Zhong and Yang *Journal of Inequalities and Applications* (2017) 2017:78
DOI 10.1186/s13660-017-1355-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

An extension of a multidimensional Hilbert-type inequality

Jianhua Zhong and Bicheng Yang*

*Correspondence:
bcyang@gdei.edu.cn
Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, P.R. China

Abstract

In this paper, by the use of weight coefficients, the transfer formula and the technique of real analysis, a new multidimensional Hilbert-type inequality with multi-parameters and a best possible constant factor is given, which is an extension of some published results. Moreover, the equivalent forms, the operator expressions and a few particular inequalities are considered.

MSC: 26D15; 47A05

Keywords: Hilbert-type inequality; weight coefficient; equivalent form; operator; norm

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, then we have the following Hardy-Hilbert inequality with the best possible constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \tag{1}$$

and the following Hilbert-type inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\|a\|_p\|b\|_q \tag{2}$$

with the best possible constant factor pq (cf. [1], Theorem 315, Theorem 341). Inequalities (1) and (2) are important in the analysis and its applications (cf. [1–3]).

Assuming that $\{\mu_m\}_{m=1}^{\infty}$, $\{\nu_n\}_{n=1}^{\infty}$ are positive sequences,

$$U_m = \sum_{i=1}^{m}\mu_i, \qquad V_n = \sum_{j=1}^{n}\nu_j \quad \bigl(m,n \in \mathbb{N} = \{1,2,\ldots\}\bigr),$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{n^{q-1}}\right)^{\frac{1}{q}}. \tag{3}$$

For $\mu_i = \nu_j = 1$ $(i, j \in \mathbb{N})$, inequality (3) reduces to (1).

In 2014, Yang and Chen [4] gave the following multidimensional Hilbert-type inequality: For $i_0, j_0 \in \mathbb{N}$, $\alpha, \beta > 0$,

$$\|x\|_\alpha := \left(\sum_{k=1}^{i_0} |x^{(k)}|^\alpha\right)^{\frac{1}{\alpha}} \quad \left(x = \left(x^{(1)}, \ldots, x^{(i_0)}\right) \in \mathbb{R}^{i_0}\right),$$

$$\|y\|_\beta := \left(\sum_{k=1}^{j_0} |y^{(k)}|^\beta\right)^{\frac{1}{\beta}} \quad \left(y = \left(y^{(1)}, \ldots, y^{(j_0)}\right) \in \mathbb{R}^{j_0}\right),$$

$0 < \lambda_1 + \eta \leq i_0$, $0 < \lambda_2 + \eta \leq j_0$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, we have

$$\begin{aligned}
&\sum_n \sum_m \frac{(\min\{\|m\|_\alpha, \|n\|_\beta\})^\eta}{(\max\{\|m\|_\alpha, \|n\|_\beta\})^{\lambda+\eta}} a_m b_n \\
&\quad < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \left[\sum_m \|m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right]^{\frac{1}{p}} \left[\sum_n \|n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{4}$$

where $\sum_m = \sum_{m_{i_0}=1}^\infty \cdots \sum_{m_1=1}^\infty$, $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$, the series on the right-hand side are positive, and the best possible constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is indicated by

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{\lambda + 2\eta}{(\lambda_1 + \eta)(\lambda_2 + \eta)}.$$

For $i_0 = j_0 = \lambda = 1$, $\eta = 0$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (4) reduces to (2). The other results on this type of inequalities were provided by [5–17].

In 2015, Shi and Yang [18] gave another extension of (2) as follows:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{\max\{U_m, V_n\}} < pq\left(\sum_{m=1}^\infty \frac{a_m^p}{m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{b_n^q}{n^{q-1}}\right)^{\frac{1}{q}}. \tag{5}$$

Some other results on Hardy-Hilbert-type inequalities were given by [19–25].

In this paper, by the use of weight coefficients, the transfer formula and the technique of real analysis, a new multidimensional Hilbert-type inequality with multi-parameters and a best possible constant factor is given, which is an extension of (4) and (5). Moreover, the equivalent forms, the operator expressions and a few particular inequalities are considered.

2 Some lemmas

If $\mu_i^{(k)} > 0$ $(k = 1, \ldots, i_0; i = 1, \ldots, m)$, $\nu_j^{(l)} > 0$ $(l = 1, \ldots, j_0; j = 1, \ldots, n)$, then we set

$$U_m^{(k)} := \sum_{i=1}^m \mu_i^{(k)} \quad (k = 1, \ldots, i_0), \qquad V_n^{(l)} := \sum_{j=1}^n \nu_j^{(l)} \quad (l = 1, \ldots, j_0), \tag{6}$$

$$U_m = \left(U_m^{(1)}, \ldots, U_m^{(i_0)}\right), \qquad V_n = \left(V_n^{(1)}, \ldots, V_n^{(j_0)}\right) \quad (m, n \in \mathbb{N}).$$

