Jiang *Journal of Inequalities and Applications* (2017) 2017:39
DOI 10.1186/s13660-017-1307-1

Journal of Inequalities and Applications
a SpringerOpen Journal

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Gevrey-smoothness of lower dimensional hyperbolic invariant tori for nearly integrable symplectic mappings

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Abstract

This paper provides a normal form for a class of lower dimensional hyperbolic invariant tori of nearly integrable symplectic mappings with generating functions. We prove the persistence and the Gevrey-smoothness of the invariant tori under some conditions.

Keywords: symplectic mappings; KAM iteration; Gevrey-smoothness; normal form

1 Introduction and main results

Area-preserving mappings have some dynamical properties similar to Hamiltonian systems, and hence become an important test ground of all kinds of theories for studying Hamiltonian systems, such as Poincaré [1, 2] on three body problem, Moser [3] on the differentiable form of KAM theory, Aubry and Mather [4–8] on Aubry-Mather theorem, Conley and Zehnder [9, 10] on symplectic topology. So area-preserving mappings have attracted many scholars' interest. We refer to [11–17]. Among all the mappings, symplectic mappings are special for their symplectic structures; we refer to [18–21] for more results on symplectic structures.

On the other hand, many mathematicians turn to the study of the connection between the KAM tori and the parameter. The first work is due to Pöschel [22] who proved that the persisting invariant tori are C^∞-smooth in the frequency parameter. Popov [23] obtained the Gevrey-smoothness, a notion intermediate between C^∞-smoothness and analyticity, of invariant tori in the frequencies under the Kolmogorov non-degeneracy condition. Xu and You [24] obtained a similar result under the Rüssmann non-degeneracy condition by an improved KAM method. For more results, we refer to [25, 26].

Motivated by [19, 24], we consider the persistence and the Gevrey-smoothness of lower dimensional hyperbolic invariant tori for symplectic mappings determined by generating functions under Rüssmann's non-degeneracy condition. We consider the following parameterized symplectic mapping:

$$\Phi(\cdot;\xi):(x,u,y,v)\in\mathbb{T}^n\times\mathcal{W}\times\mathcal{O}\times\mathcal{W}\to(\hat{x},\hat{u},\hat{y},\hat{v})\in\mathbb{T}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m,$$

where $\xi \in \Pi \subset \mathcal{O}$ is a parameter and $\mathcal{O} \subset \mathbb{R}^n$ is a bounded closed connected domain. Suppose $\Phi(\cdot;\xi)$ is implicitly defined by

$$\begin{cases} \hat{x} = \partial_{\hat{y}} H(x,u,\hat{y},\hat{v};\xi), \\ y = \partial_x H(x,u,\hat{y},\hat{v};\xi), \\ \hat{u} = \partial_{\hat{v}} H(x,u,\hat{y},\hat{v};\xi), \\ v = \partial_u H(x,u,\hat{y},\hat{v};\xi), \end{cases} \tag{1.1}$$

where

$$H(x,u,\hat{y},\hat{v};\xi) = N + P, \tag{1.2}$$

$$N(x,u,\hat{y},\hat{v};\xi) = \langle x + \omega(\xi), \hat{y} \rangle + \langle Au, \hat{v} \rangle + \frac{1}{2}\langle Bu, u \rangle + \frac{1}{2}\langle C\hat{v}, \hat{v} \rangle. \tag{1.3}$$

Suppose A is a constant matrix and B, C are symmetric. If $P = 0$, Φ can be expressed explicitly as

$$\begin{aligned} &\hat{x} = x + \omega(y), \qquad \hat{y} = y, \\ &\hat{u} = \bigl(A - C\bigl(A^T\bigr)^{-1}B\bigr)u + C\bigl(A^T\bigr)^{-1}v, \qquad \hat{v} = -\bigl(A^T\bigr)^{-1}Bu + \bigl(A^T\bigr)^{-1}v. \end{aligned} \tag{1.4}$$

We define

$$\Omega = \begin{pmatrix} A - C(A^T)^{-1}B & C(A^T)^{-1} \\ -(A^T)^{-1}B & (A^T)^{-1} \end{pmatrix}_{2m\times 2m}.$$

Denote the eigenvalues of Ω by $(\lambda_1, \lambda_2, \ldots, \lambda_{2m})$. We call the lower dimensional invariant torus elliptic if $|\lambda_i| = 1$, $\lambda_i \neq 1$, $\forall i = 1,2,\ldots,2m$ and hyperbolic if $|\lambda_i| \neq 1, \forall i = 1,2,\ldots,2m$.

We note that, although some results on symplectic mappings can be anticipated by Hamiltonian systems, there are still many differences for lower dimensional invariant tori. The first one is concerned with the relations of variables. In symplectic mappings, some variables determined by generating functions take on an implicit form and hence lead to more difficulties than in a Hamiltonian system. The second one is the non-degeneracy condition, which will result in a more complicated proof for estimate of measure.

Before presenting the main result, we give some assumptions and definitions.

Assumption 1 (Rüssmann's non-degeneracy condition) There exists an integer $\bar{n} > 1$ such that

$$\operatorname{rank}\bigl\{\partial_\xi^\beta \omega(\xi) : 1 \le |\beta| \le \bar{n}\bigr\} = n, \quad \forall \xi \in \Pi, \tag{1.5}$$

where

$$\partial_\xi^\beta \omega(\xi) = \bigl(\partial_\xi^\beta \omega_1(\xi), \partial_\xi^\beta \omega_2(\xi), \ldots, \partial_\xi^\beta \omega_n(\xi)\bigr)^T$$

with

$$\partial_\xi^\beta \omega_i(\xi) = \frac{\partial_\xi^{|\beta|} \omega_i(\xi)}{\partial \xi_1^{\beta_1} \partial \xi_1^{\beta_2} \cdots \partial \xi_n^{\beta_n}},$$

and

$$\beta = (\beta_1, \beta_2, \dots, \beta_n).$$

Assumption 2 (Hyperbolic condition) Let $A = \operatorname{diag}(a_1, a_2, \dots, a_m), B = \operatorname{diag}(b_1, b_2, \dots, b_m)$, $C = \operatorname{diag}(c_1, c_2, \dots, c_m)$. Define $\Delta_i = \frac{a_i^2 - b_i c_i + 1}{a_i}, i = 1, 2, \dots, m$. Suppose $\Delta_i^2 > 4, i = 1, 2, \dots, m$.

Remark 1.1 By direct calculation, we have the eigenvalues of Ω,

$$\lambda_i = \frac{\Delta_i \pm \sqrt{(\Delta_i)^2 - 4}}{2}, \quad i = 1, 2, \dots, m.$$

If $\Delta_i^2 > 4$, we have $|\lambda_i| \neq 1, i = 1, 2, \dots, m$, so the lower dimensional invariant torus is hyperbolic. If otherwise $\Delta_i^2 < 4$, we have $|\lambda_i| = 1$, and hence the lower dimensional invariant torus is elliptic.

