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Weighted norm inequalities for multilinear Calderón-Zygmund operators in generalized Morrey spaces

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Abstract

In this paper, the authors study the boundedness of multilinear Calderón-Zygmund singular integral operators and their commutators in generalized Morrey spaces.

MSC: 42B20

Keywords: multilinear Calderón-Zygmund operators; commutators; weighted Morrey spaces

1 Introduction

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions, i.e.,

$$T:\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n).$$

In [1], it is said that a function K belongs to the class m-CZK(A,ε) if

(1) $|K(y_0,y_1,\dots,y_m)|\le\frac{A}{(\sum_{k,l=0}^{m}|y_k-y_l|)^{mn}}$,

(2) if $|y_j-y_j'|\le\frac{1}{2}\max_{0\le k\le m}|y_j-y_k|$,

$$\bigl|K(y_0,\dots,y_j,\dots,y_m)-K\bigl(y_0,\dots,y_j',\dots,y_m\bigr)\bigr|\le\frac{A|y_j-y_j'|^{\varepsilon}}{(\sum_{k,l=0}^{m}|y_k-y_l|)^{mn+\varepsilon}}$$

for some $\varepsilon>0$ and $j=0,1,2\dots,m$.

The operator T is said to be an m-linear Calderón-Zygmund operator if there exists a function $K\in m$-CZK(A,ε) defined away from the diagonal $y_0=y_1=y_2\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$ such that

$$T(f_1,\dots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m\tag{1.1}$$

for $x\notin\bigcap_{j=1}^{m}\operatorname{supp}f_j$, and that T extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q for some $1\le q_j<\infty$ with $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$.

It was shown in [1] that if $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{r}$, then an m-linear Calderón-Zygmund operator satisfies

$$T : L^{r_1} \times \cdots \times L^{r_m} \to L^r$$

when $1 < r_j < \infty$ for $j = 1, \ldots, m$ and

$$T : L^{r_1} \times \cdots \times L^{r_m} \to L^{r,\infty}$$

when $1 \leq r_j < \infty$ for $j = 1, \ldots, m$ and at least one $r_j = 1$. In particular,

$$T : L^1 \times \cdots \times L^1 \to L^{1,\infty}.$$

The theory of multiple weight associated with m-linear Calderón-Zygmund operators was developed by Lerner et al. [2]. Let $1 < p_j < \infty$ for $j = 1, \ldots, m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{p} = (p_1, \ldots, p_m)$, we say $\vec{\omega} \in A_{\vec{p}}$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}} \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B \omega_j^{1-p'_j} \right)^{1/p'_j} < \infty,$$

where B is the ball in $\mathbb{R}^n$ and $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$. They showed that if $\vec{\omega} \in A_{\vec{p}}$ then

$$\left\| T(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}. \tag{1.2}$$

If $1 \leq p_j < \infty$ for $j = 1, \ldots, m$ and at least one of the $p_j = 1$, they also proved

$$\left\| T(\vec{f}) \right\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}. \tag{1.3}$$

Let $\vec{b} = (b_1, \ldots, b_m)$ be a vector-valued locally integrable function. If $\vec{b} = (b_1, \ldots, b_m)$ in $(BMO)^m$, we denote $\|\vec{b}\|_{(BMO)^m} = \sup_{j=1,\ldots,m} \|b_j\|_{BMO}$ (see [2]). The commutator generated by an m-linear Calderón-Zygmund operator T and a $(BMO)^m$ function $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \ldots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}), \tag{1.4}$$

where each term is the commutator of b_j and T in the jth entry of T, that is,

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \ldots, f_j, \ldots, f_m) - T(f_1, \ldots, b_j f_j, \ldots, f_m).$$

Pérez and Torres [3] proved that if $\vec{b} \in (BMO)^m$ then

$$T_{\vec{b}} : L^{p_1} \times \cdots \times L^{p_m} \to L^p$$

for $1 < p_j < \infty$ and $1 < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, where $j = 1, \ldots, m$. In [2], the authors proved that if $\vec{\omega} \in A_{\vec{p}}$ and $\vec{b} \in (BMO)^m$, then

$$\left\| T_{\vec{b}}(\vec{f}) \right\|_{L^p(\nu_{\vec{\omega}})} \leq C \|\vec{b}\|_{(BMO)^m} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)} \tag{1.5}$$

for $1 < p_j < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, where $j = 1, \ldots, m$.

