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Strong convergence of an extragradient-type algorithm for the multiple-sets split equality problem

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Abstract

This paper introduces a new extragradient-type method to solve the multiple-sets split equality problem (MSSEP). Under some suitable conditions, the strong convergence of an algorithm can be verified in the infinite-dimensional Hilbert spaces. Moreover, several numerical results are given to show the effectiveness of our algorithm.

Keywords: strong convergence; extragradient-type; multiple-sets split equality problem

1 Introduction

The split feasibility problem (SFP) was first presented by Censor *et al.* [1]; it is an inverse problem that arises in medical image reconstruction, phase retrieval, radiation therapy treatment, signal processing *etc.* The SFP can be mathematically characterized by finding a point x that satisfies the property

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

if such a point exists, where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded and linear operator.

There are various algorithms proposed to solve the SFP, see [2–4] and the references therein. In particular, Byrne [5, 6] introduced the CQ-algorithm motivated by the idea of an iterative scheme of fixed point theory. Moreover, Censor *et al.* [7] introduced an extension upon the form of SFP in 2005 with an intersection of a family of closed and convex sets instead of the convex set C, which is the original of the multiple-sets split feasibility problem (MSSFP).

Subsequently, an important extension, which goes by the name of split equality problem (SEP), was made by Moudafi [8]. It can be mathematically characterized by finding points $x \in C$ and $y \in Q$ that satisfy the property

$$Ax = By, \tag{1.2}$$

if such points exist, where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, H_3 is also a Hilbert space, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded and linear operators. When $B = I$, the SEP reduces to SFP. For more information about the methods for solving SEP, see [9, 10].

This paper considers the multiple-sets split equality problem (MSSEP) which generalizes the MSSFP and SEP and can be mathematically characterized by finding points x and y that satisfy the property

$$x \in \bigcap_{i=1}^{t} C_i \quad \text{and} \quad y \in \bigcap_{j=1}^{r} Q_j \quad \text{such that } Ax = By, \tag{1.3}$$

where r, t are positive integers, $\{C_i\}_{i=1}^{t} \in H_1$ and $\{Q_j\}_{j=1}^{r} \in H_2$ are nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, H_3 is also a Hilbert space, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded and linear operators. Obviously, if $B = I$, the MSSEP is just right MSSFP; if $t = r = 1$, the MSSEP changes into the SEP. Moreover, when $B = I$ and $t = r = 1$, the MSSEP reduces to the SFP.

One of the most important methods for computing the solution of a variational inequality and showing the quick convergence is an extragradient algorithm, which was first introduced by Korpelevich [11]. Moreover, this method was applied for finding a common element of the set of solutions for a variational inequality and the set of fixed points of a nonexpansive mapping, see Nadezhkina *et al.* [12]. Subsequently, Ceng *et al.* in [13] presented an extragradient method, and Yao *et al.* in [14] proposed a subgradient extragradient method to solve the SFP. However, all these methods to solve the problem have only weak convergence in a Hilbert space. On the other hand, a variant extragradient-type method and a subgradient extragradient method introduced by Censor *et al.* [15, 16] possess strong convergence for solving the variational inequality.

Motivated and inspired by the above works, we introduce an extragradient-type method to solve the MSSEP in this paper. Under some suitable conditions, the strong convergence of an algorithm can be verified in the infinite-dimensional Hilbert spaces. Finally, several numerical results are given to show the feasibility of our algorithm.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let I denote the identity operator on H.

Next, we recall several definitions and basic results that will be available later.

Definition 2.1 A mapping $T : H \to H$ goes by the name of

(i) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H;$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\| \le \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in H;$$

(iii) contractive on x if there exists $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \le \alpha \|x - y\|, \quad \forall x, y \in H;$$

(iv) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(v) β-inverse strongly monotone if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

The following properties of an orthogonal projection operator were introduced by Bauschke *et al.* in [17], and they will be powerful tools in our analysis.

Proposition 2.2 ([17]) *Let P_C be a mapping from H onto a closed, convex and nonempty subset C of H if*

$$P_C(x) = \arg\min_{y\in C} \|x - y\|, \quad \forall x \in H,$$

then P_C is called an orthogonal projection from H onto C. Furthermore, for any $x, y \in H$ and $z \in C$,

(i) $\langle x - P_C x, z - P_C x\rangle \leq 0$;
(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$;
(iii) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$.