Definition 1.1 Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded closed connected domain. A function $F : \mathcal{O} \to \mathbb{R}$ is said to belong to Gevrey-class $G^\mu(\mathcal{O})$ of index μ ($\mu \geq 1$) if F is $C^\infty(\mathcal{O})$-smooth and there exists a constant J such that for all $p \in \mathcal{O}$,

$$\left|\partial_p^\beta F(p)\right| \leq cJ^{|\beta|+1} \beta!^\mu,$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ and $\beta!^\mu = \beta_1! \beta_2! \cdots \beta_n!$ for $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}_+^n$.

Remark 1.2 By definition, it is easy to see that the Gevrey-smooth function class G^1 coincides with the analytic function class. Moreover, we have

$$G^1 \subset G^{\mu_1} \subset G^{\mu_2} \subset C^\infty$$

for $1 < \mu_1 < \mu_2 < \infty$.

Set

$$\mathcal{T}_s = \left\{x \in \mathbb{C}^n / 2\pi \mathbb{Z}^n : |\operatorname{Im} x|_\infty \leq s\right\},$$
$$\mathcal{B}_r = \left\{y \in \mathbb{C}^n : |y|_1 \leq r^2\right\},$$

and

$$\mathcal{W}_r = \left\{w \in \mathbb{C}^m : |w|_2 \leq r\right\}.$$

Denote by $\mathcal{D}(s, r) = \mathcal{T}_s \times \mathcal{W}_r \times \mathcal{B}_r \times \mathcal{W}_r$. Here, $|x|_\infty = \max_{1 \leq j \leq n} |x_j|$, $|y|_1 = \sum_{1 \leq j \leq n} |y_j|$ and $|w|_2 = (\sum_{1 \leq j \leq m} |w_j|^2)^{\frac{1}{2}}$.

Denote

$$\Pi = \left\{\xi \in \mathcal{O} \mid \operatorname{dist}(\xi, \partial \mathcal{O}) \geq h\right\}$$

and

$$\Pi_h = \{\xi \in \mathbb{C}^n \mid \operatorname{dist}(\xi, \Pi) \le h\}.$$

Definition 1.2 $f \in G^{1,\mu}(\mathcal{D}(s,r) \times \Pi)$ means that $f \in C^\infty(\mathcal{D}(s,r) \times \Pi)$ and $f(x,y,u,v;\xi)$ is analytic with respect to (x,y,u,v) on $\mathcal{D}(s,r)$ and G^μ-smooth in ξ on Π.

Below we define some norms. If $P(x,u,\hat{y},\hat{v};\xi)$ is analytic in $(x,u,\hat{y},\hat{v})$ on $\mathcal{D}(s,r)$ and $\bar{n}$-times continuously differentiable in ξ on Π, we have

$$P(x,u,\hat{y},\hat{v};\xi) = \sum_{k\in\mathbb{Z}^n} P_k(u,\hat{y},\hat{v};\xi)e^{\mathrm{i}\langle k,x\rangle},$$

where

$$P_k(u,\hat{y},\hat{v};\xi) = \sum_{l,i,j} P_{klij}(\xi)\hat{y}^l u^i \hat{v}^j.$$

Define

$$\|P\|_{D(s,r)\times\Pi} = \sum_{k\in\mathbb{Z}^n} |P_k|_r e^{s|k|},$$

where

$$|P_k|_r = \sup_{(u,\hat{y},\hat{v})\in\mathcal{W}_r\times\mathcal{B}_r\times\mathcal{W}_r} \sum_{i,j,l} \|P_{klij}\|_s \hat{y}^l u^i \hat{v}^j.$$

This norm is apparently stronger than the super-norm. Moreover, the Cauchy estimate of analytic functions is also valid under this norm.

Let $X_P = (-\partial_{\hat{y}}P, -\partial_{\hat{v}}P, \partial_x P, \partial_u P)$ and denote a weighted norm by

$$\begin{aligned}\|X_P\|_{r;\mathcal{D}(s,r)\times\Pi} &= \|\partial_{\hat{y}}P\|_{D(s,r)\times\Pi} + \frac{1}{r}\|\partial_{\hat{v}}P\|_{D(s,r)\times\Pi} \\ &\quad + \frac{1}{r^2}\|\partial_x P\|_{D(s,r)\times\Pi} + \frac{1}{r}\|\partial_u P\|_{D(s,r)\times\Pi},\end{aligned}$$

where

$$\|\partial_{\hat{x}}P\|_{D(s,r)\times\Pi} = \sum_j \|\partial_{\hat{x}_j}P\|_{D(s,r)\times\Pi},$$

$$\|\partial_{\hat{y}}P\|_{D(s,r)\times\Pi} = \max_j \|\partial_{\hat{y}_j}P\|_{D(s,r)\times\Pi}$$

and

$$\|\partial_u P\|_{D(s,r)\times\Pi} = \left(\sum_j \left(\|\partial_{u_j}P\|_{s,r}\right)^2\right)^{\frac{1}{2}},$$

$\|\partial_{\hat{v}}P\|_{D(s,r)\times\Pi}$ is defined similarly.

Now we introduce the main result. Let $\tau \geq n\bar{n} - 1$. For $\delta \in (0,1)$, let $\mu = \tau + \delta + 2$ and $\sigma = (\frac{3}{4})^{\frac{\delta}{\tau+1+\delta}}$.

Theorem 1.1 *Consider the symplectic mapping $\Phi(\cdot;\xi)$, which is implicitly defined by a generating function $H(\cdot;\xi)$ in* (1.2). *Let $\max_{\xi\in\Pi_h}|\frac{\partial\omega}{\partial\xi}| \leq T$. Suppose that Assumptions* 1, 2 *hold. Then there exists $\gamma > 0$ such that for any $0 < \alpha < 1$, if $\|X_P\|_{r;\mathcal{D}(s,r)\times\Pi_d} = \epsilon \leq \gamma^3\alpha^{2\bar{\nu}}\rho^{2\nu}$ with $\bar{\nu} = \bar{n}+1, \nu = \tau(\bar{n}+1)+n+\bar{n}$, the following results hold true*:

(i) *There exist a non-empty Cantor-like subset $\Pi_* \subset \Pi$ and, for $\xi \in \Pi_*$, a symplectic mapping $\Psi_*(\cdot;\xi)$, where $\Psi_* \in G^{1,\mu}$ with*

$$\left\|\partial_\xi^\beta(\Psi_* - \mathrm{id})\right\|_{r;D(\frac{s}{2},\frac{r}{2})\times\Pi_*} \leq c\rho^\nu J^{|\beta|}\beta!^\mu\gamma^{\frac{9}{4(n+1)}} \tag{1.6}$$

for $\forall\beta \in Z_n^+$ and $J = \frac{2T+1}{\alpha}[\frac{4(\mu-1)(n+1)}{3}]^{\mu-1}$. Moreover, $\Phi_* = \Psi_*^{-1}\circ\Phi\circ\Psi_*$ is generated by $H_* = N_* + P_*$ as in (1.1) *satisfying*

$$N_*(x,u,\hat{y},\hat{v};\xi) = \langle x+\omega_*,\hat{y}\rangle + \langle A_*u,\hat{v}\rangle + \frac{1}{2}\langle B_*u,u\rangle + \frac{1}{2}\langle C_*\hat{v},\hat{v}\rangle,$$

$$P_*(x,u,\hat{y},\hat{v};\xi) = \sum_{|i|+|j|+2|l|\geq 3} P_{lij}(x;\xi)\hat{y}^l u^i\hat{v}^j.$$