Feuto [4] introduced the generalized weighted Morrey space $(L^p(\omega), L^q)^\alpha$. Let $1 \leq p \leq \alpha \leq q \leq \infty$ and ω be a weight. The space $(L^p(\omega), L^q)^\alpha$ was defined to be the set of all measurable functions f satisfying $\|f\|_{(L^p(\omega),L^q)^\alpha} < \infty$, where

$$\|f\|_{(L^p(\omega),L^q)^\alpha} = \sup_{r>0} {}_r\|f\|_{(L^p(\omega),L^q)^\alpha}$$

with

$${}_r\|f\|_{(L^p(\omega),L^q)^\alpha} := \left[\int_{\mathbb{R}^n} \left(\omega\big(B(y,r)\big)^{1/\alpha - 1/p - 1/q} \|f \chi_{B(y,r)}\|_{L^p(\omega)} \right)^q \mathrm{d}y \right]^{1/q}.$$

When $\omega \equiv 1$, the space $(L^p, L^q)^\alpha$ was introduced in [5]. If $p < \alpha$ and $q = \infty$, the space $(L^p(\omega), L^\infty)^\alpha$ is just the weighted Morrey space $L^{p,\kappa}(\omega)$ with $\kappa = 1 - p/\alpha$ defined by Komori and Shirai [6].

Similarly, the weak space $(L^{p,\infty}(\omega), L^q)^\alpha$ is defined with

$${}_r\|f\|_{(L^{p,\infty}(\omega),L^q)^\alpha} := \left[\int_{\mathbb{R}^n} \left(\omega\big(B(y,r)\big)^{1/\alpha - 1/p - 1/q} \|f \chi_{B(y,r)}\|_{L^{p,\infty}(\omega)} \right)^q \mathrm{d}y \right]^{1/p}.$$

When $p = 1$, the space $(L^{1,\infty}(\omega), L^q)^\alpha$ was introduced in [4].

Feuto proved in [4] that Calderón-Zygmund singular integral operators, Marcinkiewicz operators, the maximal operators associated to Bochner-Riesz operators and their commutators are bounded on $(L^p(\omega), L^q)^\alpha$.

In this paper, we aim to study the boundedness of multilinear singular integral operators on the product of generalized Morrey spaces. Inspired by the above mentioned works, we state our main results as follows.

Theorem 1.1 *Let T be an m-linear Calderón-Zygmund operator, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{\omega} \in A_{\vec{p}}$.*

(1) *If $1 < p_j < \infty$, $j = 1, \ldots, m$ and $p \leq \alpha < q \leq \infty$, then*

$$\left\| T(\vec{f}) \right\|_{(L^p(\nu_{\vec{\omega}}),L^q)^\alpha} \lesssim \prod_{i=1}^{m} \|f_i\|_{(L^{p_i}(\omega_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

(2) *If $1 \leq p_j < \infty$, $j = 1, \ldots, m$ and at least one of $p_j = 1$, $p \leq \alpha < q \leq \infty$, then*

$$\left\| T(\vec{f}) \right\|_{(L^{p,\infty}(\nu_{\vec{\omega}}),L^q)^\alpha} \lesssim \prod_{i=1}^{m} \|f_i\|_{(L^{p_i}(\omega_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

Theorem 1.2 *Let $T_{\vec{b}}$ be a multilinear commutator, $\vec{b} \in (BMO)^m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_j < \infty$ and $\vec{\omega} \in A_{\vec{p}}$. If $p \le \alpha < q \le \infty$, then*

$$\| T_{\vec{b}}(\vec{f}) \|_{(L^p(\nu_{\vec{\omega}}),L^q)^\alpha} \lesssim \|\vec{b}\|_{(BMO)^m} \prod_{i=1}^{m} \|f_i\|_{(L^{p_i}(\omega_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

Remark 1.3 When $m = 1$, Theorem 1.1 is just Theorem 2.1 in [4] and Theorem 1.2 is just Theorem 2.5 in [4].