We also set functions $\mu_k(t) := \mu_m^{(k)}$, $t \in (m-1, m]$ $(m \in \mathbb{N})$; $\nu_l(t) := \nu_n^{(l)}$, $t \in (n-1, n]$ $(n \in \mathbb{N})$, and

$$U_k(x) := \int_0^x \mu_k(t)\,dt \quad (k = 1, \ldots, i_0), \tag{7}$$

$$V_l(y) := \int_0^y \nu_l(t)\,dt \quad (l = 1, \ldots, j_0), \tag{8}$$

$$U(x) := \bigl(U_1(x), \ldots, U_{i_0}(x)\bigr), \qquad V(y) := \bigl(V_1(y), \ldots, V_{j_0}(y)\bigr) \quad (x, y \geq 0). \tag{9}$$

It follows that $U_k(m) = U_m^{(k)}$ $(k = 1, \ldots, i_0; m \in \mathbb{N})$, $V_l(n) = V_n^{(l)}$ $(l = 1, \ldots, j_0; n \in \mathbb{N})$, and for $x \in (m-1, m)$, $U_k'(x) = \mu_k(x) = \mu_m^{(k)}$ $(k = 1, \ldots, i_0; m \in \mathbb{N})$; for $y \in (n-1, n)$, $V_l'(y) = \nu_l(y) = \nu_n^{(l)}$ $(l = 1, \ldots, j_0; n \in \mathbb{N})$.

Lemma 1 (cf. [21]) *Suppose that $g(t)$ (> 0) is decreasing in $\mathbb{R}_+$ and strictly decreasing in $[n_0, \infty)$ $(n_0 \in \mathbb{N})$, satisfying $\int_0^\infty g(t)\,dt \in \mathbb{R}_+$. We have*

$$\int_1^\infty g(t)\,dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)\,dt. \tag{10}$$

Lemma 2 *If $i_0 \in \mathbb{N}$, $\alpha, M > 0$, $\Psi(u)$ is a non-negative measurable function in $(0,1]$, and*

$$D_M := \left\{ x \in \mathbb{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha \leq 1 \right\}, \tag{11}$$

then we have the following transfer formula (cf. [26]):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_s = \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha} - 1}\,du. \tag{12}$$

Lemma 3 *For $i_0, j_0 \in \mathbb{N}$, $\mu_m^{(k)} \geq \mu_{m+1}^{(k)}$ $(m \in \mathbb{N}, k = 1, \ldots, i_0)$, $\nu_n^{(l)} \geq \nu_{n+1}^{(l)}$ $(n \in \mathbb{N}; l = 1, \ldots, j_0)$, $\alpha, \beta > 0$, $\varepsilon > 0$, we have*

$$\sum_m \|U_m\|_\alpha^{-i_0 - \varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} + O(1), \tag{13}$$

$$\sum_n \|V_n\|_\beta^{-j_0 - \varepsilon} \prod_{k=1}^{j_0} \nu_n^{(k)} \leq \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} + \tilde{O}(1) \quad \bigl(\varepsilon \to 0^+\bigr). \tag{14}$$

Proof For $M > i_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{i_0}{M^\alpha}, \\ \frac{1}{(Mu^{1/\alpha})^{i_0 + \varepsilon}}, & \frac{i_0}{M^\alpha} \leq u \leq 1. \end{cases}$$

By (12), it follows that

$$\begin{aligned} \int_{\{x \in \mathbb{R}_+^{i_0}; x_i \geq 1\}} \frac{dx}{\|x\|_\alpha^{i_0 + \varepsilon}} &= \lim_{M \to \infty} \int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_{i_0} \\ &= \lim_{M \to \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_{i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha} - 1}}{(Mu^{1/\alpha})^{i_0 + \varepsilon}}\,du = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})}. \end{aligned}$$

Then by (10) and the above result, we find

$$
\begin{aligned}
0 &< \sum_{\{m\in\mathbb{N}^{i_0};m_i\geq 2\}} \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&= \sum_{\{m\in\mathbb{N}^{i_0};m_i\geq 2\}} \int_{\{x\in\mathbb{N}^{i_0};m-1\leq x_i<m\}} \bigl\|U(m)\bigr\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}\,dx \\
&< \sum_{\{m\in\mathbb{N}^{i_0};m_i\geq 2\}} \int_{\{x\in\mathbb{N}^{i_0};m-1\leq x_i<m\}} \bigl\|U(x)\bigr\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)}(x)\,dx \\
&= \int_{\{x\in\mathbb{N}^{i_0};x_i\geq 1\}} \bigl\|U(x)\bigr\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu^{(k)}(x)\,dx \overset{\nu=U(x)}{=} \int_{\{\nu\in\mathbb{R}_+^{i_0};\nu_i\geq\mu_1^{(i)}\}} \|\nu\|_\alpha^{-i_0-\varepsilon}\,d\nu \\
&= \int_{\{\nu\in\mathbb{R}_+^{i_0};\nu_i\geq 1\}} \|\nu\|_\alpha^{-i_0-\varepsilon}\,d\nu + O_{i_0}(1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + O_{i_0}(1).
\end{aligned}
$$