(ii) *Hence, for $\xi \in \Pi_*$, the symplectic mapping $\Phi(\cdot;\xi)$ admits a lower dimensional invariant torus*

$$T_\xi = \Psi_*(T^n,0,0,0;\xi),$$

whose frequencies ω_* satisfy

$$\left|\partial_\xi^\beta(\omega_*(\xi) - \omega(\xi))\right| \leq c\rho^{2\nu}J^{|\beta|}\beta!^\mu\gamma^{\frac{9}{4(n+1)}} \tag{1.7}$$

and

$$\left|\langle\omega_*(\xi),k\rangle + 2\pi l\right| \geq \frac{\alpha}{(1+|k|)^\tau} \tag{1.8}$$

for all $\xi \in \Pi_*$, $0 \neq k \in Z^n$. Moreover, we have

$$\mathrm{meas}(\Pi\setminus\Pi_*) \leq c\alpha^{\frac{1}{m}}.$$

2 The proof of main results

We will use the idea for Hamiltonian systems in [24] to prove our results. In Section 2.1, one KAM step iteration is presented. The key lies in solving a homological equation. Then we will show the KAM step can iterate infinitely in Section 2.2. Convergence of the iteration and the estimate of measure will be presented in Sections 2.3 and 2.4.

2.1 KAM-step

Iteration Lemma *Consider a symplectic mapping $\Phi(\cdot;\xi)$ defined in Theorem* 1.1. *Let $0 < E < 1, 0 < \rho = (1-\sigma)s/10 < \frac{s}{5}$ and $0 < \eta < \frac{1}{8}$. Let $K > 0$ satisfy $\eta^2 e^{-K\rho} = E$. Let*

$$\max_{\xi\in\Pi_h}\left|\frac{\partial\omega}{\partial\xi}\right| \leq T, \qquad h = \frac{\alpha}{2(1+K)^{\tau+1}T}.$$

Moreover, $\omega(\xi)$ *satisfies that: for* $k \in \mathbb{Z}^n \setminus \{0\}$, $l \in \mathbb{Z}$,

$$\bigl|\langle k,\omega\rangle + 2\pi l\bigr| \geq \frac{2\alpha}{(1+|k|)^{\tau}}. \tag{2.1}$$

Suppose Assumptions 1, 2 *hold. Suppose that* P *satisfies*

$$\|X_P\|_{r;D(s,r)\times\Pi_d} \leq \epsilon = \eta^2\alpha^{2\bar{\nu}}\rho^{2\nu}E,$$

with $0<\alpha<1, \bar{\nu}=\bar{n}+1, \nu=\tau(\bar{n}+1)+n+\bar{n}$. *Then we have the following results:*

(1) $\forall\xi\in\Pi_h$, *there exists a symplectic diffeomorphism* $\Psi(\cdot;\xi)$ *with*

$$\|\Psi - \mathrm{id}\,\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi_h} \leq \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}},$$
$$\|D\Psi - \mathrm{id}\,\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi_h} \leq \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu+1}},$$

such that the conjugate mapping $\Phi_+(\cdot;\xi)=\Psi^{-1}\circ\Phi\circ\Psi$ *is generated by* $H_+(\cdot;\xi)=N_+ + P_+$, *where*

$$N_+ = \bigl\langle x+\omega_+(\xi),\hat{y}\bigr\rangle + \langle A_+u,\hat{v}\rangle + \frac{1}{2}\langle B_+u,u\rangle + \frac{1}{2}\langle C_+\hat{v},\hat{v}\rangle$$

and P_+ *satisfies*

$$\|X_P\|_{r_+;D(s_+,r_+)\times\Pi_d} \leq \eta_+^2\alpha_+^{2\bar{\nu}}\rho_+^{\nu}E_+ = \epsilon_+,$$

with

$$s_+ = s-5\rho, \qquad \rho_+ = \sigma\rho, \qquad \eta_+ = E_+,$$
$$r_+ = \eta r, \qquad E_+ = E^{\frac{4}{3}}, \qquad \frac{\alpha}{2} \leq \alpha_+ \leq \alpha.$$

Furthermore, we have

$$\bigl|\omega_+(\xi) - \omega(\xi)\bigr| \leq \epsilon, \quad \forall\xi\in\Pi_h. \tag{2.2}$$

(2) *Let* $\alpha_+ = \alpha - (K+1)^{\tau+1}\epsilon$,

$$\bar{\Pi} = \left\{\xi\in\Pi : \bigl|\bigl\langle k,\omega_+(\xi)\bigr\rangle\bigr| < \frac{2\alpha_+}{(1+|k|)^{\tau}}, k\in Z^n, K<|k|\leq K_+\right\}, \tag{2.3}$$

and $\Pi_+ = \Pi\setminus\bar{\Pi}$. *Then, for* $\forall\xi\in\Pi_+, \forall k\in Z^n$ *and* $0<|k|\leq K_+$, *we have*

$$\bigl|\bigl\langle k,\omega_+(\xi)\bigr\rangle\bigr| \geq \frac{2\alpha_+}{(1+|k|)^{\tau}}, \tag{2.4}$$

where $K_+>0$ *such that* $\frac{e^{-K_+\rho_+}}{\eta_+^2} = E_+$.

(3) *Let* $T_+ = T + \frac{6\epsilon}{h}$ *and* $h_+ = \frac{\alpha_+}{2(K_+ +1)^{\tau+1}T_+}$. *If* $h_+ \leq \frac{5}{6}h$, *we have* $\max_{\xi\in\Pi_{h_+}}|\frac{\partial\omega_+}{\partial\xi}| \leq T_+$, *where* Π_{h_+} *is the complex* h_+*-neighborhood of* Π_+.

A. The equivalent form of (1.2).

Let

$$P(p,\hat{q}) = P_{000}(x) + \langle P_{100}(x),\hat{y}\rangle + \langle P_{010}(x),u\rangle + \langle P_{001}(x),\hat{v}\rangle + \langle P_{011}(x)u,\hat{v}\rangle + \frac{1}{2}\langle P_{020}(x)u,u\rangle + \frac{1}{2}\langle P_{002}(x)\hat{v},\hat{v}\rangle + \sum_{|i|+|j|+2|l|\geq 3} P_{lij}\hat{y}^l u^i \hat{v}^j \tag{2.5}$$

with

$$P_{lij} = \left.\frac{\partial^{l+i+j}P}{\partial\hat{y}^l\partial u^i\partial\hat{v}^j}\right|_{u=0,\hat{y}=0,\hat{v}=0}.$$

Let

$$Q(x,u,\hat{v}) = \langle Q_2(x)u,\hat{v}\rangle + \frac{1}{2}\langle Q_1(x)u,u\rangle + \frac{1}{2}\langle Q_3(x)\hat{v},\hat{v}\rangle$$

with $Q_1(x) = P_{020}(x)$, $Q_2(x) = P_{011}(x)$, $Q_3(x) = P_{002}(x)$.