2 Notations and preliminaries

We first recall the definition of A_p weight. A nonnegative locally integrable function ω belongs to A_p $(p > 1)$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x)\,\mathrm{d}x \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'}\,\mathrm{d}x \right)^{p-1} < \infty,$$

where p' is the conjugate index of p, i.e., $1/p + 1/p' = 1$. We say that $\omega \in A_1$ if there is a constant $C > 0$ such that

$$\frac{1}{|B|} \int_B \omega(x)\,\mathrm{d}x \le C \inf_{x \in B} \omega(x)$$

for any ball B. If $\omega \in A_p$, then there exists $\delta > 0$ such that

$$\frac{\omega(E)}{\omega(B)} \lesssim \left(\frac{|E|}{|B|} \right)^\delta \tag{2.1}$$

for any measurable subset E of a ball B. Since the A_p classes are increasing with respect to p, we use the following notation $A_\infty = \bigcup_{p>1} A_p$. $A \lesssim B$ means $A \le CB$, where C is a positive constant independent of the main parameters. For $\lambda > 0$ and a ball $B \subset \mathbb{R}^n$, we write λB for the ball with same center as B and radius λ times radius of B.

Now we give the definition of $A_{\vec{p}}$ condition.

Definition 2.1 ([2]) Let $1 \le p_j < \infty$ for $j = 1, \ldots, m$, $\vec{p} = (p_1, \ldots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Given $\vec{\omega} = (\omega_1, \ldots, \omega_m)$, set

$$\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}.$$

We say that $\vec{\omega} \in A_{\vec{p}}$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \nu_{\vec{\omega}} \right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|B|} \int_B \omega_j^{1-p'_j} \right)^{1/p'_j} < \infty.$$

p' is the conjugate index of p. When $p_j = 1$, denote $p'_j = \infty$, $(\frac{1}{|B|} \int_B \omega_j^{1-p'_j})^{1/p'_j}$ is understood as $(\inf_B \omega_j)^{-1}$

Obviously, if $m = 1$, $A_{\vec{p}}$ is the classical A_p class. $A_{\vec{p}}$ has the following characterization.

Lemma 2.2 ([2]) *Let $\vec{\omega} = (\omega_1, \ldots, \omega_m)$. Then $\vec{\omega} \in A_{\vec{p}}$ if and only if*

$$\omega_j^{1-p'_j} \in A_{mp'_j} \quad \text{and} \quad \nu_{\vec{\omega}} \in A_{mp},$$

where the condition $\omega_j^{1-p'_j} \in A_{mp_j'}$ is understood as $\omega_j^{1/m} \in A_1$ in the case $p_j = 1$.

Lemma 2.3 ([2]) *Assume that $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ satisfies $A_{\vec{p}}$ condition. Then there exists a finite constant $r > 1$ such that $\vec{\omega} \in A_{\vec{p}/r}$.*

In order to prove the results for commutators, we need the following properties of BMO. For $b \in BMO$, $1 < p < \infty$ and $\omega \in A_\infty$, we get

$$\|b\|_{BMO} \sim \sup_B \left(\frac{1}{|B|} \int_B \left| b(x) - b_B \right|^p \mathrm{d}x \right)^{1/p} \tag{2.2}$$

and for all balls B,

$$\left(\frac{1}{\omega(B)} \int_B \left| b(x) - b_B \right|^p \omega(x) \, \mathrm{d}x \right)^{1/p} \le C\|b\|_{BMO}. \tag{2.3}$$

For all nonnegative integers k, we obtain

$$|b_{2^{k+1}B} - b_B| \le C(k+1)\|b\|_{BMO}, \tag{2.4}$$

where $\omega(B) = \int_B \omega(x) \, \mathrm{d}x$, $b_B = \frac{1}{|B|} \int_B b(x) \, \mathrm{d}x$ (see [4]).