For $i_0 = 1$, $0 < \sum_{\{m\in\mathbb{N}^{i_0};m_i=1\}} \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} < \infty$; for $i_0 \geq 2$, $\mu^{(i)} = \max_m \mu_m^{(i)}$, $b = \sum_{i=1}^{i_0}\mu^{(i)}$, in the same way, we find

$$
\begin{aligned}
0 &< \sum_{\{m\in\mathbb{N}^{i_0};\exists i,m_i=1\}} \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\leq \|U_1\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_1^{(k)} + \sum_{i=1}^{i_0}\mu^{(i)} \sum_{\{m\in\mathbb{N}^{i_0-1};m_j\geq 2(j\neq i)\}} \|U_m\|_\alpha^{-(i_0-1)-(\varepsilon+1)} \prod_{k=1(k\neq i)}^{i_0} \mu_m^{(k)} \\
&= O_1(1) + \frac{b\Gamma^{i_0-1}(\frac{1}{\alpha})}{(1+\varepsilon)(i_0-1)^{(1+\varepsilon)/\alpha}\alpha^{i_0-2}\Gamma(\frac{i_0-1}{\alpha})} + bO_{i_0-1}(1) < \infty.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\sum_m \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} &= \sum_{\{m\in\mathbb{N}^{i_0};\exists i,m_i=1\}} \sum_m \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\quad + \sum_{\{m\in\mathbb{N}^{i_0};m_j\geq 2\}} \|U_m\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_m^{(k)} \\
&\leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + O(1) \quad \bigl(\varepsilon\to 0^+\bigr).
\end{aligned}
$$

Hence, we have (13). In the same way, we have (14). □

Definition 1 For $\alpha,\beta > 0$, $0 < \lambda_1+\eta \leq i_0$, $0 < \lambda_2+\eta \leq j_0$, $\lambda_1+\lambda_2 = \lambda$, we define two weight coefficients $w(\lambda_1,n)$ and $W(\lambda_2,m)$ as follows:

$$
w(\lambda_1,n) := \sum_m \frac{(\min\{\|U_m\|_\alpha,\|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha,\|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|U_m\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0}\mu_m^{(k)}, \tag{15}
$$

$$W(\lambda_2, m) := \sum_n \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|U_m\|_\alpha^{\lambda_1}}{\|V_n\|_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \nu_n^{(l)}. \tag{16}$$

Example 1 With regard to the assumptions of Definition 1, we set

$$k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}} \quad (x, y > 0).$$

Then, (i) for fixed $y > 0$,

$$k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}} = \begin{cases} \frac{1}{y^{\lambda+\eta}x^{i_0-\lambda_1-\eta}}, & 0 < x < y, \\ \frac{y^\eta}{x^{i_0+\lambda_2+\eta}}, & x \geq y, \end{cases}$$

is decreasing in $\mathbb{R}_+$ and strictly decreasing in $([y]+1,\infty)$. In the same way, for fixed $x > 0$, $k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}$ is decreasing in $\mathbb{R}_+$ and strictly decreasing in $([x]+1,\infty)$. We still have

$$\begin{aligned} k(\lambda_1) &:= \int_0^\infty k_\lambda(u,1)\frac{du}{u^{1-\lambda_1}} = \int_0^\infty \frac{(\min\{u,1\})^\eta}{(\max\{u,1\})^{\lambda+\eta}}\frac{du}{u^{1-\lambda_1}} \\ &= \int_0^1 \frac{u^\eta}{u^{1-\lambda_1}}\,du + \int_1^\infty \frac{1}{u^{\lambda+\eta}}\frac{du}{u^{1-\lambda_1}} = \frac{\lambda+2\eta}{(\lambda_1+\eta)(\lambda_2+\eta)}. \end{aligned} \tag{17}$$

(ii) For $b > 0$, we have

$$\frac{d}{dx}\bigl(b+x^\alpha\bigr)^{\frac{1}{\alpha}} = \bigl(b+x^\alpha\bigr)^{\frac{1}{\alpha}-1}x^{\alpha-1} > 0 \quad (x > 0).$$

Hence, for $m-1 < x_i < m$ $(i = 1,\ldots,i_0;\ m \in \mathbb{N})$, we have $\|U(m)\|_\alpha > \|U(x)\|_\alpha$ and

$$\begin{aligned} &\frac{(\min\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{1}{\|U(m)\|_\alpha^{i_0-\lambda_1}} \\ &\quad < \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{1}{\|U(x)\|_\alpha^{i_0-\lambda_1}}; \end{aligned}$$

for $m < x_i < m+1$ $(i = 1,\ldots,i_0;\ m \in \mathbb{N})$, we have $\|U(m)\|_\alpha < \|U(x)\|_\alpha$ and

$$\begin{aligned} &\frac{(\min\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{1}{\|U(m)\|_\alpha^{i_0-\lambda_1}} \\ &\quad > \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{1}{\|U(x)\|_\alpha^{i_0-\lambda_1}}. \end{aligned}$$