Then we rewrite H as

$$H = N + Q + (P - Q),$$

where $N + Q$ is the new main term and $P - Q$ is the new small term.

Now we will study the following function which is equivalent to (1.2):

$$H(x,u,\hat{y},\hat{v};\xi) = N(x,u,\hat{y},\hat{v};\xi) + P(x,u,\hat{y},\hat{v};\xi), \tag{2.6}$$

where

$$N = \langle x + \omega(\xi),\hat{y}\rangle + \langle Au,\hat{v}\rangle + \frac{1}{2}\langle Bu,u\rangle + \frac{1}{2}\langle C\hat{v},\hat{v}\rangle + Q(x,u,\hat{v})$$

and

$$P = P_{000}(x) + \langle P_{100}(x),\hat{y}\rangle + \langle P_{010}(x),u\rangle + \langle P_{001}(x),\hat{v}\rangle + \sum_{2|l|+|i|+|j|\geq 3} P_{lij}\hat{y}^l u^i \hat{v}^j.$$

B. Generating functions of conjugate mappings.

For convenience, let $p = (x,u)$ and $q = (y,v)$. $\hat{p}$ and $\hat{q}$ have a similar meaning. The symplectic structure becomes $dp \wedge dq$ on $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$. Consider a symplectic mapping $\Phi : (p,q) \to (\hat{p},\hat{q})$ generated by

$$\hat{p} = \partial_{\hat{q}}H(p,\hat{q}), \qquad q = \partial_p H(p,\hat{q}), \tag{2.7}$$

where $H(p,\hat{q}) = N(p,\hat{q}) + P(p,\hat{q})$, where N is the main term and P is a small perturbation.

We need a symplectic transformation $\Psi : (p_+,q_+) \to (p,q)$ generated by

$$q = q_+ + F_1(p,q_+), \qquad p_+ = p + F_2(p,q_+), \tag{2.8}$$

with

$$F_1(p,q_+) = \frac{\partial F(p,q_+)}{\partial p},$$
$$F_2(p,q_+) = \frac{\partial F(p,q_+)}{\partial q_+}.$$

The generating function is $\langle p,q_+\rangle + F(p,q_+)$ with F being a small function.

By (2.7) and (2.8), we have a conjugate mapping $\Phi = \Psi^{-1} \circ \Phi \circ \Psi : (p_+,q_+) \to (\hat{p}_+,\hat{q}_+)$ implicitly by

$$\hat{p}_+ = H_2(p,\hat{q}) + F_2(\hat{p},\hat{q}_+), \qquad q_+ = H_1(p,\hat{q}) - F_1(p,q_+) \tag{2.9}$$

with

$$H_1(p,\hat{q}) = \frac{\partial H(p,\hat{q})}{\partial p},$$
$$H_2(p,\hat{q}) = \frac{\partial H(p,\hat{q})}{\partial \hat{q}}.$$

So we have the following lemma.

Lemma 2.1 ([19]) *The conjugate symplectic mapping Φ_+ can be implicitly determined by a generating function $H_+(p_+,\hat{q}_+)$ through*

$$\hat{p}_+ = \partial_{\hat{q}_+} H_+(p_+,\hat{q}_+), \qquad q_+ = \partial_{p_+} H_+(p_+,\hat{q}_+), \tag{2.10}$$

where

$$\begin{aligned} H_+(p_+,\hat{q}_+) = {} & H(p,\hat{q}) + H_1(p,\hat{q})F_2(p,q_+) - H_2(p,\hat{q})F_1(\hat{p},\hat{q}_+) \\ & + F(\hat{p},\hat{q}_+) - F(p,q_+) - F_1(p,q_+)F_2(p,q_+), \end{aligned} \tag{2.11}$$

where $p,\hat{p},\hat{q},q_+$ depend on $(p_+,\hat{q}_+)$ as explained above.

Set $z = (p_+,\hat{q}_+)$. We have

$$H_+(z) = H(z) + F\bigl(N_2(z),\hat{q}_+\bigr) - F\bigl(p_+,N_1(z)\bigr) + \Upsilon(z) \tag{2.12}$$

with

$$N_1(z) = \frac{\partial N(p_+,\hat{q}_+)}{\partial p_+}, \qquad N_2(z) = \frac{\partial N(p_+,\hat{q}_+)}{\partial \hat{q}_+}$$

and $\Upsilon(z)$ satisfying

$$|||X_\Upsilon|||_{r;\mathcal{D}(s-5\rho,r/16)} \le \frac{c\epsilon^2}{\alpha^{2\bar{\nu}}\rho^{2\nu}}, \tag{2.13}$$

where $\bar{\nu} = \bar{n}+1$ and $\nu = \tau(\bar{n}+1)+\bar{n}+n$.

C. Truncation.

Let

$$P = R + (P - R), \tag{2.14}$$

where

$$R(p,\hat{q}) = \sum_{|k|\le K} \left(P^k_{000}(x) + \langle P^k_{100}(x), \hat{y}\rangle + \langle P^k_{010}(x), u\rangle + \langle P^k_{001}(x), \hat{v}\rangle\right) \tag{2.15}$$

$$= R_{000}(x) + \langle R_{100}(x), \hat{y}\rangle + \langle R_{010}(x), u\rangle + \langle R_{001}(x), \hat{v}\rangle, \tag{2.16}$$

and

$$P - R = \sum_{|l|+|i|+|j|=1} P_{lij}\hat{y}^l u^i \hat{v}^j + \sum_{2|l|+|i|+|j|\ge 3} P_{lij}\hat{y}^l u^i \hat{v}^j.$$

Let $F(p,\hat{q})$ possess the same form as (2.15).

D. Extending the small divisor estimate.

$\forall \xi \in \Pi_h$, there exists $\xi_0 \in \Pi$ such that $|\xi - \xi_0| < h$. So we have, for $0 < |k| \le K$,

$$\left|\langle k, \omega(\xi)\rangle + 2\pi l\right| \ge \left|\langle k, \omega(\xi_0)\rangle + 2\pi l\right| - \left|\langle k, \omega(\xi) - \omega(\xi_0)\rangle\right| \ge \frac{\alpha}{(1+|k|)^\tau}.$$

E. Homological equation.

By (2.12), it follows that

$$\begin{aligned} &N(p_+, \hat{q}_+) + P(p_+, \hat{q}_+) - F\left(p_+, N_p(p_+, \hat{q}_+)\right) + F\left(N_q(p_+, \hat{q}_+), \hat{q}_+\right) + \Upsilon(z) \\ &\quad = \bar{N}(p_+, \hat{q}_+) + \bar{P}(p_+, \hat{q}_+). \end{aligned} \tag{2.17}$$

For simplicity, below we drop the subscripts '+' in p_+ and $\hat{q}_+$.