The following lemmas provide the main mathematical results in the sequel.

Lemma 2.3 ([18]) *Let C be a nonempty closed convex subset of a real Hilbert space H, let $T : C \to H$ be α-inverse strongly monotone, and let $r > 0$ be a constant. Then, for anyx, $y \in C$,*

$$\left\|(I - rT)x - (I - rT)y\right\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\left\|T(x) - T(y)\right\|^2.$$

Moreover, when $0 < r < 2\alpha$, $I - rT$ is nonexpansive.

Lemma 2.4 ([19]) *Let $\{x^k\}$ and $\{y^k\}$ be bounded sequences in a Hilbert space H, and let $\{\beta_k\}$ be a sequence in $[0,1]$ which satisfies the condition $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$. Suppose that $x^{k+1} = (1-\beta_k)y^k + \beta_k x^k$ for all $k \geq 0$ and $\limsup_{k\to\infty}(\|y^{k+1} - y^k\| - \|x^{k+1} - x^k\|) \leq 0$. Then $\lim_{k\to\infty} \|y^k - x^k\| = 0$.*

The lemma below will be a powerful tool in our analysis.

Lemma 2.5 ([20]) *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the condition $a_{k+1} \leq (1 - m_k)a_k + m_k\delta_k, \forall k \geq 0$, where $\{m_k\}$, $\{\delta_k\}$ are sequences of real numbers such that*

(i) *$\{m_k\} \in [0,1]$ and $\sum_{k=0}^{\infty} m_k = \infty$ or, equivalently,*

$$\prod_{k=0}^{\infty}(1 - m_k) = \lim_{k\to\infty} \prod_{j=0}^{k}(1 - m_j) = 0;$$

(ii) $\limsup_{k\to\infty} \delta_k \le 0$ *or*

(ii)' $\sum_{k=0}^{\infty} \delta_k m_k$ *is convergent. Then* $\lim_{k\to\infty} a_k = 0$.

3 Main results

In this section, we propose a formal statement of our present algorithm. Review the multiple-sets split equality problem (MSSEP), without loss of generality, suppose $t > r$ in (1.3) and define $Q_{r+1} = Q_{r+2} = \cdots = Q_t = H_2$. Hence, MSSEP (1.3) is equivalent to the following problem:

$$\text{find} \quad x \in \bigcap_{i=1}^{t} C_i \quad \text{and} \quad y \in \bigcap_{j=1}^{t} Q_j \quad \text{such that } Ax = By. \tag{3.1}$$

Moreover, set $S_i = C_i \times Q_i \in H = H_1 \times H_2$ $(i = 1, 2, \ldots, t)$, $S = \bigcap_{i=1}^{t} S_i$, $G = [A, -B] : H \to H_3$, the adjoint operator of G is denoted by G^*, then the original problem (3.1) reduces to

$$\text{finding} \quad w = (x, y) \in S \quad \text{such that } Gw = 0. \tag{3.2}$$

Theorem 3.1 *Let $\Omega \neq \emptyset$ be the solution set of MSSEP* (3.2). *For an arbitrary initial point $w_0 \in S$, the iterative sequence $\{w_n\}$ can be given as follows*:

$$\begin{cases} v_n = P_S\{(1-\alpha_n)w_n - \gamma_n G^* G w_n\}, \\ w_{n+1} = P_S\{w_n - \mu_n G^* G v_n + \lambda_n (v_n - w_n)\}, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\gamma_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ are sequences in H satisfying the following conditions:

$$\begin{cases} \gamma_n \in (0, \frac{2}{\rho(G^*G)}), & \lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0; \\ \lambda_n \in (0,1), & \lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0; \\ \mu_n \le \frac{2}{\rho(G^*G)}\lambda_n, & \lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0; \\ \sum_{n=1}^{\infty}(\frac{\gamma_n}{\lambda_n}) < \infty. \end{cases} \tag{3.4}$$

Then $\{w_n\}$ converges strongly to a solution of MSSEP (3.2).