3 Proof of the main results

Proof of Theorem 1.1 (1) Let $B = B(y, r)$ be a ball of $\mathbb{R}^n$, $f_i = f_i \chi_{2B} + f_i \chi_{(2B)^c}$ and denote $f_i \chi_{2B}$ by f_i^0 and $f_i \chi_{(2B)^c}$ by f_i^∞ ($i = 1, \ldots, m$), χ_E denotes the characteristic function of set E. For $x \in B(y, r)$, we have

$$\begin{aligned} \left| T\vec{f}(x) \right| &\le \left| T\left(f_1^0, \ldots, f_m^0\right)(x) \right| + \sum_{\alpha_1, \ldots, \alpha_m \in \{0, \infty\}} \left| T\left(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(x) \right| \\ &\quad + \left| T\left(f_1^\infty, \ldots, f_m^\infty\right)(x) \right| \\ &= I + II + III, \end{aligned}$$

where $\alpha_1, \ldots, \alpha_m$ are not all equal to 0 or ∞ at the same time. We first estimate III. Since $2^{k-1} r \le |x - y_i| \le 2^{k+2} r$, we have

$$\begin{aligned} III &= \left| \int_{(\mathbb{R}^n \setminus 2B)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(\sum_{i=1}^m |x - y_i|)^{mn}} \, \mathrm{d}\vec{y} \right| \\ &\le \int_{(\mathbb{R}^n \setminus 2B)^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(\sum_{i=1}^m |x - y_i|)^{mn}} \, \mathrm{d}\vec{y} \\ &= \sum_{k=1}^\infty \int_{(2^{k+1}B \setminus 2^k B)^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(\sum_{i=1}^m |x - y_i|)^{mn}} \, \mathrm{d}\vec{y} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{\infty} \prod_{i=1}^{m} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_i(y_i)|}{|x-y_i|^n} \, \mathrm{d}y_i \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^{m} \int_{2^{k+1}B} \left|f_i(y_i)\right| \mathrm{d}y_i, \qquad (3.1)
\end{aligned}
$$

the Hölder inequality gives us that

$$
\int_{2^{k+1}B} \left|f_i(y_i)\right| \mathrm{d}y_i \le \left(\int_{2^{k+1}B} \left|f(y_i)\right|^{p_i} \omega_i(y_i) \, \mathrm{d}y_i \right)^{1/p_i} \left(\int_{2^{k+1}B} \omega(y_i)^{-p'{}_i/p_i} \, \mathrm{d}y_i \right)^{1/p'{}_i}. \qquad (3.2)
$$

By the definition of $A_{\vec{p}}$ condition, we obtain

$$
III \lesssim \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \qquad (3.3)
$$

For II, we just consider this case: $\alpha_i = \infty$ for $i = 1, \ldots, l$ and $\alpha_j = 0$ for $j = l+1, \ldots, m$,

$$
\begin{aligned}
&\left|T\left(f_1^{\infty}, \ldots, f_l^{\infty}, f_{l+1}^0, \ldots, f_m^0\right)(x)\right| \\
&= \left| \int_{(\mathbb{R}^n \setminus 2B)^l} \int_{(2B)^{m-l}} \frac{f_1(y_1) \cdots f_m(y_m)}{(\sum_{i=1}^m |x-y_i|)^{mn}} \, \mathrm{d}\vec{y} \right| \\
&\le \int_{(\mathbb{R}^n \setminus 2B)^l} \int_{(2B)^{m-l}} \frac{|f_1(y_1) \cdots f_m(y_m)|}{(\sum_{i=1}^m |x-y_i|)^{mn}} \, \mathrm{d}\vec{y} \\
&\lesssim \prod_{i=l+1}^{m} \int_{2B} \left|f_i(y_i)\right| \mathrm{d}y_i \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^{l} \int_{2^{k+1}B \setminus 2^k B} \left|f_i(y_i)\right| \mathrm{d}y_i \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^{m} \int_{2^{k+1}B} \left|f_i(y_i)\right| \mathrm{d}y_i. \qquad (3.4)
\end{aligned}
$$

By (3.2) and the definition of $A_{\vec{p}}$ condition, we have

$$
II \lesssim \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \qquad (3.5)
$$

Combining all the cases together, we obtain

$$
\begin{aligned}
\left|T\vec{f}(x)\right| \lesssim& \left| \int_{(2B)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) \, \mathrm{d}\vec{y} \right| \\
&+ \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \qquad (3.6)
\end{aligned}
$$

Taking $L^p(\nu_{\vec{\omega}})$ norm on the ball $B(y,r)$ on both sides of (3.6), by (1.2), we get

$$
\begin{aligned}
\|T\vec{f} \chi_{B(y,r)}\|_{L^p(\nu_{\vec{\omega}})} \lesssim& \prod_{i=1}^{m} \|f_i \chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)} \\
&+ \sum_{k=1}^{\infty} \frac{(\int_B \nu_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \qquad (3.7)
\end{aligned}
$$