Similar to the discussion in [27], we get the homological equations.

$$-F\left(N_q(p,\hat{q}), \hat{q}\right) + F\left(p, N_p(p,\hat{q})\right) = R - [R]. \tag{2.18}$$

To solve (2.18), we need some preparations.

Let $x + \omega = \tilde{x}$. Since

$$\hat{p} = N_{\hat{q}}(p,\hat{p}) = \left(\tilde{x}, (A + Q_2)u + (C + Q_3)v\right)$$

and

$$q = N_p(p,\hat{q}) = \left(\hat{y} + Q_x, (A + Q_2)\hat{v} + (B + Q_1)u\right),$$

we have

$$\begin{aligned} F\left(p, N_p(p,\hat{q})\right) &= F_{000}(x) + \langle F_{100}(x), \hat{y} + Q_x\rangle + \langle F_{010}(x), u\rangle \\ &\quad + \langle F_{001}(x), (A + Q_2)^T \hat{v} + (B + Q_1)u\rangle \end{aligned} \tag{2.19}$$

and

$$F\big(N_{\hat{q}}(p,\hat{q}),\hat{q}\big) = F_{000}(\tilde{x}) + \langle F_{100}(\tilde{x}),\hat{y}\rangle + \langle F_{001}(\tilde{x}),\hat{v}\rangle \\ + \langle F_{010}(\tilde{x}),(A+Q_2)^T u + (C+Q_3)\hat{v}\rangle. \tag{2.20}$$

So we get

$$F\big(N_{\hat{q}}(p,\hat{q}),\hat{q}\big) - F\big(p,N_p(p,\hat{q})\big) = L_0 + L_{10} + L_{01} - \langle F_{100}(x),Q_x\rangle,$$

where

$$L_0 = \big(F_{000}(\tilde{x}) - F_{000}(x)\big) + \langle F_{100}(\tilde{x}) - F_{100}(x),\hat{y}\rangle,$$
$$L_{10} = \langle A^T F_{010}(\tilde{x}) - F_{010}(x) - BF_{001}(x),u\rangle + \langle Q_2^T F_{010}(\tilde{x}) - Q_1^T F_{010}(x),u\rangle$$

and

$$L_{01} = \langle C^T F_{010}(\tilde{x}) + F_{010}(x) - AF_{001}(x),u\rangle + \langle Q_3^T F_{010}(\tilde{x}) - Q_2 F_{010}(x),u\rangle.$$

After these preparations, we can solve (2.18) which is equivalent to solving the following:

$$\begin{cases} F_{000}(\tilde{x}) - F_{000}(x) = R_{000}(x) - [R_{000}], \\ F_{100}(\tilde{x}) - F_{100}(x) = R_{100}(x) - [R_{100}], \end{cases} \tag{2.21}$$

and

$$\begin{cases} A^T F_{010}(\tilde{x}) - F_{010}(x) - BF_{001}(x) + Q_2^T F_{010}(\tilde{x}) - Q_1^T F_{010}(x) = R_{010}(x), \\ C^T F_{010}(\tilde{x}) + F_{010}(x) - AF_{001}(x) + Q_3^T F_{010}(\tilde{x}) - Q_2 F_{010}(x) = R_{001}(x). \end{cases} \tag{2.22}$$

Firstly, we solve (2.21) for F_{000} and F_{100}. Expand $F_{000}(x)$ and $R_{000}(x)$:

$$F_{000}(x) = \sum_{k\in\mathbb{Z}^n} F_{k000} e^{\mathrm{i}\langle k,x\rangle},$$
$$R_{000}(x) = \sum_{k\in\mathbb{Z}^n} R_{k000} e^{\mathrm{i}\langle k,x\rangle}.$$

Then we get

$$F_{kj00} = \frac{1}{e_k - 1} R_{kj00} \tag{2.23}$$

with $e_k = e^{\mathrm{i}\langle k,\omega\rangle}$, $k \neq 0$. By Assumption 2, we have the following estimate:

$$\|F_{k000}\|_{s-\rho} \le \frac{c\|R_{k000}\|_s}{\alpha\rho^{\tau+n}}.$$

So we have

$$\|X_{F_{000}}\|_{r;D(s-\rho)\times\Pi} \le \frac{c\|R_{000}\|_s}{\alpha^{\bar{\nu}}\rho^{\nu}}.$$

Similarly we have

$$\|X_{F_{100}}\|_{r;D(s-\rho)\times\Pi} \leq \frac{c\|R_{100}\|_s}{\alpha^{\bar{\nu}}\rho^{\nu}}.$$

Next we will get F_{010} and F_{001} from (2.22). Let $F_{010} = (F_{010}^1, \dots, F_{010}^m)$ and $F_{001} = (F_{001}^1, \dots, F_{001}^m)$. Expand $F_{0i'j'}^l(x)$ and $R_{0i'j'}^l(x)$:

$$F_{0i'j'}^l(x) = \sum_{k\in\mathbb{Z}^n} F_{k0i'j'}^l e^{\mathrm{i}\langle k,x\rangle},$$
$$R_{0i'j'}^l(x) = \sum_{k\in\mathbb{Z}^n} R_{k0i'j'}^l e^{\mathrm{i}\langle k,x\rangle}$$

with $l = 1, 2, \dots, m$ and $(i', j') = (0,1), (1,0)$.

Let

$$X = \begin{pmatrix} F_{010}^l \\ F_{001}^l \end{pmatrix}, \qquad Y = \begin{pmatrix} R_{010}^l \\ R_{001}^l \end{pmatrix},$$
$$M_k = \begin{pmatrix} ae_k - 1 & -b \\ be_k & e_k - a \end{pmatrix}.$$

To get the estimate of $F_{0i'j'}^l(x)$, we rewrite (2.22) as the following form:

$$\sum_{k\in Z^n} M_k X_k e^{\mathrm{i}\langle k,x\rangle} = \sum_{k\in Z^n} Y_k e^{\mathrm{i}\langle k,x\rangle} + \sum_{k\in Z^n} N_k X_k e^{\mathrm{i}\langle k,x\rangle}, \tag{2.24}$$

where N_k is composed of the components of $Q_j, j = 1, 2, 3, 4$. We can set $|N_k| \leq \epsilon_0$.

