Deshwal et al. *Journal of Inequalities and Applications* (2017) 2017:57
DOI 10.1186/s13660-017-1328-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Modified Stancu operators based on inverse Polya Eggenberger distribution

Sheetal Deshwal[1], PN Agrawal[1] and Serkan Araci[2*]

*Correspondence:
mtsrkn@hotmail.com
[2]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, 27410, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we construct a sequence of modified Stancu-Baskakov operators for a real valued function bounded on $[0,\infty)$, based on a function $\tau(x)$. This function $\tau(x)$ is infinite times continuously differentiable on $[0,\infty)$ and satisfy the conditions $\tau(0)=0$, $\tau'(x)>0$ and $\tau''(x)$ is bounded for all $x\in[0,\infty)$. We study the degree of approximation of these operators by means of the Peetre K-functional and the Ditzian-Totik modulus of smoothness. The quantitative Voronovskaja-type theorems are also established in terms of the first order Ditzian-Totik modulus of smoothness.

MSC: 41A36; 41A25

Keywords: Baskakov operator; Ditzian-Totik modulus of smoothness; rate of convergence; Voronovskaja-type theorem

1 Introduction

In 1923, Eggenberger and Pólya [1] were the first to introduce Pólya-Eggenberger distribution. After that, in 1969, Johnson and Kotz [2] gave a short discussion of Pólya-Eggenberger distribution.

The Pólya-Eggenberger distribution X [2] is defined by

$$Pr(X=k)=\binom{n}{k}\frac{a(a+s)\cdots(a+(u-1)s)b(b+s)\cdots(b+(n-u-1)s)}{(a+b)(a+b+s)\cdots(a+b+(n-1)s)},$$
$$k=0,1,2,\ldots,n. \tag{1.1}$$

The inverse Pólya-Eggenberger distribution N is defined by

$$Pr(N=n+k)=\binom{(n+k-1)}{k}\frac{a(a+s)\cdots(a+(n-1)s)b(b+s)\cdots(b+(k-1)s)}{(a+b)(a+b+s)\cdots(a+b+(n+k-1)s)},$$
$$k=0,1,2,\ldots,n. \tag{1.2}$$

In 1970, Stancu [3] introduced a generalization of the Baskakov operators based on inverse Pólya-Eggenberger distribution for a real valued bounded function on $[0,\infty)$, defined by

$$V_n^{[\alpha]}(f;x)=\sum_{k=0}^{\infty}v_{n,k}(x,\alpha)f\left(\frac{k}{n}\right)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}f\left(\frac{k}{n}\right), \tag{1.3}$$

© The Author(s) 2017. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

where α is a non-negative parameter which may depend only on $n \in \mathbb{N}$ and $a^{[n,h]} = a(a-h)(a-2h)\cdots(a-(n-1)h), a^{[0,h]} = 1$ is known as a factorial power of a with increment h. For $\alpha = 0$, the operator (1.3) reduces to Baskakov operators [4].

In 1989, Razi [5] studied convergence properties of Stancu-Kantorovich operators based on Pólya-Eggenberger distribution. Very recently, Deo *et al.* [6] introduced a Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution and studied some of its convergence properties. For some other relevant research in this direction we refer the reader to [7–9].

Now, for $\alpha = \frac{1}{n}$, we get a special case of Stancu-Baskakov operators (1.3) defined as

$$V_n^{\langle \frac{1}{n} \rangle}(f;x) = \frac{(2n-1)!}{(n-1)!} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(nx)_k}{(n+nx)_{n+k}} f\left(\frac{k}{n}\right), \tag{1.4}$$

where $(a)_n := a^{[n,-1]} = a(a+1)\cdots(a+(n-1))$ is called the Pochhammer symbol.

For the Lupas operator, given by

$$L_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} f\left(\frac{k}{n}\right), \tag{1.5}$$

let $\mu_{n,m}(x) = L_n(t^m;x), m \in \mathbb{N} \cup \{0\}$ be the mth order moment.

Lemma 1 *For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$ and we have the recurrence relation*

$$x(1+x)\mu'_{n,m}(x) = n\mu_{n,m+1}(x) - nx\mu_{n,m}(x), \quad m \in \mathbb{N} \cup \{0\}, \tag{1.6}$$

where $\mu'_{n,m}(x)$ is the derivative of $\mu_{n,m}(x)$.

Proof On differentiating $\mu_{n,m}(x)$ with respect to x, the proof of the recurrence relation easily follows; hence the details are omitted. □

Remark 1 From Lemma 1, we have

$$\mu_{n,1}(x) = x, \qquad \mu_{n,2}(x) = \frac{x+(n+1)x^2}{n}, \qquad \mu_{n,3}(x) = \frac{(n+1)(n+2)x^3 + 3(n+1)x^2 + x}{n^2}.$$

The values of the Stancu-Baskakov operators (1.4) for the test functions $e_i(t) = t^i$, $i = 0,1,2$, are given in the following lemma.