Lemma 4 *With regard to the assumptions of Definition* 1, (i) *we have*

$$w(\lambda_1, n) < K_2(\lambda_1) \quad \bigl(n \in \mathbb{N}^{j_0}\bigr), \tag{18}$$

$$W(\lambda_2, m) < K_1(\lambda_1) \quad \bigl(m \in \mathbb{N}^{i_0}\bigr), \tag{19}$$

where

$$K_1(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k(\lambda_1), \qquad K_2(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1); \tag{20}$$

(ii) *for* $\mu_m^{(k)} \ge \mu_{m+1}^{(k)}$ $(m \in \mathbb{N})$, $\nu_n^{(l)} \ge \nu_{n+1}^{(l)}$ $(n \in \mathbb{N})$, $U_\infty^{(k)} = V_\infty^{(l)}$ $(k = 1, \ldots, i_0,\ l = 1, \ldots, j_0)$, $0 < \lambda_1 + \eta \le i_0$, $\lambda_2 + \eta > 0$, $0 < \varepsilon < p\lambda_1$ $(p > 1)$, *we have*

$$0 < K_2(\lambda_1)\bigl(1 - \theta_\lambda(n)\bigr) < w(\lambda_1, n) \quad \bigl(n \in \mathbb{N}^{j_0}\bigr), \tag{21}$$

where, for $c := \max_{1 \le k \le i_0}\{\mu_1^{(k)}\}$ (> 0),

$$\theta_\lambda(n) := \frac{1}{k(\lambda_1)} \int_0^{c i_0^{1/\alpha}/\|V_n\|_\beta} \frac{(\min\{v,1\})^\eta v^{\lambda_1 - 1}}{(\max\{v,1\})^{\lambda+\eta}}\, dv = O\biggl(\frac{1}{\|V_n\|_\beta^{\lambda_1+\eta}}\biggr). \tag{22}$$

Proof (i) By (10), (12) and Example 1(ii), for $0 < \lambda_1 + \eta \le i_0$, $\lambda > 0$, it follows that

$$\begin{aligned}
w(\lambda_1, n) &= \sum_m \int_{\{x \in \mathbb{N}^{i_0}; m_i - 1 \le x_i \le m_i\}} \frac{(\min\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \\
&\quad \times \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(m)\|_\alpha^{i_0 - \lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}\, dx \\
&< \sum_m \int_{\{x \in \mathbb{N}^{i_0}; m_i - 1 \le x_i \le m_i\}} \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \\
&\quad \times \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0 - \lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}(x)\, dx \\
&= \int_{\mathbb{R}_+^{i_0}} \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0 - \lambda_1}} \prod_{k=1}^{i_0} \mu^{(k)}(x)\, dx \\
&\overset{u=U(x)}{\le} \int_{\mathbb{R}_+^{i_0}} \frac{(\min\{\|u\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|u\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|u\|_\alpha^{i_0 - \lambda_1}}\, du \\
&= \lim_{M \to \infty} \int_{D_M} \frac{(\min\{M[\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha]^{1/\alpha}, \|V_n\|_\beta\})^\eta}{(\max\{M[\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha]^{1/\alpha}, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{M^{\lambda_1 - i_0}\|V_n\|_\beta^{\lambda_2}\, du}{[\sum_{i=1}^{i_0}(\frac{u_i}{M})^\alpha]^{(i_0 - \lambda_1)/\alpha}} \\
&= \lim_{M \to \infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{(\min\{Mu^{1/\alpha}, \|V_n\|_\beta\})^\eta \|V_n\|_\beta^{\lambda_2}}{(\max\{Mu^{1/\alpha}, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{u^{\frac{i_0}{\alpha}-1}\, du}{M^{i_0 - \lambda_1} u^{(i_0 - \lambda_1)/\alpha}} \\
&= \lim_{M \to \infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{(\min\{Mu^{1/\alpha}, \|V_n\|_\beta\})^\eta \|V_n\|_\beta^{\lambda_2}}{(\max\{Mu^{1/\alpha}, \|V_n\|_\beta\})^{\lambda+\eta}} u^{\frac{\lambda_1}{\alpha}-1}\, du \\
&\overset{v=\frac{Mu^{1/\alpha}}{\|V_n\|_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty \frac{(\min\{v,1\})^\eta v^{\lambda_1 - 1}}{(\max\{v,1\})^{\lambda+\eta}}\, dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1}\Gamma(\frac{i_0}{\alpha})} \frac{\lambda + 2\eta}{(\lambda_1 + \eta)(\lambda_2 + \eta)} = K_2(\lambda_1).
\end{aligned}$$

Hence, we have (18). In the same way, we have (19).