By a direct calculation, we have

$$|M_k| = |e_k - \lambda_i| \left|e_k - \lambda_i'\right|,$$

where

$$\lambda_i = \frac{\Delta_i + \sqrt{(\Delta_i)^2 - 4}}{2}, \qquad \lambda_i = \frac{\Delta_i - \sqrt{(\Delta_i)^2 - 4}}{2}, \quad i = 1, 2, \dots, m,$$

are the eigenvalues of Ω. By Assumption 2, we have $|\lambda_i| \neq 1, |\lambda_i'| \neq 1$. Since $|e_k| = 1$, it follows that $|M_k| > c_0 > 0$. We rewrite (2.24) as

$$\Lambda X = Y + \Lambda_1 X.$$

Since $|M_k| > 0$, we have the operator Λ is invertible and hence $X = \Lambda^{-1}(Y + \Lambda_1 X)$. Set $\Xi X = X - \Lambda^{-1}\Lambda_1 X$, then we have $\Xi X = L^{-1} Y$. So

$$\begin{aligned} \|\Xi X_1 - \Xi X_2\| &= \left\|\Lambda^{-1}\Lambda_1 X_1 - \Lambda^{-1}\Lambda_1 X_1\right\| \\ &\leq \left\|\Lambda^{-1}\right\| \cdot \left\|\Lambda_1\right\| \cdot \left\|X_1 - X_2\right\|. \end{aligned}$$

Set $\epsilon_0 = \frac{c_0}{2}$, then we have

$$\|\Xi X_1 - \Xi X_2\| \le \frac{1}{2}\|X_1 - X_2\|.$$

By the implicit function theorem, we have $\|X\| \le c\|Y\|$, with c depending on A, B, C. So

$$\|F_{0i'j'}\|_{D(s-\rho,r)\times\Pi} \le \frac{cr\|R_{0i'j'}\|_{D(s,r)\times\Pi}}{\alpha^{\bar{\nu}}\rho^{\nu}}$$

with $(i',j') = (0,1),(1,0)$.

From the above discussion, we have

$$\|X_F\|_{r;D(s-\rho,r)\times\Pi} \le \frac{c}{\alpha^{\bar{\nu}}\rho^{\nu}}\|X_R\|_{r;D(s,r)\times\Pi} \le \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}}, \tag{2.25}$$

where $\bar{\nu} = \bar{n} + 1$ and $\nu = \tau(\bar{n}+1) + \bar{n} + n$.

By (2.8) and (2.25), we obtain

$$\|\Psi - \mathrm{id}\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi} \le \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}},$$
$$\|D\Psi - \mathrm{id}\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi} \le \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu+1}}.$$

From the above discussion, we get the conjugate mapping $\Phi_+(\cdot;\xi) = \Psi^{-1}\circ\Phi\circ\Psi$ generated by $H_+ = \bar{N} + \bar{P}$, where

$$\bar{N} = [R_{000}] + \langle x + \omega(\xi) + R_{100}, \hat{y}\rangle + \langle Au, \hat{v}\rangle + \frac{1}{2}\langle Bu, u\rangle + \frac{1}{2}\langle C\hat{v}, \hat{v}\rangle + Q(x,u,\hat{v}),$$

and

$$\bar{P} = \Upsilon + (P - R) - \langle F_{100}(x), Q_x\rangle.$$

Recalling (2.6), we find there are second order terms of $u, \hat{v}$ in P_+, so we will put these terms into the main term. Let $Q_+ = -\langle F_{100}(x), Q_x\rangle + \Upsilon_1$, where Υ_1 contains the second order term on $u, \hat{v}$ in Υ.

Then we get $H_+ = N_+ + P_+$, where

$$N_+ = [R_{000}] + \langle x + \omega(\xi) + R_{100}, \hat{y}\rangle + \langle Au, \hat{v}\rangle + \frac{1}{2}\langle Bu, u\rangle + \frac{1}{2}\langle C\hat{v}, \hat{v}\rangle + Q(x,u,\hat{v}) + Q_+$$

and

$$P_+ = (P - R) + \Upsilon - \Upsilon_1.$$

We note that N_+ has the same form as N.

Since

$$P - R = \sum_{|l|+|i|+|j|=1} P_{lij}\hat{y}^l u^i \hat{v}^j + \sum_{2|l|+|i|+|j|\geq 3} P_{lij}\hat{y}^l u^i \hat{v}^j,$$

we have

$$\|X_{P-R}\|_{\eta r;\mathcal{D}(s-5\rho,\eta r)\times\Pi} \leq c\cdot\epsilon\left(\eta + \frac{e^{-K\rho}}{\eta^2}\right). \tag{2.26}$$

By (2.13) and (2.26), we have

$$\|X_{P_+}\|_{\eta r;D(s-5\rho,\eta r)\times\Pi} \leq c\cdot\epsilon\left(\eta + \frac{e^{-K\rho}}{\eta^2}\right) + \frac{c\epsilon^2}{\eta^2\alpha^{2\bar{\nu}}\rho^{2\nu}}.$$

F. Choice of parameters.

We choose $0 < E < 1$ and set

$$\eta = E, \qquad \epsilon = \eta^2\alpha^{2\overline{\nu}}\rho^{2\nu}E, \qquad \frac{e^{-K\rho}}{\eta^2} = E, \qquad h = \frac{\alpha}{2(K+1)^{\tau+1}T}.$$

Fix $\sigma \in (0,1)$. We define

$$\rho_+ = \sigma\rho, \qquad s_+ = s - 5\rho, \qquad r_+ = \eta r,$$
$$\alpha_+ = \alpha - (K+1)^{\tau+1}\epsilon, \qquad \epsilon_+ = c\eta\epsilon, \qquad E_+ = cE^{\frac{4}{3}}.$$

By the estimate of P_+, supposing $\alpha < 2\alpha_+$, we have

$$\begin{aligned}\|X_{P_+}\|_{\eta r;D(s-5\rho,\eta r)\times\Pi_+} &\leq c\cdot\epsilon\left(\eta + \frac{e^{-K\rho}}{\eta^2}\right) + \frac{c\epsilon^2}{\eta^2\alpha^{2\bar{\nu}}\rho^{2\nu}}\\ &\leq c\eta\epsilon = c\alpha^{2\overline{\nu}}\rho^{2\nu}E^4\\ &\leq c\alpha_+^{2\bar{\nu}}\rho_+^{2\tau}E_+^3.\end{aligned}$$

Setting $\epsilon_+ = c\alpha_+\rho_+^\tau E_+^3$, we arrive at

$$\|X_{P_+}\|_{r_+;\mathcal{D}(s_+,r_+)\times\Pi_+} \leq \epsilon_+.$$

By Iteration Lemma, we have

$$\begin{aligned}\left|\langle k,\omega_+(\xi)\rangle + 2\pi l\right| &\geq \left|\langle k,\omega(\xi) + 2\pi l\rangle\right| - \left|\langle k,\omega_+(\xi) - \omega(\xi)\rangle\right|\\ &\geq \frac{2}{(1+|k|)^\tau}\left[\alpha - (1+K)^{\tau+1}\epsilon\right]\\ &= \frac{2\alpha_+}{(1+|k|)^\tau},\end{aligned}$$

where $\xi \in \Pi_+$ and $0 \neq k \leq K$. So we choose $\alpha_+ = \alpha - (1+K)^{\tau+1}\epsilon$. By the choice of α_+, the definition of $\bar{\Pi}$ in (2.3) and $\Pi_+ = \Pi \setminus \bar{\Pi}$, it follows, for $\forall\xi \in \Pi_+$,

$$\left|\langle k,\omega_+(\xi)\rangle + 2\pi l\right| \geq \frac{2\alpha_+}{(1+|k|)^\tau}, \quad \forall k \in \mathbb{Z}^n, 0 < |k| \leq K_+.$$