Lemma 2 ([10]) *The Stancu-Baskakov operators* (1.4) *verify*:

(i) $V_n^{\langle \frac{1}{n} \rangle}(1;x) = 1$,

(ii) $V_n^{\langle \frac{1}{n} \rangle}(t;x) = \frac{nx}{n-1}$,

(iii) $V_n^{\langle \frac{1}{n} \rangle}(t^2;x) = \frac{n^2}{(n-1)(n-2)}[x^2 + \frac{x(x+1)}{n} + \frac{1}{n}(1-\frac{1}{n})x]$.

(iv) $V_n^{\langle \frac{1}{n} \rangle}(t^3;x) = \frac{n^3}{(n-1)(n-2)(n-3)}[\frac{(n+1)(n+2)}{n^2}x^3 + \frac{3(2n^2+n-1)}{n^3}x^2 + \frac{(2n-1)(3n-1)}{n^4}x]$

(v) $V_n^{\langle \frac{1}{n} \rangle}(t^4;x) = \frac{n^4}{(n-1)(n-2)(n-3)(n-4)}[\frac{(n+1)(n+2)(n+3)}{n^3}x^4 + \frac{6(n+1)(n+2)(2n-1)}{n^4}x^3 + \frac{6(6n^3+n^2-4n+1)}{n^5}x^2 + \frac{26n^2-27n+7}{n^5}x]$.

Proof The identities (i)-(iii) are proved in [10], hence we give the proof of the identity (iv). The identity (v) can be proved in a similar manner.

We have

$$V_n^{\langle\frac{1}{n}\rangle}\left(t^3;x\right) = \frac{(2n-1!)}{(n-1)!}\sum_{k=0}^{\infty}\binom{(n+k-1)}{k}\frac{(nx)_k}{(n+nx)_{n+k}}\left(\frac{k}{n}\right)^3$$
$$= \frac{1}{B(nx,n)}\int_0^{\infty}\frac{t^{nx-1}}{(1+t)^{nx+n}}\mu_{n,3}(t)\,dt,$$

where $B(nx,n)$ is the Beta function.

Therefore using Remark 1, we get

$$V_n^{\langle\frac{1}{n}\rangle}\left(t^3;x\right) = \frac{1}{B(nx,n)}\int_0^{\infty}\frac{t^{nx-1}}{(1+t)^{nx+n}}\left[\frac{(n+1)(n+2)t^3+3(n+1)t^2+t}{n^2}\right]dt.$$

Now, by a simple calculation, we get the required result. □

As a consequence of Lemma 2, we obtain the following.

Lemma 3 *For the Stancu-Baskakov operator* (1.4), *the following equalities hold*:

(i) $V_n^{\langle\frac{1}{n}\rangle}((t-x);x) = \frac{x}{n-1}$,

(ii) $V_n^{\langle\frac{1}{n}\rangle}((t-x)^2;x) = \frac{2nx(x+1)+(2x-1)x}{(n-1)(n-2)}$,

(iii) $V_n^{\langle\frac{1}{n}\rangle}((t-x)^4;x) = \frac{1}{n(n-1)(n-2)(n-3)(n-4)}[12n(n^2-13n+2)x^3(x+1)+12n(n^2+8n-13)x^2(x+1)+(26n^2+48n-22)x(x+1)+(29-75n)x]$.

Let $0 \le r_n(x) \le 1$ be a sequence of continuous functions for each $x \in [0,1]$ and $n \in \mathbb{N}$. Using this sequence $r_n(x)$, for any $f \in C[0,1]$, King [11] proposed the following modification of the Bernstein polynomial for a better approximation:

$$\left((B_nf)\circ r_n\right)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\left(r_n(x)\right)^k\left(1-r_n(x)\right)^{n-k}.$$

Gonska *et al.* [12] introduced a sequence of King-type operators $D_n^{\tau}: C[0,1] \to C[0,1]$ defined as

$$D_n^{\tau}f = (B_nf)\circ(B_n\tau)^{-1}\circ\tau,$$

where $\tau \in C[0,1]$ such that $\tau(0)=0, \tau(1)=1$ and $\tau'(x)>0$ for each $x \in [0,1]$ and studied global smoothness preservation, the approximation of decreasing and convex functions, the validity of a Voronovskaja-type theorem and a recursion formula generalizing a corresponding result for the classical Bernstein operators.

Motivated by the above work, in the present paper we introduce modified Stancu-Baskakov operators based on a function $\tau(x)$ and obtain the rate of approximation of these operators with the help of Peetre's K-functional and the Ditzian-Totik modulus of smoothness. Also, we prove a quantitative Voronovskaja-type theorem by using the first order Ditzian-Totik modulus of smoothness.

Throughout this paper, we assume that C denotes a constant not necessarily the same at each occurence.

