1 = 0.$$

Or

$$\mathbb{E}\left(\sum_{1 \le j < k \le N_t^*} \xi_j(\xi_k - \mathbb{E}\xi_k)\right) = 0.$$

Hence,

$$\mathbb{E}\left(\sum_{1 \le j < k \le N_t^*} \xi_j \xi_k\right) = \mathbb{E}\xi \cdot \mathbb{E}\left(\sum_{1 \le j < k \le N_t^*} \xi_j\right) = \mathbb{E}\xi \cdot \mathbb{E}\left(\sum_{j=1}^{N_t^*-1} \xi_j \big(N_t^* - j\big)\right).$$

Write $Z_n = \sum_{j=1}^{n} \xi_j$. Notice that

$$\sum_{j=1}^{N_t^*-1} \xi_j \big(N_t^* - j\big) = \sum_{n=1}^{N_t^*-1} Z_n.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\sum_{j=1}^{N_t^*-1} \xi_j \big(N_t^* - j\big)\right) &= \mathbb{E}\left(\sum_{n=1}^{N_t^*-1} Z_n\right) \\ &= \sum_{n=1}^{\infty} \mathbb{E}(Z_n I_{\{N_t^* \ge n+1\}}) \\ &= \sum_{n=1}^{\infty} \mathbb{E}(Z_n I_{\{\tau_{n+1} \le t\}}).\end{aligned} \tag{3.3}$$

For each n,

$$\begin{aligned}\mathbb{E}(Z_n I_{\{\tau_{n+1} \le t\}}) &= \sum_{j=1}^{n} \mathbb{E}(\xi_j \cdot I_{\{\tau_{n+1} \le t\}}) \\ &\le \sum_{j=1}^{n} \mathbb{E}(\xi_j \cdot I_{\{\sum_{k \ne j}^{n+1} \theta_k \le t\}}) \\ &= \sum_{j=1}^{n} \mathbb{E}\xi \cdot \mathbb{P}\left\{\sum_{k=1}^{n} \theta_k \le t\right\} = n\mathbb{E}\xi \cdot \mathbb{P}\big\{N_t^* \ge n\big\}.\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), $\exists c > 0$ such that

$$\mathbb{E}\left(\sum_{j=1}^{N_t^*-1} \xi_j\left(N_t^*-j\right)\right) \le \mathbb{E}\xi \cdot \sum_{n=1}^{\infty} n\mathbb{P}\left\{N_t^* \ge n\right\} \le c \cdot \mathbb{E}\xi \cdot \mathbb{E}\left(N_t^*\right)^2.$$

Combining our computation and Lemma 2.1,

$$\mathbb{E}\left(\sum_{1\le j<k\le N_t^*} \xi_j\xi_k\right) \le c \cdot (\mathbb{E}\xi)^2 \cdot \mathbb{E}\left(N_t^*\right)^2 \le c\left(\lambda t \cdot \overline{F}(x)\right)^2 = o\left(\lambda t\overline{F}(x)\right).$$

By (3.2), therefore,

$$\mathbb{P}\left\{\max_{k\le N_t^*}(X_k-\mu) > x\right\} \gtrsim \lambda t\overline{F}(x) \quad \text{for } t,x \to \infty. \tag{3.5}$$

Hence, by (3.1) and (3.5), we have

$$\mathbb{P}\left\{\max_{k\le N_t^*}(X_k-\mu) > x\right\} \sim \lambda t\overline{F}(x) \quad \text{for } t,x \to \infty.$$

4 Conclusions

As was remarked by a few researchers in the area, precise large-deviation results of size-dependent renewal risk models are particularly useful for evaluating some risk measures such as the conditional tail expectation of the aggregate amount of claims from a large insurance portfolio. Finally, we would like to point out that equation (1.5) agrees with existing ones in the literature. This indicates that the aggregate amount of S_t^* defined by (1.1) does not affect the asymptotic behavior of the large deviations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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