Now we give the choice of T_+. Suppose $h_+ \leq \frac{5}{6}h$. By the Cauchy estimate, for $\xi \in \Pi^+_{h_+}$, we have

$$\left|\partial\left(\omega_+(\xi)-\omega(\xi)\right)/\partial\xi\right|_{h_+} \leq \frac{|\omega_+(\xi)-\omega(\xi)|_h}{h-h_+} \leq \frac{6\epsilon}{h}.$$

Define $T_+ = T + \frac{6\epsilon}{h}$ and $h_+ = \frac{\alpha_+}{T_+(1+K_+)^{\tau+1}}$, then we have

$$\max_{\xi\in\Pi_{h_+}} |\partial\omega_+/\partial\xi| \leq \max_{\xi\in\Pi_{h_+}} \left|\partial\left(\omega_+-\omega(\xi)\right)/\partial\xi\right| + \max_{\xi\in\Pi_{h_+}} |\partial\omega/\partial\xi| \leq T_+.$$

Thus all the parameters for H_+ are defined, and so Iteration Lemma is proved.

2.2 Iteration

Define inductive sequences

$$\begin{aligned}
&\rho_{j+1} = \sigma\rho_j, \qquad s_{j+1} = s_j - 5\rho, \qquad r_{j+1} = \eta_j r_j,\\
&\alpha_{j+1} = \alpha_j - (1+K_j)^{\tau+1}\epsilon_j, \qquad E_{j+1} = cE_j^{\frac{4}{3}}, \qquad T_{j+1} = T_j + \frac{6\epsilon_j}{d_j},\\
&\eta_{j+1} = E_{j+1}, \qquad \epsilon_{j+1} = \alpha_j^{2\bar{\nu}}\rho_{j+1}^{2\nu}E_{j+1}\eta_{j+1}^2,
\end{aligned}$$

and

$$\frac{e^{-K_{j+1}\rho_{j+1}}}{\eta_{j+1}^2} = E_{j+1}, \qquad h_{j+1} = \frac{\alpha_{j+1}}{(1+K)_j^{\tau+1}T_{j+1}}.$$

Define

$$\Pi_{j+1} = \left\{\xi \in \Pi_j : \left|\langle\omega_j(\xi)+2\pi l, k\rangle\right| \geq \frac{2\alpha_j}{(|k|+1)^\tau}, K_j < |k| \leq K_{j+1}\right\}$$

and

$$\Pi_{h_{j+1}} = \left\{\xi \in C^n : \operatorname{dist}(\xi, \Pi_{j+1}) \leq h_{j+1}\right\}.$$

In the following we give some estimates for Gevrey-smoothness.

Let $\gamma_j = K_j\rho_j = -\ln E_j^3$. We have $\frac{K_{j+1}}{K_j} = \frac{1}{2}\frac{\ln c}{\ln E_j} + \frac{4}{3\sigma}$, and hence $\frac{4}{3} \leq \frac{K_{j+1}}{K_j} \leq \frac{4}{3}\frac{1}{\rho}$ for E_0 small enough. If $12 < K_j < K_{j+1}$, we have $\frac{h_{j+1}}{h_j} = \frac{\alpha_{j+1}}{\alpha_j}\frac{T_j}{T_{j+1}}\frac{(1+K_j)^\tau}{(1+K_{j+1})^\tau} \leq \frac{5}{6}$, and hence $h_{j+1} \leq \frac{5}{6}h_j$, which means $h_+ \leq \frac{5}{6}h$ holds. Suppose $\max_{\xi\in\Pi_{h_j}} |\frac{\partial\omega_j}{\partial\xi}| \leq T_j$. Let $T_{j+1} = T_j + \frac{6\epsilon_j}{d_j}$. Then we have $|\frac{\partial\omega_{j+1}}{\partial\xi}| = |\frac{\partial(\omega_{j+1}-\omega_j+\omega_j)}{\partial\xi}| \leq T_{j+1}$. By the choice of σ, we can easily get that $\rho_{j+1}\gamma_{j+1}^{\frac{\delta}{\tau+1}} \geq \rho_j\gamma_j^{\frac{\delta}{\tau+1}}$. Since $\rho_0\gamma_0^{\frac{\delta}{\tau+1}} \geq 1$, we have $\rho_j\gamma_j^{\frac{\delta}{\tau+1}} \geq 1$ for all $j > 1$.

By the definitions of T_j, h_j and ϵ_j, we have $T_{j+1} = T_j + \frac{6\epsilon_j}{d_j} = T_0 + 6\sum_{i=0}^{j}(\gamma_i)^\tau e^{-\gamma_i}T_i$. Noting $\gamma_j = -\ln E_j^3$ and $E_j \leq (cE_0)^{(\frac{4}{3})^j}$, we can choose E_0 to be sufficiently small such that $\sum_{i=0}^{j}(\gamma_i)^\tau e^{-\gamma_i}T_i \leq \frac{1}{6}$, then we have $T_0 \leq T_j \leq T_0 + 1$. Similarly, we have $\frac{1}{2}\alpha_j \leq \alpha_{j+1} \leq \alpha_j$.

By Iteration Lemma, there exists a sequence of symplectic mappings $\{\Psi_j(\cdot;\xi)\}$, generated by $\langle p, q_+\rangle + F_j(p, q_+)$, satisfying

$$\|\Psi_j - \mathrm{id}\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\epsilon_j}{\alpha_j^{\bar{\nu}}\rho_j^{\nu}},$$

$$\|D\Psi_j - \mathrm{Id}\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\epsilon_j}{\alpha_j^{\bar{\nu}}\rho_j^{\nu+1}}.$$

Define $\Psi^j = \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_j$. Then we have a sequence of symplectic mappings $\{\Phi_{j+1}(\cdot;\xi) = (\Psi^j)^{-1} \circ \Phi_j \circ \Psi^j\}$, generated by $H_{j+1}(\cdot;\xi) = N_{j+1} + P_{j+1}$, where

$$N_{j+1} = \bigl\langle x + \omega_{j+1}(\xi), \hat{y}\bigr\rangle + \langle A_{j+1}u, \hat{v}\rangle + \frac{1}{2}\langle B_{j+1}u, u\rangle + \frac{1}{2}\langle C_{j+1}\hat{v}, \hat{v}\rangle$$

with

$$|\omega_{j+1} - \omega_j| \le c\epsilon_j, \quad \forall j \ge 1,$$

and

$$\|X_{P_{j+1}}\|_{r_{j+1};\mathcal{D}(s_{j+1},r_{j+1})\times\Pi_{h_{j+1}}} \le \epsilon_{j+1}.$$

2.3 The convergence of the KAM iteration

Now we prove the convergence of the KAM iteration. Similar to [27], we have

$$\bigl\|\Psi^j - \Psi^{j-1}\bigr\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le c\alpha_{j-1}^{\bar{\nu}}\rho_{j-1}^{\nu}E_{j-1}^3,$$

and

$$\bigl\|D\bigl(\Psi^j - \Psi^{j-1}\bigr)\bigr\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \alpha_{j-1}^{\bar{\nu}}\rho_{j-1}^{\nu+1}E_{j-1}^3.$$