Proof In view of the property of the projection, we infer $\hat{w} = P_S(\hat{w} - tG^*G\hat{w})$ for any $t > 0$. Further, from the condition in (3.4), we get that $\mu_n \le \frac{2}{\rho(G^*G)}\lambda_n$, it follows that $I - \frac{\mu_n}{\lambda_n}G^*G$ is nonexpansive. Hence,

$$\begin{aligned} &\|w_{n+1} - \hat{w}\| \\ &\quad = \bigl\|P_S\bigl\{w_n - \mu_n G^*Gv_n + \lambda_n(v_n - w_n)\bigr\} - P_S\bigl\{\hat{w} - tG^*G\hat{w}\bigr\}\bigr\| \\ &\quad = \left\|P_S\left\{(1-\lambda_n)w_n + \lambda_n\left(I - \frac{\mu_n}{\lambda_n}G^*G\right)v_n\right\} - P_S\left\{(1-\lambda_n)\hat{w} + \lambda_n\left(I - \frac{\mu_n}{\lambda_n}G^*G\right)\hat{w}\right\}\right\| \\ &\quad \le (1-\lambda_n)\|w_n - \hat{w}\| + \lambda_n\left\|\left(I - \frac{\mu_n}{\lambda_n}G^*G\right)v_n - \left(I - \frac{\mu_n}{\lambda_n}G^*G\right)\hat{w}\right\| \\ &\quad \le (1-\lambda_n)\|w_n - \hat{w}\| + \lambda_n\|v_n - \hat{w}\|. \end{aligned} \tag{3.5}$$

Since $\alpha_n \to 0$ as $n \to \infty$ and from the condition in (3.4), $\gamma_n \in (0, \frac{2}{\rho(G^*G)})$, it follows that $\alpha_n \le 1 - \frac{\gamma_n \rho(G^*G)}{2}$ as $n \to \infty$, that is, $\frac{\gamma_n}{1-\alpha_n} \in (0, \frac{2}{\rho(G^*G)})$. We deduce that

$$\begin{aligned} &\|v_n - \hat{w}\| \\ &\quad = \big\|P_S\big\{(1-\alpha_n)w_n - \gamma_n G^*Gw_n\big\} - P_S\big(\hat{w} - tG^*G\hat{w}\big)\big\| \\ &\quad \le (1-\alpha_n)\left(w_n - \frac{\gamma_n}{1-\alpha_n}G^*Gw_n\right) - \left\{\alpha_n\hat{w} + (1-\alpha_n)\left(\hat{w} - \frac{\gamma_n}{1-\alpha_n}G^*G\hat{w}\right)\right\} \\ &\quad \le \left\|-\alpha_n\hat{w} + (1-\alpha_n)\left[w_n - \frac{\gamma_n}{1-\alpha_n}G^*Gw_n - \hat{w} + \frac{\gamma_n}{1-\alpha_n}G^*G\hat{w}\right]\right\|, \end{aligned} \tag{3.6}$$

which is equivalent to

$$\|v_n - \hat{w}\| \le \alpha_n\|-\hat{w}\| + (1-\alpha_n)\|w_n - \hat{w}\|. \tag{3.7}$$

Substituting (3.7) in (3.5), we obtain

$$\begin{aligned} \|w_n - \hat{w}\| &\le (1-\lambda_n)\|w_n - \hat{w}\| + \lambda_n\big(\alpha_n\|-\hat{w}\| + (1-\alpha_n)\|w_n - \hat{w}\|\big) \\ &\le (1-\lambda_n\alpha_n)\|w_n - \hat{w}\| + \lambda_n\alpha_n\|-\hat{w}\| \\ &\le \max\big\{\|w_n - \hat{w}\|, \|-\hat{w}\|\big\}. \end{aligned}$$

By induction,

$$\|w_n - \hat{w}\| \le \max\big\{\|w_0 - \hat{w}\|, \|-\hat{w}\|\big\}.$$

Consequently, $\{w_n\}$ is bounded, and so is $\{v_n\}$.

Let $T = 2P_S - I$. From Proposition 2.2, one can know that the projection operator P_S is monotone and nonexpansive, and $2P_S - I$ is nonexpansive.