Multiplying both sides of (3.7) by $\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}$, by Lemma 2.2 and (2.1), we obtain

$$\begin{aligned}&\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}\|T\vec{f}\chi_{B(y,r)}\|_{L^p(\nu_{\vec{\omega}})}\\&\quad\lesssim\sum_{k=0}^{\infty}\frac{1}{2^{nk\delta(1/\alpha-1/q)}}\nu_{\vec{\omega}}\big(2^{k+1}B\big)^{1/\alpha-1/q-1/p}\prod_{i=1}^{m}\|f_i\chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i)}.\end{aligned}\tag{3.8}$$

For $\frac{1}{p_1/p}+\frac{1}{p_2/p}+\cdots+\frac{1}{p_m/p}=1$, we have

$$\begin{aligned}&\big\|\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}\|T\vec{f}\chi_{B(y,r)}\|_{L^p(\nu_{\vec{\omega}})}\big\|_{L^q(R^n)}\\&\quad\lesssim\sum_{k=0}^{\infty}\frac{1}{2^{nk\delta(1/\alpha-1/q)}}\prod_{i=1}^{m}\big\|\omega_i\big(2^{k+1}B\big)^{p/\alpha p_i-1/p_i-p/qp_i}\|f_i\chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i)}\big\|_{L^{qp_i/p}(R^n)}\end{aligned}$$

by the Hölder inequality. Since $\sum_{k=0}^{\infty}\frac{1}{2^{nk\delta(1/\alpha-1/q)}}<\infty$, we obtain the expected result

$$\|T\vec{f}\|_{(L^p(\nu_{\vec{\omega}}),L^q)^\alpha}\lesssim\prod_{i=1}^{m}\|f_i\|_{(L^{p_i}(\omega_i),L^{qp_i/p})^{\alpha p_i/p}}.$$

(2) For $\lambda>0$, by (3.6) and (1.3), we get

$$\begin{aligned}&\lambda\nu_{\vec{\omega}}\big(x\in B(y,r):\big|T\vec{f}(x)\big|>\lambda\big)^{1/p}\\&\quad\lesssim\prod_{i=1}^{m}\|f_i\chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)}+\sum_{k=1}^{\infty}\frac{(\int_B\nu_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.\end{aligned}$$

That is,

$$\begin{aligned}&\|Tf\chi_{B(y,r)}\|_{L^{p,\infty}(\nu_{\vec{\omega}})}\\&\quad\lesssim\prod_{i=1}^{m}\|f_i\chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)}+\sum_{k=1}^{\infty}\frac{(\int_B\nu_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.\end{aligned}\tag{3.9}$$

Multiplying both sides of (3.9) by $\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}$, we conclude as in the case (1). □

Proof of Theorem 1.2 It suffices to prove $T^j_{\vec{b}}$. For $B=B(y,r)$, $x\in B$

$$\begin{aligned}T^j_{\vec{b}}(\vec{f})(x)&=T^j_{\vec{b}}(\vec{f}\chi_{2B})(x)+\sum_{\alpha_1,\ldots,\alpha_m\in\{0,\infty\}}\big(b_j(x)T\big(f_1^{\alpha_1},\ldots,f_j^{\alpha_j},\ldots,f_m^{\alpha_m}\big)\\&\qquad-T\big(f_1^{\alpha_1},\ldots,b_jf_j^{\alpha_j},\ldots,f_m^{\alpha_m}\big)(x)\big)\\&\qquad+b_j(x)T\big(f_1^{\infty},\ldots,f_j^{\infty},\ldots,f_m^{\infty}\big)-T\big(f_1^{\infty},\ldots,b_jf_j^{\infty},\ldots,f_m^{\infty}\big)(x)\\&=I'+II'+III',\end{aligned}$$

where $\alpha_1,\ldots,\alpha_m$ are not all equal to 0 or ∞ at the same time. We first deal with III'.