(ii) By (10) and in the same way, for $c = \max_{1 \le k \le i_0}\{\mu_1^{(k)}\}$ (> 0), we have

$$\begin{aligned}
w(\lambda_1, n) &\ge \sum_m \int_{\{x \in \mathbb{N}^{i_0}; m_i \le x_i \le m_i + 1\}} \frac{(\min\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(m)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \\
&\quad \times \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(m)\|_\alpha^{i_0 - \lambda_1}} \prod_{k=1}^{i_0} \mu_{m+1}^{(k)}\, dx
\end{aligned}$$

$$
\begin{aligned}
&> \sum_m \int_{\{x\in\mathbb{N}^{i_0};m_i\le x_i\le m_i+1\}} \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \\
&\quad\times \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu^{(k)}(x)\,dx \\
&= \int_{[1,\infty)^{i_0}} \frac{(\min\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U(x)\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu^{(k)}(x)\,dx \\
&\overset{\nu=U(x)}{\ge} \int_{[c,\infty)^{i_0}} \frac{(\min\{\|\nu\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|\nu\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|\nu\|_\alpha^{i_0-\lambda_1}}\,d\nu.
\end{aligned}
$$

For $M > ci_0^{1/\alpha}$, we set

$$
\Psi(u) = \begin{cases} 0, & 0 < u < \frac{c^\alpha i_0}{M^\alpha}, \\ \frac{(\min\{Mu^{1/\alpha},\|V_n\|_\beta\})^\eta}{(\max\{Mu^{1/\alpha},\|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{(Mu^{1/\alpha})^{i_0-\lambda_1}}, & \frac{c^\alpha i_0}{M^\alpha} \le u \le 1. \end{cases}
$$

By (12), it follows that

$$
\begin{aligned}
&\int_{\{x\in\mathbb{R}_+^{i_0};x_i\ge c\}} \frac{(\min\{\|x\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|x\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2}}{\|x\|_\alpha^{i_0-\lambda_1}}\,dx \\
&= \lim_{M\to\infty} \int\cdots\int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1\cdots dx_{i_0} \\
&= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{c^\alpha i_0/M^\alpha}^1 \frac{(\min\{Mu^{\frac{1}{\alpha}}, \|V_n\|_\beta\})^\eta}{(\max\{Mu^{\frac{1}{\alpha}}, \|V_n\|_\beta\})^{\lambda+\eta}} \frac{\|V_n\|_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1}\,du}{(Mu^{\frac{1}{\alpha}})^{i_0-\lambda_1}} \\
&\overset{\nu=\frac{Mu^{1/\alpha}}{\|V_n\|_\beta}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{ci_0^{1/\alpha}/\|V_n\|_\beta}^1 \frac{(\min\{\nu,1\})^\eta \nu^{\lambda_1-1}}{(\max\{\nu,1\})^{\lambda+\eta}}\,d\nu.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
w(\lambda_1, n) &> \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{ci_0^{1/\alpha}/\|V_n\|_\beta}^1 \frac{(\min\{\nu,1\})^\eta \nu^{\lambda_1-1}}{(\max\{\nu,1\})^{\lambda+\eta}}\,d\nu \\
&= K_2(\lambda_1)\bigl(1-\theta_\lambda(n)\bigr) > 0.
\end{aligned}
$$

For $\|V_n\|_\beta \ge ci_0^{1/\alpha}$, we obtain

$$
\begin{aligned}
0 < \theta_\lambda(n) &= \frac{1}{k(\lambda_1)} \int_0^{ci_0^{1/\alpha}/\|V_n\|_\beta} \frac{(\min\{\nu,1\})^\eta \nu^{\lambda_1-1}}{(\max\{\nu,1\})^{\lambda+\eta}}\,d\nu \\
&= \frac{1}{k(\lambda_1)} \int_0^{ci_0^{1/\alpha}/\|V_n\|_\beta} \nu^{\lambda_1+\eta-1}\,d\nu = \frac{1}{(\lambda_1+\eta)k(\lambda_1)} \left(\frac{ci_0^{1/\alpha}}{\|V_n\|_\beta}\right)^{\lambda_1+\eta},
\end{aligned}
$$

and then (21) and (22) follow. □

3 Main results

Setting functions

$$\Phi(m) := \frac{\|U_m\|_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{p-1}} \quad (m \in \mathbb{N}^{i_0}),$$
$$\Psi(n) := \frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0}\nu_n^{(l)})^{q-1}} \quad (n \in \mathbb{N}^{j_0}),$$

and the following normed spaces:

$$l_{p,\Phi} := \left\{a = \{a_m\}; \|a\|_{p,\Phi} := \left\{\sum_m \Phi(m)|a_m|^p\right\}^{\frac{1}{p}} < \infty\right\},$$
$$l_{q,\Psi} := \left\{b = \{b_n\}; \|b\|_{q,\Psi} := \left\{\sum_n \Psi(n)|b_n|^q\right\}^{\frac{1}{q}} < \infty\right\},$$
$$l_{p,\Psi^{1-p}} := \left\{c = \{c_n\}; \|c\|_{p,\Psi^{1-p}} := \left\{\sum_n \Psi^{1-p}(n)|c_n|^p\right\}^{\frac{1}{p}} < \infty\right\},$$

we have the following.