Therefore,

$$\begin{aligned} w_{n+1} &= \frac{I+T}{2}\left[(1-\lambda_n)w_n + \lambda_n\left(1 - \frac{\mu_n}{\lambda_n}G^*G\right)v_n\right] \\ &= \frac{I-\lambda_n}{2}w_n + \frac{\lambda_n}{2}\left(I - \frac{\mu_n}{\lambda_n}G^*G\right)v_n + \frac{T}{2}\left[(1-\lambda_n)w_n + \lambda_n\left(I - \frac{\mu_n}{\lambda_n}G^*G\right)v_n\right], \end{aligned}$$

that is,

$$w_{n+1} = \frac{1-\lambda_n}{2}w_n + \frac{1+\lambda_n}{2}b_n, \tag{3.8}$$

where $b_n = \frac{\lambda_n(I-\frac{\mu_n}{\lambda_n}G^*G)v_n + T[(1-\lambda_n)w_n + \lambda_n(I-\frac{\mu_n}{\lambda_n}G^*G)v_n]}{1+\lambda_n}$.

Indeed,

$$\begin{aligned} &\|b_{n+1} - b_n\| \\ &\quad \le \frac{\lambda_{n+1}}{1+\lambda_{n+1}}\left\|\left(I - \frac{\mu_{n+1}}{\lambda_{n+1}}G^*G\right)v_{n+1} - \left(I - \frac{\mu_n}{\lambda_n}G^*G\right)v_n\right\| + \left|\frac{\lambda_{n+1}}{1+\lambda_{n+1}} - \frac{\lambda_n}{1+\lambda_n}\right| \end{aligned}$$

$$\begin{aligned}
&\times \left\| \left(I - \frac{\mu_n}{\lambda_n} G^* G\right) v_n \right\| + \frac{T}{1+\lambda_{n+1}} \left\{ (1-\lambda_{n+1}) w_{n+1} + \lambda_{n+1} \left(I - \frac{\mu_{n+1}}{\lambda_{n+1}} G^* G\right) v_{n+1} \right. \\
&\left. - \left[(1-\lambda_n) w_n + \lambda_n \left(I - \frac{\mu_n}{\lambda_n} G^* G\right) v_n \right] \right\} + \left| \frac{1}{1+\lambda_{n+1}} - \frac{1}{1+\lambda_n} \right| \\
&\times \left\| T\left[(1-\lambda_n) w_n + \lambda_n \left(I - \frac{\mu_n}{\lambda_n} G^* G\right) v_n \right] \right\|. \qquad (3.9)
\end{aligned}$$

For convenience, let $c_n = (I - \frac{\mu_n}{\lambda_n} G^* G) v_n$. By Lemma 2.5 in Shi *et al.* [1], it follows that $(I - \frac{\mu_n}{\lambda_n} G^* G)$ is nonexpansive and averaged. Hence,

$$\begin{aligned}
&\|b_{n+1} - b_n\| \\
&\quad \le \frac{\lambda_{n+1}}{1+\lambda_{n+1}} \|c_{n+1} - c_n\| + \left| \frac{\lambda_{n+1}}{1+\lambda_{n+1}} - \frac{\lambda_n}{1+\lambda_n} \right| \|c_n\| \\
&\qquad + \frac{T}{1+\lambda_{n+1}} \left\{ (1-\lambda_{n+1}) w_{n+1} + \lambda_{n+1} c_{n+1} - \left[(1-\lambda_n) w_n + \lambda_n c_n \right] \right\} \\
&\qquad + \left| \frac{1}{1+\lambda_{n+1}} - \frac{1}{1+\lambda_n} \right| \left\| T\left[(1-\lambda_n) w_n + \lambda_n c_n \right] \right\| \\
&\quad \le \frac{\lambda_{n+1}}{1+\lambda_{n+1}} \|c_{n+1} - c_n\| + \left| \frac{\lambda_{n+1}}{1+\lambda_{n+1}} - \frac{\lambda_n}{1+\lambda_n} \right| \|c_n\| \\
&\qquad + \frac{1-\lambda_{n+1}}{1+\lambda_{n+1}} \|w_{n+1} - w_n\| + \frac{\lambda_{n+1}}{1+\lambda_{n+1}} \|c_{n+1} - c_n\| + \frac{\lambda_n - \lambda_{n+1}}{1+\lambda_{n+1}} \|w_n\| \\
&\qquad + \frac{\lambda_{n+1} - \lambda_n}{1+\lambda_{n+1}} \|c_n\| + \left| \frac{1}{1+\lambda_{n+1}} - \frac{1}{1+\lambda_n} \right| \left\| T\left[(1-\lambda_n) w_n + \lambda_n c_n \right] \right\|. \qquad (3.10)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|c_{n+1} - c_n\| \\
&\quad = \left\| \left(I - \frac{\mu_{n+1}}{\lambda_{n+1}} G^* G\right) v_{n+1} - \left(I - \frac{\mu_n}{\lambda_n} G^* G\right) v_n \right\| \\
&\quad \le \|v_{n+1} - v_n\| \\
&\quad = \left\| P_S\left[(1-\alpha_{n+1}) w_{n+1} - \gamma_n G^* G w_{n+1} \right] - P_S\left[(1-\alpha_n) w_n - \gamma_n G^* G w_n \right] \right\| \\
&\quad \le \left\| \left(I - \gamma_{n+1} G^* G\right) w_{n+1} - \left(I - \gamma_{n+1} G^* G\right) w_n + (\gamma_n - \gamma_{n+1}) G^* G w_n \right\| \\
&\qquad + \alpha_{n+1} \|-w_{n+1}\| + \alpha_n \|w_n\| \\
&\quad \le \|w_{n+1} - w_n\| + |\gamma_n - \gamma_{n+1}| \left\| G^* G w_n \right\| + \alpha_{n+1} \|-w_{n+1}\| + \alpha_n \|w_n\|. \qquad (3.11)
\end{aligned}$$