$$\begin{aligned}\big|III'\big|&\le\big|\big(b_j(x)-b_B\big)T\big(f_1^{\infty},\ldots,f_j^{\infty},\ldots,f_m^{\infty}\big)\big|\\&\quad+\big|T\big(f_1^{\infty},\ldots,(b_j-b_B)f_j^{\infty},\ldots,f_m^{\infty}\big)(x)\big|\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\big(b_j(x)-b_B\big)\right| \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad + \sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \int_{(2^{k+1}B)^m} \prod_{i=1,i\neq j}^{m} \left|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)\right| \,d\vec{y}.
\end{aligned}
\tag{3.10}
$$

Select suitable $s>1$ to raise $\int_{(2^{k+1}B)^m} \prod_{i=1,i\neq j}^{m} |f_i(y_i)f_j(y_j)(b_j(y_j)-b_{2^{k+1}B})| \,d\vec{y}$ by the Hölder inequality and such that $\vec{\omega}\in A_{\vec{p}/s}$ by Lemma 2.3. Then characterization $A_{\vec{p}/s}$ and (2.3) yield

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \int_{(2^{k+1}B)^m} \prod_{i=1,i\neq j}^{m} \left|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)\right| \,d\vec{y} \\
&\quad\leq \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{m/s}} \left(\prod_{i=1,i\neq j}^{m} \int_{2^{k+1}B} \left|f_i(y_i)\right|^s \,dy_i\right)^{1/s} \\
&\qquad\times \left(\int_{2^{k+1}B} \left|\big(b_j(y_j)-b_{2^{k+1}B}\big)f_j(y_j)\right|^s \,dy_j\right)^{1/s} \\
&\quad\leq \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{m/s}} \prod_{i=1,i\neq j}^{m} \left(\int_{2^{k+1}B} \left|f_i(y_i)\right|^{p_i} \omega_i(y_i) \,dy_i\right)^{1/p_i} \\
&\qquad\times \left(\int_{2^{k+1}B} \omega_i(y_i)^{-s/(p_i-s)} \,dy_i\right)^{(p_i-s)/p_is} \\
&\qquad\times \left(\int_{2^{k+1}B} \left|b_j(y_j)-b_{2^{k+1}B}\right|^{p_js/(p_j-s)} \omega_i(y_j)^{-s/(p_j-s)} \,dy_j\right)^{1/s} \\
&\qquad\times \left(\int_{2^{k+1}B} \left|f_j(y_j)\right|^{p_j} \omega_j(y_j) \,dy_j\right)^{1/p_j} \\
&\quad\lesssim \|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.
\end{aligned}
\tag{3.11}
$$

So we have

$$
\begin{aligned}
\left|III'\right| &\leq \left|\big(b_j(x)-b_B\big)\right| \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad + \sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad + \|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} \nu_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.
\end{aligned}
\tag{3.12}
$$

For II', we just consider this case: $\alpha_i=\infty$ for $i=1,\dots,l$ and $\alpha_j=0$ for $j=l+1,\dots,m$. There are two cases:

$$
\begin{aligned}
&b_j(x)T\big(f_1^{\infty},\dots,f_j^{\infty},\dots,f_l^{\infty},f_{l+1}^{0},\dots,f_m^{0}\big) \\
&\quad - T\big(f_1^{\infty},\dots,b_jf_j^{\infty},\dots,f_l^{\infty},f_{l+1}^{0},\dots,f_m^{0}\big)(x)
\end{aligned}
$$

or

$$\begin{aligned}&b_j(x)T\big(f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_j^0,\dots,f_m\big)\\&\quad-T\big(f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,b_jf_j^0,\dots,f_m^0\big)(x).\end{aligned}$$

We just consider the following case:

$$\begin{aligned}&\big|b_j(x)T\big(f_1^\infty,\dots,f_j^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)\\&\qquad-T\big(f_1^\infty,\dots,b_jf_j^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)(x)\big|\\&\quad\le\big|\big(b_j(x)-b_B\big)T\big(f_1^\infty,\dots,f_j^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)\big|\\&\qquad+\big|T\big(f_1^\infty,\dots,(b_j-b_B)f_j^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0\big)(x)\big|\\&\quad\le\big|\big(b_j(x)-b_B\big)\big|\sum_{k=1}^\infty\frac{1}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}\\&\qquad+\sum_{k=1}^\infty\frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}\\&\qquad+\sum_{k=1}^\infty\frac{\prod_{i=l+1}^m\int_{2B}|f_i(y_i)|\,\mathrm{d}y_i}{|2^{k+1}B|^m}\int_{(2^{k+1}B)^m}\prod_{i=1,i\neq j}^l\big|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)\big|\,\mathrm{d}\vec{y}.\end{aligned}\tag{3.13}$$