Theorem 1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta > 0$, $\lambda > 0$, $0 < \lambda_1 + \eta \le i_0$, $0 < \lambda_2 + \eta \le j_0$, $\lambda_1 + \lambda_2 = \lambda$, then for $a_m, b_n \ge 0$, $a = \{a_m\} \in l_{p,\Phi}$, $b = \{b_n\} \in l_{q,\Psi}$, $\|a\|_{p,\Phi}, \|b\|_{q,\Psi} > 0$, we have the following equivalent inequalities*:

$$I := \sum_n \sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} a_m b_n < K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{23}$$

$$J := \left\{\sum_n \frac{\prod_{k=1}^{j_0}\nu_n^{(k)}}{\|V_n\|_\beta^{j_0-p\lambda_2}}\left[\sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta a_m}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}}\right]^p\right\}^{\frac{1}{p}}$$
$$< K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\Phi}, \tag{24}$$

where

$$K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1). \tag{25}$$

Proof By Hölder's inequality with weight (cf. [27]), we have

$$I = \sum_n \sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}}$$
$$\times \left[\frac{\|U_m\|_\alpha^{\frac{i_0-\lambda_1}{q}}}{\|V_n\|_\beta^{\frac{j_0-\lambda_2}{p}}} \frac{(\prod_{l=1}^{j_0}\nu_n^{(l)})^{\frac{1}{p}} a_m}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{\frac{1}{q}}}\right]\left[\frac{\|V_n\|_\beta^{\frac{j_0-\lambda_2}{p}}}{\|U_m\|_\alpha^{\frac{i_0-\lambda_1}{q}}} \frac{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{\frac{1}{q}} b_n}{(\prod_{l=1}^{j_0}\nu_n^{(l)})^{\frac{1}{p}}}\right]$$
$$\le \left[\sum_m W(\lambda_2, m)\frac{\|U_m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p}{(\prod_{k=1}^{i_0}\mu_m^{(k)})^{p-1}}\right]^{\frac{1}{p}}\left[\sum_n w(\lambda_1, n)\frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{l=1}^{j_0}\nu_n^{(l)})^{q-1}}\right]^{\frac{1}{q}}.$$

Then by (18) and (19), we have (23). We set

$$b_n := \frac{\prod_{l=1}^{j_0} \nu_n^{(l)}}{\|V_n\|_\beta^{j_0-p\lambda_2}} \left[\sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta a_m}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \right]^{p-1}, \quad n \in \mathbb{N}^{j_0}.$$

Then we have $J = \|b\|_{q,\Psi}^{q-1}$. Since the right-hand side of (24) is finite, it follows $J < \infty$. If $J = 0$, then (24) is trivially valid; if $J > 0$, then by (23), we have

$$\|b\|_{q,\Psi}^q = J^p = I < K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi} \|b\|_{q,\Psi},$$
$$\|b\|_{q,\Psi}^{q-1} = J < K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi},$$

namely (24) follows.

On the other hand, assuming that (24) is valid, by Hölder's inequality (cf. [27]), we have

$$\begin{aligned} I &= \sum_n \frac{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{1/p}}{\|V_n\|_\beta^{(j_0/p)-\lambda_2}} \sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} a_m \\ &\quad \times \frac{\|V_n\|_\beta^{(j_0/p)-\lambda_2}}{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{1/p}} b_n \le J \|b\|_{q,\Psi}. \end{aligned} \tag{26}$$

Then by (24) we have (23), which is equivalent to (24). □

Theorem 2 *With regard to the assumptions of Theorem* 1, *if* $\mu_m^{(k)} \ge \mu_{m+1}^{(k)}$ $(m \in \mathbb{N})$, $\nu_n^{(l)} \ge \nu_{n+1}^{(l)}$ $(n \in \mathbb{N})$, $U_\infty^{(k)} = V_\infty^{(l)} = \infty$ $(k = 1, \ldots, i_0,\ l = 1, \ldots, j_0)$, *then the constant factor* $K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1)$ *in* (23) *and* (24) *is the best possible.*

Proof For $0 < \varepsilon < p(\lambda_1 + \eta)$, $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (-\eta, -\eta + i_0))$, $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(> -\eta)$, we set

$$\tilde{a} = \{\tilde{a}_m\}, \qquad \tilde{a}_m := \|U_m\|_\alpha^{-i_0+\tilde{\lambda}_1} \prod_{k=1}^{i_0} \mu_m^{(k)} \quad \left(m \in \mathbb{N}^{i_0}\right),$$
$$\tilde{b} = \{\tilde{b}_n\}, \qquad \tilde{b}_n := \|V_n\|_\beta^{-j_0+\tilde{\lambda}_2-\varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \quad \left(n \in \mathbb{N}^{j_0}\right).$$

Then by (13) and (14), we obtain

$$\begin{aligned} \|\tilde{a}\|_{p,\Phi} \|\tilde{b}\|_{q,\Psi} &= \left[\sum_m \frac{\|U_m\|_\alpha^{p(i_0-\lambda_1)-i_0} \tilde{a}_m^p}{(\prod_{k=1}^{i_0} \mu_m^{(k)})^{p-1}} \right]^{\frac{1}{p}} \left[\sum_n \frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0} \tilde{b}_n^q}{(\prod_{l=1}^{j_0} \nu_n^{(l)})^{q-1}} \right]^{\frac{1}{q}} \\ &= \left(\sum_m \|U_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \right)^{\frac{1}{p}} \left(\sum_n \|V_n\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \right)^{\frac{1}{q}} \\ &\le \frac{1}{\varepsilon} \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1) \right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon \tilde{O}(1) \right)^{\frac{1}{q}}. \end{aligned}$$