Substituting (3.11) in (3.10), we infer that

$$\begin{aligned}
&\|b_{n+1} - b_n\| \\
&\quad \le \left| \frac{\lambda_{n+1}}{1+\lambda_{n+1}} - \frac{\lambda_n}{1+\lambda_n} \right| \|c_n\| + \frac{\lambda_n - \lambda_{n+1}}{1+\lambda_{n+1}} \|w_n\| + \frac{\lambda_{n+1} - \lambda_n}{1+\lambda_{n+1}} \|c_n\| \\
&\qquad + \|w_{n+1} - w_n\| + \left| \frac{1}{1+\lambda_{n+1}} - \frac{1}{1+\lambda_n} \right| \left\| T\left[(1-\lambda_n) w_n + \lambda_n c_n \right] \right\| \\
&\qquad + |\gamma_n - \gamma_{n+1}| \|w_n\| + \alpha_{n+1} \|-w_{n+1}\| + \alpha_n \|w_n\|. \qquad (3.12)
\end{aligned}$$

By virtue of $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=0$, it follows that $\lim_{n\to\infty}|\frac{\lambda_{n+1}}{1+\lambda_{n+1}}-\frac{\lambda_n}{1+\lambda_n}|=0$. Moreover, $\{w_n\}$ and $\{v_n\}$ are bounded, and so is $\{c_n\}$. Therefore, (3.12) reduces to

$$\limsup_{n\to\infty}\big(\|b_{n+1}-b_n\|-\|w_{n+1}-w_n\|\big)\le 0. \tag{3.13}$$

Applying (3.13) and Lemma 2.4, we get

$$\lim_{n\to\infty}\|b_n-w_n\|=0. \tag{3.14}$$

Combining (3.14) with (3.8), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Using the convexity of the norm and (3.5), we deduce that

$$\begin{aligned}
&\|w_{n+1}-\hat{w}\|^2\\
&\quad\le(1-\lambda_n)\|w_n-\hat{w}\|^2+\lambda_n\|v_n-\hat{w}\|^2\\
&\quad\le(1-\lambda_n)\|w_n-\hat{w}\|^2+\lambda_n\bigg\|-\alpha_n\hat{w}\\
&\qquad+(1-\alpha_n)\bigg[w_n-\frac{\gamma_n}{1-\alpha_n}G^*Gw_n-\bigg(\hat{w}-\frac{\gamma_n}{1-\alpha_n}G^*G\hat{w}\bigg)\bigg]\bigg\|^2\\
&\quad\le(1-\lambda_n)\|w_n-\hat{w}\|^2+\lambda_n\alpha_n\|-\hat{w}\|^2\\
&\qquad+(1-\alpha_n)\lambda_n\bigg[\|w_n-\hat{w}\|^2+\frac{\gamma_n}{1-\alpha_n}\bigg(\frac{\gamma_n}{1-\alpha_n}-\frac{2}{\rho(G^*G)}\bigg)\big\|G^*Gw_n-G^*G\hat{w}\big\|^2\bigg]\\
&\quad\le\|w_n-\hat{w}\|^2+\lambda_n\alpha_n\|-\hat{w}\|^2+\lambda_n\gamma_n\bigg(\frac{\gamma_n}{1-\alpha_n}-\frac{2}{\rho(G^*G)}\bigg)\big\|G^*Gw_n-G^*G\hat{w}\big\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\lambda_n\gamma_n\bigg(\frac{2}{\rho(G^*G)}-\frac{\gamma_n}{1-\alpha_n}\bigg)\big\|G^*Gw_n-G^*G\hat{w}\big\|^2\\
&\quad\le\|w_n-\hat{w}\|^2-\|w_{n+1}-\hat{w}\|^2+\lambda_n\alpha_n\|-\hat{w}\|^2\\
&\quad\le\|w_{n+1}-w_n\|\big(\|w_n-\hat{w}\|+\|w_{n+1}-\hat{w}\|\big)+\lambda_n\alpha_n\|-\hat{w}\|^2.
\end{aligned}$$