2 Modified Stancu-Baskakov operators

Let $\tau(x)$ be continuously differentiable ∞ times on $[0,\infty)$, such that $\tau(0)=0, \tau'(x)>0$ and $\tau''(x)$ is bounded for all $x\in[0,\infty)$. We introduce a sequence of Stancu-Baskakov operators for $f\in C_B[0,\infty)$, the space of all continuous and bounded functions on $[0,\infty)$, endowed with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$, by

$$V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)=\sum_{k=0}^{\infty}p_{n,k}^{\langle\frac{1}{n},\tau\rangle}(x)\big(f\circ\tau^{-1}\big)\bigg(\frac{k}{n}\bigg),\quad x\in[0,\infty), \tag{2.1}$$

where

$$p_{n,k}^{\langle\frac{1}{n},\tau\rangle}(x)=\frac{(2n-1)!}{(n-1)!}\binom{n+k-1}{k}\frac{(n\tau(x))_k}{(n(1+\tau(x)))_{n+k}}.$$

Lemma 4 *The operator defined by* (2.1) *satisfies the following equalities*:

(i) $V_n^{\langle\frac{1}{n},\tau\rangle}(1;x)=1$,

(ii) $V_n^{\langle\frac{1}{n},\tau\rangle}(\tau(t);x)=\frac{n\tau(x)}{n-1}$,

(iii) $V_n^{\langle\frac{1}{n},\tau\rangle}(\tau^2(t);x)=\frac{n^2}{(n-1)(n-2)}[\tau^2(x)+\frac{\tau(x)(\tau(x)+1)}{n}+\frac{1}{n}(1-\frac{1}{n})\tau(x)]$,

(iv) $V_n^{\langle\frac{1}{n}\rangle}(\tau^3(t);x)=\frac{n^3}{(n-1)(n-2)(n-3)}[\frac{(n+1)(n+2)}{n^2}\tau^3(x)+\frac{3(2n^2+n-1)}{n^3}\tau^2(x)+\frac{(2n-1)(3n-1)}{n^4}\tau(x)]$,

(v) $V_n^{\langle\frac{1}{n}\rangle}(\tau^4(t);x)=\frac{n^4}{(n-1)(n-2)(n-3)(n-4)}[\frac{(n+1)(n+2)(n+3)}{n^3}\tau^4(x)+\frac{6(n+1)(n+2)(2n-1)}{n^4}\tau^3(x)+\frac{6(6n^3+n^2-4n+1)}{n^5}\tau^2(x)+\frac{26n^2-27n+7}{n^5}\tau(x)]$.

Proof The proof of lemma is straightforward on using Lemma 2. Hence we omit the details. □

Let the mth order central moment for the operators given by (2.1) be defined as

$$\mu_{n,m}^{\tau}(x)=V_n^{\langle\frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big)^m;x\big).$$

Lemma 5 *For the central moment operator* $\mu_{n,m}^{\tau}(x)$, *the following equalities hold*:

(i) $\mu_{n,1}^{\tau}(x)=\frac{\tau(x)}{n-1}$,

(ii) $\mu_{n,2}^{\tau}(x)=\frac{2n\phi_\tau^2(x)+(2\tau(x)-1)\tau(x)}{(n-1)(n-2)}$,

(iii) $\mu_{n,4}^{\tau}(x)=\frac{1}{n(n-1)(n-2)(n-3)(n-4)}[12n(n^2-13n+2)\tau^2(x)\phi_\tau^2(x)+12n(n^2+8n-13)\tau(x)\phi_\tau^2(x)+(26n^2+48n-22)\phi_\tau^2(x)+(29-75n)\tau(x)]$,

where $\phi_{\tau(x)}^2(x)=\tau(x)(\tau(x)+1)$.

Proof Using the definition (2.1) of the modified Stancu-Baskakov operators and Lemma 4, the proof of the lemma easily follows. Hence, the details are omitted. □

Let

$$W^2=\big\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\big\}.$$

For $f\in C_B[0,\infty)$ and $\delta>0$, the Peetre K-functional [13] is defined by

$$K(f;\delta)=\inf_{g\in W^2}\big\{\|f-g\|+\delta\|g\|_{W^2}\big\},$$

where

$$\|g\|_{W^2} = \|g\| + \|g'\| + \|g''\|.$$

From [14], Proposition 3.4.1, there exists a constant $C > 0$ independent of f and δ such that

$$K(f;\delta) \le C\big(\omega_2(f;\sqrt{\delta}) + \min\{1,\delta\}\|f\|\big), \tag{2.2}$$

where ω_2 is the second order modulus of smoothness of $f \in C_B[0,\infty)$ and is defined as

$$\omega_2(f;\delta) = \sup_{0<|h|\le\delta} \sup_{x,x+2h\in[0,\infty)} \big|f(x+2h) - 2f(x+h) + f(x)\big|.$$

In the following, we assume that $\inf_{x\in[0,\infty)} \tau'(x) \ge a, a \in \mathbb{R}^+ := (0,\infty)$.