By the Cauchy estimate, we have

$$\bigl\|\partial_\xi^\beta\bigl(\Psi^j - \Psi^{j-1}\bigr)\bigr\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\alpha_{j-1}^{\bar{\nu}}\rho_{j-1}^{\nu}E_{j-1}^3\beta!}{h_j^{|\beta!|}},$$

$$\bigl\|\partial_\xi^\beta D\bigl(\Psi^j - \Psi^{j-1}\bigr)\bigr\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\alpha_{j-1}^{\bar{\nu}}\rho_{j-1}^{\nu+1}E_{j-1}^3\beta!}{h_j^{|\beta!|}}$$

and

$$\bigl\|\partial_\xi^\beta\bigl(\omega^j - \omega^{j-1}\bigr)\bigr\|_{\Pi_j} \le \frac{c\epsilon_{j-1}\beta!}{h_j^{|\beta!|}}.$$

Let $U_j^\beta = \frac{c\alpha_{j-1}^{\bar{\nu}}\rho_{j-1}^{\nu}E_{j-1}^3\beta!}{h_j^{|\beta!|}}$ and $G_j^\beta = \frac{c\epsilon_{j-1}\beta!}{h_j^{|\beta!|}}$. Now we estimate U_j^β and G_j^β for $\beta \in Z_n^+$.

Since $\rho_j\gamma_j^{\frac{\delta}{\tau+1}} \ge 1$ for all $j > 1$, we have $\frac{1}{\rho_j} \le \gamma_j^{\frac{\delta}{\tau+1}}$. Then we have $K_j = \frac{\gamma_j}{\rho_j} \le \gamma_j^{1+\frac{\delta}{\tau+1}}$, which means that $K_j^{\tau+1} \le \gamma_j^{\tau+1+\delta}$. Noting that $h_j = \frac{\alpha_j}{2(K+1)_j^{\tau+1}T_j}$, $T_j < T + 1$, $\frac{1}{2}\alpha \le \alpha_j$ and $E_{j-1} = E_j^{\frac{3}{4}} =$

$e^{-\frac{\gamma_j}{4}}$, we have

$$\begin{aligned} U_j^\beta &\leq c\alpha^{\bar{\nu}}\rho_j^\nu \beta!\left(\frac{T+1}{\frac{\alpha}{2}}\right)^{|\beta|}\left(\gamma_j^{\tau+1+\delta}\right)^{|\beta|}e^{-\frac{3\gamma_j}{4}} \\ &\leq c\rho_j^\nu\left(\frac{2(T+1)}{\alpha}\right)^{|\beta|}\beta!\left[\gamma_j^{\beta_1}e^{-\frac{3\gamma_j}{4}\frac{1}{(\tau+\delta)(n+1)}}\cdots\gamma_j^{\beta_n}e^{-\frac{3\gamma_j}{4}\frac{1}{(\tau+\delta)(n+1)}}\right]^{\tau+\delta}e^{-\frac{3\gamma_j}{4}\frac{1}{n+1}} \\ &\leq c\rho_j^\nu J^{|\beta|}\beta!^\mu J_j^{\frac{9}{4(n+1)}}, \end{aligned}$$

where $J = \frac{2T+1}{\alpha}[\frac{4(\mu-1)(n+1)}{3}]^{\mu-1}$, $\mu = \tau + \delta$, and c only depends on n, α, μ.

In the same way, we have

$$G_j^\beta \leq c\rho_j^{2\nu}J^{|\beta|}\beta!^\mu E_j^{\frac{9}{4(n+1)}}.$$

Note that $s_j \to \frac{s}{2}, r_j \to 0, h_j \to 0$ as $j \to \infty$. Let $D_* = D(\frac{s}{2}, 0)$, $\Pi_* = \bigcap_{j\geq 0}\Pi^j$ and $\Psi_* = \lim_{j\to\infty}\Psi^j$. Then we have

$$\left\|\partial_\xi^\beta(\Psi_* - \mathrm{id})\right\|_{\frac{r}{2};D(*)\times\Pi_*} \leq c\rho_0^\nu J^{|\beta|}\beta!^\mu E_0^{\frac{9}{4(n+1)}}, \quad \forall\beta \in Z_n^+.$$

Since Ψ_j is affine in y, Ψ^j is also affine in y, and hence we have the convergence of $\partial_\xi^\beta\Psi^j$ to $\partial_\xi^\beta\Psi^*$ on $D(\frac{s}{2}, \frac{r}{2})$ and

$$\left\|\partial_\xi^\beta(\Psi_* - \mathrm{id})\right\|_{\frac{r}{2};D(\frac{s}{2},\frac{r}{2})\times\Pi_*} \leq c\rho_0^\nu J^{|\beta|}\beta!^\mu E_0^{\frac{9}{4(n+1)}},$$

$\forall\beta \in Z_n^+$. Thus we proved (1.6).

Let $\omega_* = \lim_{j\to\infty}\omega_j$. Similarly, it follows that

$$\left|\partial_\xi^\beta\left(\omega_*(\xi) - \omega\right)(\xi)\right|_{\Pi_*} \leq c\rho_0^{2\nu}J^{|\beta|}\beta!^\mu E_0^{\frac{9}{4(n+1)}}, \quad \forall\beta \in Z_n^+.$$

Moreover we have

$$\left|\langle\omega_*(\xi), k\rangle\right| \geq \frac{\alpha_*}{(1+|k|)^\tau}$$

for all $\xi \in \prod_*$ and $0 \neq k \in Z^n$, where $\alpha_* = \lim_{j\to\infty}\alpha^j$, with $\frac{\alpha}{2} \leq \alpha_* \leq \alpha$. Thus we proved (1.7) and (1.8).

2.4 Estimate of measure

We note that $\beta \geq 1$ in Assumption 1 for symplectic mappings , while $\beta \geq 0$ in Hamiltonian systems [24, 25]. So the non-degeneracy condition in symplectic mappings and that in Hamiltonian systems are different. It means that the estimate of measure is different in two cases. But the proof for symplectic mappings is similar to [24, 25], so we omit the details.

Competing interests

The author declares that they have no competing interests.

Author's contributions
The article is a work of the author who contributed to the final version of the paper. The author read and approved the final manuscript.

Acknowledgements
The paper was completed during the author's visit to the Department of Mathematics of Pennsylvania State University, which was supported by Nanjing Tech University. The author thanks Professor Mark Levi for his invitation, hospitality and valuable discussions. The work is supported by the Natural Science Foundation of Jiangsu Higher Education Institutions of China (14KJB110009). The work is partly supported by the Natural Science Foundation of Jiangsu Province (BK20140927) and (BK20150934). The work is also in part supported by the Natural Science Foundation of China (11301263).

Received: 25 September 2016 Accepted: 26 January 2017 Published online: 07 February 2017

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