Since $\liminf_{n\to\infty}\lambda_n\gamma_n(\frac{2}{\rho(G^*G)}-\frac{\gamma_n}{1-\alpha_n})>0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|w_{n+1}-w_n\|=0$, we infer that

$$\lim_{n\to\infty}\big\|G^*Gw_n-G^*G\hat{w}\big\|=0. \tag{3.15}$$

Applying Proposition 2.2 and the property of the projection P_S, one can easily show that

$$\begin{aligned}
&\|v_n-\hat{w}\|^2\\
&\quad=\big\|P_S\big[(1-\alpha_n)w_n-\gamma_nG^*Gw_n\big]-P_S\big[\hat{w}-\gamma_nG^*G\hat{w}\big]\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \bigl\langle (1-\alpha_n)w_n - \gamma_n G^* G w_n - \bigl(\hat{w} - \gamma_n G^* G \hat{w}\bigr), v_n - \hat{w}\bigr\rangle \\
&= \frac{1}{2}\bigl\{ \bigl\| w_n - \gamma_n G^* G w_n - \bigl(\hat{w} - \gamma_n G^* G \hat{w}\bigr) - \alpha_n w_n \bigr\|^2 + \|v_n - \hat{w}\|^2 \\
&\quad - \bigl\| (1-\alpha_n) w_n - \gamma_n G^* G w_n - \bigl(\hat{w} - \gamma_n G^* G \hat{w}\bigr) - v_n + \hat{w} \bigr\|^2 \bigr\} \\
&\leq \frac{1}{2}\bigl\{ \|w_n - \hat{w}\|^2 + 2\alpha_n \|-w_n\| \bigl\| w_n - \gamma_n G^* G w_n - \bigl(\hat{w} - \gamma_n G^* G \hat{w}\bigr) - \alpha_n w_n \bigr\| \\
&\quad + \|v_n - \hat{w}\|^2 - \bigl\| w_n - v_n - \gamma_n G^* G(w_n - \hat{w}) - \alpha_n w_n \bigr\|^2 \bigr\} \\
&\leq \frac{1}{2}\bigl\{ \|w_n - \hat{w}\|^2 + \alpha_n M + \|v_n - \hat{w}\|^2 - \|w_n - v_n\|^2 \\
&\quad + 2\gamma_n \bigl\langle w_n - v_n, G^* G(w_n - \hat{w}) \bigr\rangle \\
&\quad + 2\alpha_n \langle w_n, w_n - v_n\rangle - \bigl\| \gamma_n G^* G(w_n - \hat{w}) + \alpha_n w_n \bigr\|^2 \bigr\} \\
&\leq \frac{1}{2}\bigl\{ \|w_n - \hat{w}\|^2 + \alpha_n M + \|v_n - \hat{w}\|^2 \\
&\quad - \|w_n - v_n\|^2 + 2\gamma_n \|w_n - v_n\| \bigl\| G^* G(w_n - \hat{w}) \bigr\| \\
&\quad + 2\alpha_n \|w_n\| \|w_n - v_n\| \bigr\} \\
&\leq \|w_n - \hat{w}\|^2 + \alpha_n M - \|w_n - v_n\|^2 + 4\gamma_n \|w_n - v_n\| \bigl\| G^* G(w_n - \hat{w}) \bigr\| \\
&\quad + 4\alpha_n \|w_n\| \|w_n - v_n\|,
\end{aligned}
\tag{3.16}
$$

where $M > 0$ satisfies

$$
M \geq \sup_k \bigl\{ 2\|-w_n\| \bigl\| w_n - \gamma_n G^* G w_n - \bigl(\hat{w} - \gamma_n G^* G \hat{w}\bigr) - \alpha_n w_n \bigr\| \bigr\}.
$$