Next, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [15]. Let $\phi_\tau(x) := \sqrt{\tau(x)(1+\tau(x))}$ and $f \in C_B[0,\infty)$. The first order modulus of smoothness is given by

$$\omega_{\phi_\tau}(f;t) = \sup_{0<h\le t} \left\{ \left| f\left(x + \frac{h\phi_\tau(x)}{2}\right) - f\left(x - \frac{h\phi_\tau(x)}{2}\right)\right|, x \pm \frac{h\phi_\tau(x)}{2} \in [0,\infty)\right\}.$$

Further, the appropriate K-functional is defined by

$$K_{\phi_\tau}(f;t) = \inf_{g\in W_{\phi_\tau}[0,\infty)} \left\{\|f-g\| + t\left\|\phi_\tau g'\right\|\right\} \quad (t>0),$$

where $W_{\phi_\tau}[0,\infty) = \{g : g \in AC_{\mathrm{loc}}[0,\infty), \|\phi_\tau g'\| < \infty\}$ and $g \in AC_{\mathrm{loc}}[0,\infty)$ means that g is absolutely continuous on every interval $[a,b] \subset [0,\infty)$. It is well known [15], p.11, that there exists a constant $C > 0$ such that

$$K_{\phi_\tau}(f;t) \le C\omega_{\phi_\tau}(f;t). \tag{2.3}$$

Theorem 1 *If $f \in C_B[0,\infty)$, then*

$$\left\|V_n^{\langle\frac{1}{n},\tau\rangle} f\right\| \le \|f\|.$$

Proof By the definition of the modified Stancu-Baskakov operators (2.1) and using Lemma 4 we have

$$\left|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)\right| \le \sum_{k=0}^{n} p_{n,k}^{\langle\frac{1}{n},\tau\rangle}(x)\left|\left(f\circ\tau^{-1}\right)\left(\frac{k}{n}\right)\right| \le \left\|f\circ\tau^{-1}\right\| V_n^{\langle\frac{1}{n},\tau\rangle}(1;x) = \|f\|,$$

for every $x \in [0,\infty)$. Hence the required result is immediate. □

Theorem 2 *Let $f \in C_B[0,\infty)$. Then, for $n \ge 3$, there exists a constant $C > 0$ such that*

$$\left|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x) - f(x)\right| \le C\left\{\omega_2\left(f;\frac{\phi_\tau(x)}{\sqrt{n-2}}\right) + \frac{\phi_\tau^2(x)}{n-2}\|f\|\right\} + \omega\left(f\circ\tau^{-1};\left(\frac{\tau(x)}{n-1}\right)\right),$$

on each compact subset of $[0,\infty)$.

Proof Let U be a compact subset of $[0,\infty)$. For each $x \in U$, first we define an auxiliary operator as

$$V_n^{*\langle \frac{1}{n},\tau\rangle}(f;x) = V_n^{\langle \frac{1}{n},\tau\rangle}(f;x) - f\circ\tau^{-1}\left(\frac{n\tau(x)}{n-1}\right) + f(x). \tag{2.4}$$

Now, using Lemma 4, we have

$$V_n^{*\langle \frac{1}{n},\tau\rangle}(1;x) = 1 \quad \text{and} \quad V_n^{*\langle \frac{1}{n},\tau\rangle}\big(\tau(t);x\big) = \tau(x) \quad \text{hence } V_n^{*\langle \frac{1}{n},\tau\rangle}\big(\tau(t)-\tau(x);x\big) = 0.$$

Let $g \in W^2$, $x \in U$ and $t \in [0,\infty)$. Then by Taylor's expansion, and using results in [16], p.32, we get

$$\begin{aligned}
g(t) &= \big(g\circ\tau^{-1}\big)\big(\tau(t)\big) \\
&= \big(g\circ\tau^{-1}\big)\big(\tau(x)\big) + \big(g\circ\tau^{-1}\big)'\big(\tau(x)\big)\big(\tau(t)-\tau(x)\big) + \int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\big(g\circ\tau^{-1}\big)''(u)\,du \\
&= g(x) + \big(g\circ\tau^{-1}\big)'\big(\tau(x)\big)\big(\tau(t)-\tau(x)\big) + \int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}\,du \\
&\quad - \int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}\,du.
\end{aligned} \tag{2.5}$$

Now, applying the operator $V_n^{*\langle \frac{1}{n},\tau\rangle}(\cdot;x)$ to both sides of the above equality, we get

$$\begin{aligned}
&V_n^{*\langle \frac{1}{n},\tau\rangle}(g;x) - g(x) \\
&\quad = \big(g\circ\tau^{-1}\big)'\big(\tau(x)\big)\,V_n^{*\langle \frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big);x\big) \\
&\qquad + V_n^{*\langle \frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}\,du;x\right) \\
&\qquad - V_n^{*\langle \frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}\,du;x\right) \\
&\quad = V_n^{\langle \frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}\,du;x\right) \\
&\qquad - \int_{\tau(x)}^{\frac{n\tau(x)}{n-1}} \left(\frac{n\tau(x)}{n-1}-u\right)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}\,du \\
&\qquad - V_n^{\langle \frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)} \big(\tau(t)-u\big)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}\,du;x\right) \\
&\qquad + \int_{\tau(x)}^{\frac{n\tau(x)}{n-1}} \left(\frac{n\tau(x)}{n-1}-u\right)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}\,du.
\end{aligned} \tag{2.6}$$