The estimate for

$$\sum_{k=1}^\infty\frac{\prod_{i=l+1}^m\int_{2B}|f_i(y_i)|\,\mathrm{d}y_i}{|2^{k+1}B|^m}\int_{(2^{k+1}B)^m}\prod_{i=1,i\neq j}^l\big|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)\big|\,\mathrm{d}\vec{y}$$

is similar to (3.11). We get

$$\begin{aligned}&\sum_{k=1}^\infty\frac{\prod_{i=l+1}^m\int_{2B}|f_i(y_i)|\,\mathrm{d}y_i}{|2^{k+1}B|^m}\int_{(2^{k+1}B)^m}\prod_{i=1,i\neq j}^l\big|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)\big|\,\mathrm{d}\vec{y}\\&\quad\le\sum_{k=1}^\infty\frac{1}{|2^{k+1}B|^m}\prod_{i=1,i\neq j}^m\int_{2^{k+1}B}\big|f_i(y_i)\big|\,\mathrm{d}y_i\times\int_{2^{k+1}B}\big|b_j(y_j)-b_{2^{k+1}B}\big|f(y_j)\,\mathrm{d}y_j\\&\quad\lesssim\|b_j\|_{BMO}\sum_{k=1}^\infty\frac{1}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)},\end{aligned}\tag{3.14}$$

so we have

$$\begin{aligned}\big|T_{\vec{b}}^j(\vec{f})(x)\big|&\le\big|T_{\vec{b}}^j(\vec{f}\chi_{2B})(x)\big|+\big|\big(b_j(x)-b_B\big)\big|\sum_{k=1}^\infty\frac{1}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f\chi_{2^{k+1}B}\|_{L^{p^i}(\omega_i)}\\&\quad+\sum_{k=1}^\infty\frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f_i\chi_{2^{k+1}B}\|_{L^{p^i}(\omega_i)}\\&\quad+\|b_j\|_{BMO}\sum_{k=1}^\infty\frac{1}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m\|f_i\chi_{2^{k+1}B}\|_{L^{p^i}(\omega_i)}.\end{aligned}\tag{3.15}$$

Take $L^p(\nu_{\vec{\omega}})$ norm on the ball $B(y,r)$ on both sides of (3.15). By (1.5), (2.3), (2.4), we have

$$\begin{aligned} &\big\| T^j_{\vec{b}}(\vec{f})\chi_{B(y,r)}\big\|_{L^p(\nu_{\vec{\omega}})} \\ &\quad\lesssim \|b_j\|_{BMO}\prod_{i=1}^m \|f_i\chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)} \\ &\qquad + \|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{k(\int_B \nu_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B}\nu_{\vec{\omega}})^{1/p}}\prod_{i=1}^m \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.16}$$

Multiplying both sides of (3.16) by $\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}$, by Lemma 2.2 and (2.1), we obtain

$$\begin{aligned} &\nu_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}\big\| T^j_{\vec{b}}(\vec{f})\chi_{B(y,r)}\big\|_{L^p(\nu_{\vec{\omega}})} \\ &\quad\lesssim \sum_{k=0}^{\infty}\frac{(k+1)\|b_j\|_{BMO}}{2^{nk\delta(1/\alpha-1/q)}}\nu_{\vec{\omega}}\big(2^{k+1}B\big)^{1/\alpha-1/q-1/p}\prod_{i=1}^m \|f_i\chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.17}$$

Next the proof is similar to Theorem 1.1 of (1), we get

$$\big\| T^j_{\vec{b}}(\vec{f})\big\|_{(L^p(\nu_{\vec{\omega}}),L^q)^\alpha} \lesssim \|\vec{b}\|_{(BMO)^m}\prod_{i=1}^m \|f_i\|_{(L^{p_i}(\omega_i),L^{qp_i/p})^{\alpha p_i/p}}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PW put forward the ideas of the paper, and the authors completed the paper together. They also read and approved the final manuscript.

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