From (3.5) and (3.16), we get

$$
\begin{aligned}
&\|w_{n+1} - \hat{w}\|^2 \\
&\quad \leq (1-\lambda_n)\|w_n - \hat{w}\|^2 + \lambda_n \|v_n - \hat{w}\|^2 \\
&\quad \leq \|w_n - \hat{w}\|^2 - \lambda_n \|w_n - v_n\|^2 + \alpha_n M + 4\gamma_n \|w_n - v_n\| \bigl\| \gamma_n G^* G(w_n - \hat{w}) \bigr\| \\
&\qquad + 4\alpha_n \|w_n\| \|w_n - v_n\|,
\end{aligned}
$$

which means that

$$
\begin{aligned}
\lambda_n \|w_n - v_n\|^2 \leq{} & \|w_{n+1} - w_n\| \bigl(\|w_n - \hat{w}\| + \|w_{n+1} - \hat{w}\| \bigr) + \alpha_n M \\
& + 4\gamma_n \|w_n - v_n\| \bigl\| \gamma_n G^* G(w_n - \hat{w}) \bigr\| \\
& + 4\alpha_n \|w_n\| \|w_n - v_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|w_{n+1} - w_n\| = 0$ and $\lim_{n\to\infty} \|G^* G w_n - G^* G \hat{w}\| = 0$, we infer that

$$
\lim_{n\to\infty} \|w_n - v_n\| = 0.
$$

Finally, we show that $w_n \to \hat{w}$. Using the property of the projection P_S, we derive

$$\begin{aligned}
&\|v_n - \hat{w}\|^2 \\
&\quad = \left\| P_S\left[(1-\alpha_n)\left(w_n - \frac{\gamma_n}{1-\alpha_n} G^* G w_n\right)\right] \right. \\
&\qquad \left. - P_S\left[\alpha_n \hat{w} + (1-\alpha_n)\left(\hat{w} - \frac{\gamma_n}{1-\alpha_n} G^* G \hat{w}\right)\right] \right\|^2 \\
&\quad \le \left\langle (1-\alpha_n)\left(I - \frac{\gamma_n}{1-\alpha_n} G^* G\right)(w_n - \hat{w}) - \alpha_n \hat{w}, v_n - \hat{w} \right\rangle \\
&\quad \le (1-\alpha_n)\|w_n - \hat{w}\|\|v_n - \hat{w}\| + \alpha_n \langle \hat{w}, \hat{w} - v_n\rangle \\
&\quad \le \frac{1-\alpha_n}{2}\left(\|w_n - \hat{w}\|^2 + \|v_n - \hat{w}\|^2\right) + \alpha_n \langle \hat{w}, \hat{w} - v_n\rangle,
\end{aligned}$$

which equals

$$\|v_n - \hat{w}\|^2 \le \frac{1-\alpha_n}{1+\alpha_n}\|w_n - \hat{w}\|^2 + \frac{2\alpha_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle. \tag{3.17}$$

It follows from (3.5) and (3.17) that

$$\begin{aligned}
&\|w_{n+1} - \hat{w}\|^2 \\
&\quad \le (1-\lambda_n)\|w_n - \hat{w}\|^2 + \lambda_n \|v_n - \hat{w}\|^2 \\
&\quad \le (1-\lambda_n)\|w_n - \hat{w}\|^2 + \lambda_n \left\{ \frac{1-\alpha_n}{1+\alpha_n}\|w_n - \hat{w}\|^2 + \frac{2\alpha_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle \right\} \\
&\quad \le \left(1 - \frac{2\alpha_n \lambda_n}{1+\alpha_n}\right)\|w_n - \hat{w}\|^2 + \frac{2\alpha_n \lambda_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle.
\end{aligned} \tag{3.18}$$