Again, for each $x \in U$, we have

$$\begin{aligned}
&\big|V_n^{*\langle \frac{1}{n},\tau\rangle}(g;x) - g(x)\big| \\
&\quad \le \frac{1}{2}\frac{\|g''\|}{a^2}V_n^{\langle \frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big)^2;x\big) + \frac{1}{2}\frac{\|g''\|}{a^2}\left(\frac{n\tau(x)}{n-1}-\tau(x)\right)^2
\end{aligned}$$

$$+\frac{1}{2}\frac{\|g'\|\|\tau''\|}{a^3}V_n^{\langle\frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big)^2;x\big)+\frac{1}{2}\frac{\|g'\|\|\tau''\|}{a^3}\left(\frac{n\tau(x)}{n-1}-\tau(x)\right)^2$$
$$=\frac{1}{2}\left(\frac{\|g''\|}{a^2}+\frac{\|g'\|\|\tau''\|}{a^3}\right)\left[V_n^{\langle\frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big)^2;x\big)+\left(\frac{\tau(x)}{n-1}\right)^2\right]. \tag{2.7}$$

Now, using the definition of the auxiliary operators, Theorem 1 and inequality (2.7), for each $x\in U$ we have

$$\begin{aligned}
&\big|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)\big|\\
&\quad\le\big|V_n^{*\langle\frac{1}{n},\tau\rangle}(f-g;x)\big|+\big|V_n^{*\langle\frac{1}{n},\tau\rangle}(g;x)-g(x)\big|\\
&\qquad+\big|g(x)-f(x)\big|+\left|f\circ\tau^{-1}\left(\frac{n\tau(x)}{n-1}\right)-f\circ\tau^{-1}\big(\tau(x)\big)\right|\\
&\quad\le 4\|f-g\|+\frac{1}{2}\left(\frac{\|g''\|}{a^2}+\frac{\|g'\|\|\tau''\|}{a^3}\right)\left[V_n^{\langle\frac{1}{n},\tau\rangle}\big(\big(\tau(t)-\tau(x)\big)^2;x\big)+\left(\frac{\tau(x)}{n-1}\right)^2\right]\\
&\qquad+\omega\left(f\circ\tau^{-1};\left(\frac{\tau(x)}{n-1}\right)\right).
\end{aligned}\tag{2.8}$$

Let $C=\max(4,\frac{4}{a^2},\frac{4}{a^3}\|\tau''\|)$, we get

$$\big|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)\big|\le C\left(\|f-g\|+\|g\|_{W^2}\frac{\phi_\tau^2(x)}{n-2}\right)+\omega\left(f\circ\tau^{-1};\left(\frac{\tau(x)}{n-1}\right)\right). \tag{2.9}$$

Taking the infimum on the right side of the above inequality over all $g\in W^2$ and for all $x\in U$, we have

$$\big|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)\big|\le CK\left(f;\frac{\phi_\tau^2(x)}{n-2}\right)+\omega\left(f\circ\tau^{-1};\left(\frac{\tau(x)}{n-1}\right)\right), \tag{2.10}$$

using equation (2.2), we get the required result. □

Theorem 3 *Let $f\in C_B[0,\infty)$. Then for every $x\in[0,\infty)$, and $n\ge 3$ we have*

$$\big|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)\big|\le C\omega_{\phi_\tau}\left(f;\frac{\sqrt{6}c(x)}{a\sqrt{(n-2)}}\right).$$

Proof For any $g\in W_{\phi_\tau}[0,\infty)$, by Taylor's expansion, we have

$$g(t)=\big(g\circ\tau^{-1}\big)\big(\tau(t)\big)=\big(g\circ\tau^{-1}\big)\big(\tau(x)\big)+\int_{\tau(x)}^{\tau(t)}\big(g\circ\tau^{-1}\big)'(u)\,du.$$

Applying the operator $V_n^{\langle\frac{1}{n},\tau\rangle}(\cdot;x)$ on both sides of the above equality, we get

$$\big|V_n^{\langle\frac{1}{n},\tau\rangle}(g;x)-g(x)\big|=\left|V_n^{\langle\frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)}\big(g\circ\tau^{-1}\big)'(u)\,du\right)\right|. \tag{2.11}$$