By (21) and (22), we find

$$\begin{aligned}\tilde{I} &:= \sum_n \biggl[\sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} \tilde{a}_m\biggr]\tilde{b}_n \\ &= \sum_n w(\tilde{\lambda}_1, n)\|V_n\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \\ &> K_2(\tilde{\lambda}_1)\sum_n \biggl(1 - O\biggl(\frac{1}{\|V_n\|_\beta^{\lambda_1+\eta}}\biggr)\biggr)\|V_n\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \nu_n^{(l)} \\ &= K_2(\tilde{\lambda}_1)\biggl(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \tilde{O}(1) - O_1(1)\biggr).\end{aligned}$$

If there exists a constant $K \le K_1^{\frac{1}{p}}(\lambda_1)K_2^{\frac{1}{q}}(\lambda_1)$ such that (23) is valid when replacing $K_1^{\frac{1}{p}}(\lambda_1)K_2^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon\tilde{I} < \varepsilon K\|\tilde{a}\|_{p,\Phi}\|\tilde{b}\|_{q,\Psi}$, namely

$$\begin{aligned}&K_2\biggl(\lambda_1 - \frac{\varepsilon}{p}\biggr)\biggl(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\tilde{O}(1) - \varepsilon O_1(1)\biggr) \\ &\quad< K\biggl(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\biggr)^{\frac{1}{p}}\biggl(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\tilde{O}(1)\biggr)^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon \to 0^+$, we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})k(\lambda_1)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \le K\biggl[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\biggr]^{\frac{1}{p}}\biggl[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\biggr]^{\frac{1}{q}},$$

and then $K_1^{\frac{1}{p}}(\lambda_1)K_2^{\frac{1}{q}}(\lambda_1) \le K$. Hence, $K = K_1^{\frac{1}{p}}(\lambda_1)K_2^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (23). The constant factor in (24) is still the best possible. Otherwise, we would reach a contradiction by (26) that the constant factor in (23) is not the best possible. □

4 Operator expressions

With regard to the assumptions of Theorem 2, in view of

$$\begin{aligned}&c_n := \frac{\prod_{k=1}^{j_0}\nu_n^{(k)}}{\|V_n\|_\beta^{j_0-p\lambda_2}}\biggl[\sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} a_m\biggr]^{p-1}, \quad n \in \mathbb{N}^{j_0}, \\ &c = \{c_n\}, \qquad \|c\|_{p,\Psi^{1-p}} = J < K_1^{\frac{1}{p}}(\lambda_1)K_2^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\Phi} < \infty,\end{aligned}$$

we can set the following definition.

Definition 2 Define a multidimensional Hilbert's operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For any $a \in l_{p,\Phi}$, there exists a unique representation $Ta = c \in l_{p,\Psi^{1-p}}$, satisfying

$$Ta(n) := \sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} a_m \quad \bigl(n \in \mathbb{N}^{j_0}\bigr). \tag{27}$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \left[\sum_m \frac{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\eta}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^{\lambda+\eta}} a_m \right] b_n. \tag{28}$$

Then by Theorems 1 and 2, we have the following equivalent inequalities:

$$(Ta, b) < K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \tag{29}$$

$$\|Ta\|_{p,\Psi^{1-p}} < K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi}. \tag{30}$$

It follows that T is bounded with

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}} \leq K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1). \tag{31}$$

Since the constant factor $K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1)$ in (30) is the best possible, we have

$$\|T\| = K_1^{\frac{1}{p}}(\lambda_1) K_2^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \tag{32}$$

Remark 1 (i) For $\mu_i = \nu_j = 1$ ($i, j \in \mathbb{N}$), (23) reduces to (4). Hence, (23) is an extension of (4).

(ii) For $\eta = 0$, $0 < \lambda_1 \leq i_0$, $0 < \lambda_2 \leq j_0$, (23) reduces to the following inequality:

$$\begin{aligned} &\sum_n \sum_m \frac{1}{(\max\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\lambda} a_m b_n \\ &\quad < \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\lambda}{\lambda_1 \lambda_2} \|a\|_{p,\Phi} \|b\|_{q,\Psi}. \end{aligned} \tag{33}$$

In particular, for $i_0 = j_0 = \lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (33) reduces to (5). Hence, (33) is also an extension of (5); so is (23).