Since $\frac{\gamma_n}{1-\alpha_n} \in (0, \frac{2}{\rho(G^*G)})$, we observe that $\alpha_n \in (0, 1 - \frac{\gamma_n \rho(G^*G)}{2})$, then

$$\frac{2\alpha_n \lambda_n}{1-\alpha_n} \in \left(0, \frac{2\lambda_n(2 - \gamma_n \rho(G^*G))}{\gamma_n \rho(G^*G)}\right),$$

that is to say,

$$\frac{2\alpha_n \lambda_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle \le \frac{2\lambda_n(2 - \gamma_n \rho(G^*G))}{\gamma_n \rho(G^*G)}\langle \hat{w}, \hat{w} - v_n\rangle.$$

By virtue of $\sum_{n=1}^{\infty}(\frac{\lambda_n}{\gamma_n}) < \infty$, $\gamma_n \in (0, \frac{2}{\rho(G^*G)})$ and $\langle \hat{w}, \hat{w} - v_n\rangle$ is bounded, we obtain $\sum_{n=1}^{\infty}(\frac{2\lambda_n(2-\gamma_n \rho(G^*G))}{\gamma_n \rho_n(G^*G)})\langle \hat{w}, \hat{w} - v_n\rangle < \infty$, which implies that

$$\sum_{n=1}^{\infty} \frac{2\alpha_n \lambda_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle \le \infty.$$

Moreover,

$$\sum_{n=1}^{\infty} \frac{2\alpha_n \lambda_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle = \sum_{n=1}^{\infty} \frac{2\alpha_n \lambda_n}{1+\alpha_n}\frac{1+\alpha_n}{1-\alpha_n}\langle \hat{w}, \hat{w} - v_n\rangle, \tag{3.19}$$

Table 1 $\epsilon = 10^{-5}, P = 3, M = 3, N = 3$

	n	t
Sequence (3.3)	60	0.078
Tian's (3.15)'	117	0.093
Byrne's (1.2)	1,845	1.125

Table 2 $\epsilon = 10^{-10}, P = 3, M = 3, N = 3$

	n	t
Sequence (3.3)	120	0.156
Tian's (3.15)	294	0.29
Byrne's (1.2)	8,533	2.734

Table 3 $\epsilon = 10^{-5}, P = 10, M = 10, N = 10$

	n	t
Sequence (3.3)	63	0.093
Tian's (3.15)	426	0.469
Byrne's (1.2)	2,287	1.313

Table 4 $\epsilon = 10^{-10}, P = 10, M = 10, N = 10$

	n	t
Sequence (3.3)	123	0.25
Tian's (3.15)	948	0.906
Byrne's (1.2)	13,496	2.437

it follows that all the conditions of Lemma 2.5 are satisfied. Combining (3.18), (3.19) and Lemma 2.5, we can show that $w_n \to \hat{w}$. This completes the proof. □

4 Numerical experiments

In this section, we provide several numerical results and compare them with Tian's [21] algorithm (3.15)' and Byrne's [22] algorithm (1.2) to show the effectiveness of our proposed algorithm. Moreover, the sequence given by our algorithm in this paper has strong convergence for the multiple-sets split equality problem. The whole program was written in Wolfram Mathematica (version 9.0). All the numerical results were carried out on a personal Lenovo computer with Intel(R)Pentium(R) N3540 CPU 2.16 GHz and RAM 4.00 GB.

In the numerical results, $A = (a_{ij})_{P\times N}$, $B = (b_{ij})_{P\times M}$, where $a_{ij} \in [0,1]$, $b_{ij} \in [0,1]$ are all given randomly, P, M, N are positive integers. The initial point $x_0 = (1,1,\ldots,1)$, and $y_0 = (0,0,\ldots,0)$, $\alpha_n = 0.1$, $\lambda_n = 0.1$, $\gamma_n = \frac{0.2}{\rho(G^*G)}$, $\mu_n = \frac{0.2}{\rho(G^*G)}$ in Theorem 3.1, $\rho_1^n = \rho_2^n = 0.1$ in Tian's (3.15)' and $\gamma_n = 0.01$ in Byrne's (1.2). The termination condition is $\|Ax - By\| < \epsilon$. In Tables 1-4, the iterative steps and CPU are denoted by n and t, respectively.

Competing interests
The authors declare that there are no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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