From [16], we have

$$\begin{aligned}\left|\int_{\tau(x)}^{\tau(t)}\left(g\circ\tau^{-1}\right)'(u)\,du\right| &= \left|\int_x^t \frac{g'(y)}{\tau'(y)}\tau'(y)\,dy\right| = \left|\int_x^t \frac{\phi_\tau(y)}{\phi_\tau(y)}\frac{g'(y)}{\tau'(y)}\tau'(y)\,dy\right| \\ &\le \frac{\|\phi_\tau g'\|}{a}\left|\int_x^t \frac{\tau'(y)}{\phi_\tau(y)}\,dy\right| \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\left|\int_x^t \frac{\tau'(y)}{\phi_\tau(y)}\,dy\right| &\le \left|\int_x^t\left(\frac{1}{\sqrt{\tau(y)}}+\frac{1}{\sqrt{1+\tau(y)}}\right)\tau'(y)\,dy\right| \\ &\le \left|\int_x^t \frac{1}{\sqrt{\tau(y)}}\tau'(y)\,dy\right| + \left|\int_x^t \frac{1}{\sqrt{1+\tau(y)}}\tau'(y)\,dy\right| \\ &= 2\left\{\left|\sqrt{\tau(t)}-\sqrt{\tau(x)}\right| + \left|\sqrt{1+\tau(t)}-\sqrt{1+\tau(x)}\right|\right\} \\ &< 2\left|\tau(t)-\tau(x)\right|\left(\frac{1}{\sqrt{\tau(x)}}+\frac{1}{\sqrt{1+\tau(x)}}\right) \\ &= \frac{2|\tau(t)-\tau(x)|}{\sqrt{\tau(x)(1+\tau(x))}}\left(\sqrt{1+\tau(x)}+\sqrt{\tau(x)}\right) \\ &= \frac{2|\tau(t)-\tau(x)|}{\sqrt{\tau(x)(1+\tau(x))}}c(x) \\ &= \frac{2c(x)|\tau(t)-\tau(x)|}{\phi_\tau(x)}. \end{aligned} \tag{2.13}$$

Now, from equations (2.12)-(2.13) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|V_n^{\langle\frac{1}{n},\tau\rangle}(g;x)-g(x)\right| &\le \frac{2c(x)\|\phi_\tau g'\|}{a\phi_\tau(x)}V_n^{\langle\frac{1}{n},\tau\rangle}\left(\left|\tau(t)-\tau(x)\right|;x\right) \\ &\le \frac{2c(x)\|\phi_\tau g'\|}{a\phi_\tau(x)}V_n^{\langle\frac{1}{n},\tau\rangle}\left(\left(\tau(t)-\tau(x)\right)^2;x\right)^{\frac{1}{2}} \\ &= \frac{2c(x)\|\phi_\tau g'\|}{a\phi_\tau(x)}\left[\frac{2n\phi_\tau^2(x)+(2\tau(x)-1)\tau(x)}{(n-1)(n-2)}\right]^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

Thus, for $f\in C_B[0,\infty)$ and any $g\in W_{\phi_\tau}[0,\infty)$, we have

$$\begin{aligned}\left|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)\right| &\le \left|V_n^{\langle\frac{1}{n},\tau\rangle}(f-g;x)\right| + \left|f(x)-g(x)\right| + \left|V_n^{\langle\frac{1}{n},\tau\rangle}(g;x)-g(x)\right| \\ &\le 2\|f-g\| + \frac{2c(x)\|\phi_\tau g'\|}{a\phi_\tau(x)}\left[\frac{2n\phi_\tau^2(x)+(2\tau(x)-1)\tau(x)}{(n-1)(n-2)}\right]^{\frac{1}{2}} \\ &= \frac{2c(x)\|\phi_\tau g'\|}{a}\left[\frac{2(n+1)}{(n-1)(n-2)}\right]^{\frac{1}{2}} + 2\|f-g\| \\ &\le 2\left\{\|f-g\| + \frac{\sqrt{6}c(x)}{a\sqrt{(n-2)}}\|\phi_\tau g'\|\right\}. \end{aligned} \tag{2.15}$$

Taking the infimum on the right side of the above inequality over all $g \in W_{\phi_\tau}[0,\infty)$, we get

$$\left|V_n^{\langle \frac{1}{n},\tau\rangle}(f;x)-f(x)\right| \le 2K_{\phi_\tau}\left(f;\frac{\sqrt{6}c(x)}{a\sqrt{(n-2)}}\right).$$

Finally, using equation (2.3), the theorem is immediate. □

Theorem 4 *For any $f \in C^2[0,\infty)$ and $x \in [0,\infty)$, the following inequality hold*:

$$\begin{aligned}&\left|V_n^{\langle \frac{1}{n},\tau\rangle}(f;x)-f(x)-\frac{f'(x)}{\tau'(x)}\mu_{n,1}^{\tau}(x)-\frac{1}{2}\left[\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\mu_{n,2}^{\tau}(x)\right| \\ &\quad\le \left(\mu_{n,2}^{\tau}(x)\right)^{\frac{1}{2}}\left[\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\mu_{n,2}^{\tau}(x)\right)^{\frac{1}{2}}+\frac{2\|\phi_\tau g'\|}{a}\phi_\tau^{-1}(x)\left(\mu_{n,4}^{\tau}(x)\right)^{1/2}\right].\end{aligned}$$