(iii) For $\eta = -\lambda$, $\lambda_1, \lambda_2 < 0$, (23) reduces to the following inequality:

$$\begin{aligned} &\sum_n \sum_m \frac{1}{(\min\{\|U_m\|_\alpha, \|V_n\|_\beta\})^\lambda} a_m b_n \\ &\quad < \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{(-\lambda)}{\lambda_1 \lambda_2} \|a\|_{p,\Phi} \|b\|_{q,\Psi}. \end{aligned} \tag{34}$$

(iv) For $\lambda = 0$, $\lambda_2 = -\lambda_1$ ($-\eta < \lambda_1 < \eta$), (23) reduces to the following inequality:

$$\begin{aligned} &\sum_n \sum_m \left(\frac{\min\{\|U_m\|_\alpha, \|V_n\|_\beta\}}{\max\{\|U_m\|_\alpha, \|V_n\|_\beta\}} \right)^\eta a_m b_n \\ &\quad < \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{2\eta}{\eta^2 - \lambda_1^2} \|a\|_{p,\Phi} \|b\|_{q,\Psi}. \end{aligned} \tag{35}$$

The above particular inequalities are also with the best possible constant factors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. JZ participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

Acknowledgements
This work is supported by the National Natural Science Foundation (No. 61370186, No. 61640222), and Appropriative Researching Fund for Professors and Doctors, Guangdong University of Education (No. 2015ARF25). We are grateful for their help.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 20 January 2017 Accepted: 23 March 2017 Published online: 18 April 2017

References
1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, DS, Pečarić, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Boston (1991)
3. Yang, BC: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijin (2009) (in Chinese)
4. Yang, BC, Chen, Q: A multidimensional discrete Hilbert-type inequality. J. Math. Inequal. **8**(2), 267-277 (2014)
5. Hong, Y: On Hardy-Hilbert integral inequalities with some parameters. J. Inequal. Pure Appl. Math. **6**(4), Article ID 92 (2005)
6. Zhong, WY, Yang, BC: On multiple Hardy-Hilbert's integral inequality with kernel. J. Inequal. Appl. **2007**, Article ID 27962 (2007)
7. Yang, BC, Krnić, M: On the norm of a multi-dimensional Hilbert-type operator. Sarajevo J. Math. **7**(20), 223-243 (2011)
8. Krnić, M, Pečarić, JE, Vuković, P: On some higher-dimensional Hilbert's and Hardy-Hilbert's type integral inequalities with parameters. Math. Inequal. Appl. **11**, 701-716 (2008)
9. Krnić, M, Vuković, P: On a multidimensional version of the Hilbert-type inequality. Anal. Math. **38**, 291-303 (2012)
10. Rassias, M, Yang, BC: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Math. Comput. **225**, 263-277 (2013)
11. Yang, BC: A multidimensional discrete Hilbert-type inequality. Int. J. Nonlinear Anal. Appl. **5**(1), 80-88 (2014)
12. Chen, Q, Yang, BC: On a more accurate multidimensional Mulholland-type inequality. J. Inequal. Appl. **2014**, 322 (2014)
13. Rassias, M, Yang, BC: On a multidimensional Hilbert-type integral inequality associated to the gamma function. Appl. Math. Comput. **249**, 408-418 (2014)
14. Yang, BC: On a more accurate multidimensional Hilbert-type inequality with parameters. Math. Inequal. Appl. **18**(2), 429-441 (2015)
15. Huang, ZX, Yang, BC: A multidimensional Hilbert-type integral inequality. J. Inequal. Appl. **2015**, 151 (2015)
16. Liu, T, Yang, BC, He, LP: On a multidimensional Hilbert-type integral inequality with logarithm function. Math. Inequal. Appl. **18**(4), 1219-1234 (2015)
17. Shi, YP, Yang, BC: On a multidimensional Hilbert-type inequality with parameters. J. Inequal. Appl. **2015**, 371 (2015)
18. Shi, YP, Yang, BC: A new Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor. J. Inequal. Appl. **2015**, 380 (2015)
19. Huang, QL: A new extension of Hardy-Hilbert-type inequality. J. Inequal. Appl. **2015**, 397 (2015)
20. Wang, AZ, Huang, QL, Yang, BC: A strengthened Mulholland-type inequality with parameters. J. Inequal. Appl. **2015**, 329 (2015)
21. Yang, BC, Chen, Q: On a Hardy-Hilbert-type inequality with parameters. J. Inequal. Appl. **2015**, 339 (2015)
22. Li, AH, Yang, BC, He, LP: On a new Hardy-Mulholland-type inequality and its more accurate form. J. Inequal. Appl. **2016**, 69 (2016)
23. Rassias, M, Yang, BC: On a Hardy-Hilbert-type inequality with a general homogeneous kernel. Int. J. Nonlinear Anal. Appl. **7**(1), 249-269 (2016)
24. Chen, Q, Shi, YP, Yang, BC: A relation between two simple Hardy-Mulholland-type inequalities with parameters. J. Inequal. Appl. **2016**, 75 (2016)
25. Yang, BC, Chen, Q: On a more accurate Hardy-Mulholland-type inequality. J. Inequal. Appl. **2016**, 82 (2016)
26. Yang, BC: Hilbert-type integral operators: norms and inequalities. In: Pardalos, PM, Georgiev, PG, Srivastava, HM (eds.) Nonlinear Analysis, Stability, Approximation, and Inequalities, pp. 771-859. Springer, New York (2012)
27. Kuang, JC: Applied Inequalities. Shangdong Science Technic Press, Jinan, China (2004)