Proof Let $f \in C^2[0,\infty)$ and $x,t \in [0,\infty)$. Then by Taylor's expansion, we have

$$\begin{aligned}f(t)&=\left(f\circ\tau^{-1}\right)\left(\tau(t)\right)\\ &=\left(f\circ\tau^{-1}\right)\left(\tau(x)\right)+\left(f\circ\tau^{-1}\right)'\left(\tau(x)\right)\left(\tau(t)-\tau(x)\right)+\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)\,du.\end{aligned}$$

Hence,

$$\begin{aligned}&f(t)-f(x)-\left(f\circ\tau^{-1}\right)'\left(\tau(x)\right)\left(\tau(t)-\tau(x)\right)-\frac{1}{2}\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\left(\tau(t)-\tau(x)\right)^2\\ &\quad=\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''(u)\,du-\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)du\\ &\quad=\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right]du.\end{aligned}$$

Applying $V_n^{\langle \frac{1}{n},\tau\rangle}$ to both sides of the above relation, we get

$$\begin{aligned}&\left|V_n^{\langle \frac{1}{n},\tau\rangle}(f;x)-f(x)-\frac{f'(x)}{\tau'(x)}\mu_{n,1}^{\tau}(x)-\frac{1}{2}\left[\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\mu_{n,2}^{\tau}(x)\right|\\ &\quad=\left|V_n^{\langle \frac{1}{n},\tau\rangle}\left(\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left[\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right]du;x\right)\right|\\ &\quad\le V_n^{\langle \frac{1}{n},\tau\rangle}\left(\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|du\right|;x\right).\end{aligned}\tag{2.16}$$

For $g \in W_{\phi_\tau[0,\infty)}$, we have

$$\begin{aligned}&\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|du\right|\\ &\quad\le\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(g\circ\tau^{-1}\right)(u)\right|du\right|\\ &\qquad+\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(g\circ\tau^{-1}\right)(u)-\left(g\circ\tau^{-1}\right)\left(\tau(x)\right)\right|du\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(g\circ\tau^{-1}\right)\left(\tau(x)\right)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|du\right|\\
&=\left|\int_x^t\left|\left(f\circ\tau^{-1}\right)''\left(\tau(y)\right)-g(y)\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\quad+\left|\int_x^t\left|g(y)-g(x)\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\quad+\left|\int_x^t\left|g(x)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\le 2\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left|\int_x^t\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\quad+\left|\int_x^t\left|\int_x^y\left|g'(v)\right|dv\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\tau(t)-\tau(x)\right)^2+\left\|\phi_\tau g'\right\|\left|\int_x^t\left|\int_x^y\frac{dv}{\phi_\tau(v)}\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|.
\end{aligned}
$$

Using the inequality

$$
\frac{|y-v|}{v(1+v)}\le\frac{|y-x|}{x(1+x)},\quad x<v<y,
$$

we can write

$$
\frac{|\tau(y)-\tau(v)|}{\tau(v)(1+\tau(v))}\le\frac{|\tau(y)-\tau(x)|}{\tau(x)(1+\tau(x))}.
$$

Therefore,

$$
\begin{aligned}
&\left|\int_{\tau(x)}^{\tau(t)}\left|\tau(t)-u\right|\left|\left(f\circ\tau^{-1}\right)''(u)-\left(f\circ\tau^{-1}\right)''\left(\tau(x)\right)\right|du\right|\\
&\quad\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\tau(t)-\tau(x)\right)^2\\
&\qquad+\left\|\phi_\tau g'\right\|\left|\int_x^t\left|\int_x^y\frac{|\tau(y)-\tau(x)|^{1/2}}{\tau'(v)\phi_\tau(x)}\cdot\frac{\tau'(v)}{|\tau(y)-\tau(v)|^{1/2}}\,dv\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\quad\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\tau(t)-\tau(x)\right)^2\\
&\qquad+2\frac{\|\phi_\tau g'\|}{a}\phi_\tau^{-1}(x)\left|\int_x^t\left|\tau(y)-\tau(x)\right|\left|\tau(t)-\tau(y)\right|\tau'(y)\,dy\right|\\
&\quad\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\tau(t)-\tau(x)\right)^2+2\frac{\|\phi_\tau g'\|}{a}\phi_\tau^{-1}(x)\left|\int_x^t\left(\tau(t)-\tau(x)\right)^2\tau'(y)\,dy\right|\\
&\quad\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|\left(\tau(t)-\tau(x)\right)^2+2\frac{\|\phi_\tau g'\|}{a}\phi_\tau^{-1}(x)\left|\tau(t)-\tau(x)\right|^3.
\end{aligned}
\tag{2.17}
$$

Now combining equations (2.16)-(2.17), applying Lemma 3 and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\left|V_n^{\langle\frac{1}{n},\tau\rangle}(f;x)-f(x)-\frac{f'(x)}{\tau'(x)}\mu_{n,1}^{\tau}(x)-\frac{1}{2}\left[\frac{f''(x)}{(\tau'(x))^2}-f'(x)\frac{\tau''(x)}{(\tau'(x))^3}\right]\mu_{n,2}^{\tau}(x)\right|\\
&\quad\le\left\|\left(f\circ\tau^{-1}\right)''-g\right\|V_n^{\langle\frac{1}{n},\tau\rangle}\left(\left(\tau(t)-\tau(x)\right)^2;x\right)+2\frac{\|\phi_\tau g'\|}{a}\phi_\tau^{-1}(x)V_n^{\langle\frac{1}{n},\tau\rangle}\left(\left|\tau(t)-\tau(x)\right|^3;x\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\| \left(f \circ \tau^{-1}\right)'' - g \right\| V_n^{\langle \frac{1}{n}, \tau \rangle}\left(\left(\tau(t) - \tau(x)\right)^2; x\right) \\
&\quad + \frac{2\|\phi_\tau g'\|}{a} \phi_\tau^{-1}(x)\left[V_n^{\langle \frac{1}{n}, \tau \rangle}\left(\left(\tau(t) - \tau(x)\right)^2; x\right)\right]^{1/2}\left[V_n^{\langle \frac{1}{n}, \tau \rangle}\left(\left(\tau(t) - \tau(x)\right)^4; x\right)\right]^{1/2} \\
&= \left\| \left(f \circ \tau^{-1}\right)'' - g \right\| \mu_{n,2}^{\tau}(x) + \frac{2\|\phi_\tau g'\|}{a} \phi_\tau^{-1}(x)\left(\mu_{n,2}^{\tau}(x)\right)^{1/2}\left(\mu_{n,4}^{\tau}(x)\right)^{1/2} \\
&= \left(\mu_{n,2}^{\tau}(x)\right)^{\frac{1}{2}}\left[\left\| \left(f \circ \tau^{-1}\right)'' - g \right\| \left(\mu_{n,2}^{\tau}(x)\right)^{\frac{1}{2}} + \frac{2\|\phi_\tau g'\|}{a} \phi_\tau^{-1}(x)\left(\mu_{n,4}^{\tau}(x)\right)^{1/2} \right].
\end{aligned}
$$

This completes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors equally contributed to this work. All authors read and approved the final manuscript.

Author details
[1]Indian Institute of Techology Roorkee, Roorkee, 247667, India. [2]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, 27410, Turkey.

Acknowledgements
The first author is thankful to The Ministry of Human Resource and Development, India, for the financial support to carry out the above work.

Received: 15 December 2016 Accepted: 20 February 2017 Published online: 06 March 2017

References

1. Eggenberger, F, Polya, G: Über die statistik verkerter vorgänge. Z. Angew. Math. Mech. **1**, 279-289 (1923)
2. Johnson, NL, Kotz, S: Discrete Distributions. Houghton Mifflin, Boston (1969)
3. Stancu, DD: Two classes of positive linear operators. An. Univ. Vest. Timiş., Ser. Mat.-Inform. **8**, 213-220 (1970)
4. Baskakov, VA: A sequence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR **113**, 249-251 (1957)
5. Razi, Q: Approximation of a function by Kantrovich type operators. Mat. Vesn. **41**, 183-192 (1989)
6. Deo, N, Dhamija, M, Miclăus, D: Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distributions. Appl. Math. Comput. **273**, 281-289 (2016)
7. Acar, T: Asymptotic formulas for generalized Szasz-Mirakyan operators. Appl. Math. Comput. **263**, 233-239 (2015)
8. Acar, T: (p,q)-Generalization of Szasz-Mirakan operators. Math. Methods Appl. Sci. **39**(10), 2685-2695 (2016)
9. Acar, T: Quantitative q-Voronovskaya and q-Grüss-Voronovskaya-type results for q-Szász operators. Georgian Math. J. **23**(4), 459-468 (2016)
10. Stancu, DD: Approximation of functions by a new class of linear polynomial operators. Rev. Roum. Math. Pures Appl. **13**, 1173-1194 (1968)
11. King, JP: Positive linear operators which preserve x^2. Acta Math. Hung. **99**, 203-208 (2003)
12. Gonska, H, Piţul, P, Raşa, I: General King-type operators. Results Math. **53**, 279-286 (2009)
13. Peetre, J: Theory of Interpolation of Normed Spaces, 86 pp. Notas de Matematica, vol. 39. Rio de Janeiro (1963)
14. Butzer, PL, Berens, H: Semi-Groups of Operators and Approximation. Springer, Berlin (1967)
15. Ditzian, Z, Totik, V: Moduli of Smoothness. Springer, New York (1987)
16. Acar, T, Aral, A, Raşa, I: Modified Bernstein-Durrmeyer operators. Gen. Math. **22**(1), 27-41 (2014)