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Bonnesen-style Wulff isoperimetric inequality

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Abstract

The Wulff isoperimetric inequality is a natural extension of the classical isoperimetric inequality (Green and Osher in Asian J. Math. 3:659-676 1999). In this paper, we establish some Bonnesen-style Wulff isoperimetric inequalities and reverse Bonnesen-style Wulff isoperimetric inequalities. Those inequalities obtained are extensions of known Bonnesen-style inequalities and reverse Bonnesen-style inequalities.

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1 Introduction and main results

In this paper, we restrict ourselves to considering convex bodies in the Euclidean plane $\mathbb{R}^2$. A set K is convex if, for any two points $x, y \in K$, the line segment $\overline{xy}$ joining x and y is contained in K, that is,

$$(1-t)x + ty \in K, \quad 0 \le t \le 1.$$

A compact convex set with non-empty interior is called a convex body. A convex body without corner points and line segments on the boundary is called an oval body.

Let K, W be two convex bodies and $t \in \mathbb{R}$, the Minkowski addition $K + W$ of K and W is

$$K + W = \{x + y : x \in K, y \in W\},$$

and the scalar multiplication tK of K is

$$tK = \{tx : x \in K\}.$$

The convex bodies K and W are homothetic if $K = x + tW$ with some $t > 0$ and some $x \in \mathbb{R}^2$.

Being of considerable interest, for example, in studying the growth of crystals, the Wulff flow originated from the unit-speed outward normal flow. Given a planar convex domain

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K of area A_K and perimeter L_K, by growing in the unit-speed along the direction of the outward normal, the area of the corresponding domain, which is denoted by $A_K(t)$, is a polynomial in t, which is known as the Steiner polynomial, that is,

$$A_K(t) = A_K + L_K t + \pi t^2. \tag{1}$$

The discriminant of $A_K(t) = 0$ is

$$\Delta(K) = L_K^2 - 4\pi A_K.$$

$\Delta(K)$ is the isoperimetric deficit of K, which is non-negative by the following classical isoperimetric inequality:

$$L_K^2 - 4\pi A_K \geq 0, \tag{2}$$

where the equality holds if and only if K is a disc. One can find some simplified and beautiful proofs of (2) that lead to generalizations to higher dimensions and applications to other branches of mathematics (*cf.* [1–26]).

During the 1920s, Bonnesen proved a series of inequalities of the form

$$L_K^2 - 4\pi A_K \geq B_K, \tag{3}$$

where B_K is a non-negative geometric invariant and vanishes only when K is a disc. The inequality of the form (3) is known as the Bonnesen-style inequality, and the typical one was proved by Bonnesen himself (*cf.* [5, 6]):

$$L_K^2 - 4\pi A_K \geq \pi^2 (R - r)^2, \tag{4}$$

where R and r are the radius of the minimum circumscribed disc and the radius of the maximum inscribed disc of K, respectively.

Many B_Ks were found in the last century, and mathematicians are still working on those unknown isoperimetric deficit lower limits of geometric significance. For more details, see references [20, 24–26].

If instead K grows by varying the outward normal speed to be a function $p_W(\theta)$ of the direction of the unit normal, one has the Wulff flow. The area of the domain when the initial domain K is convex and the function $p_W(\theta)$ is a support function of the convex body W with area A_W is a polynomial in t in this flow, called the Wulff-Steiner polynomial, that is (*cf.* [1]),

$$A_{K,W}(t) = A_K + L_{K,W} t + A_W t^2, \tag{5}$$

where $L_{K,W}$ is the Wulff length of ∂K with respect to W, and namely,

$$L_{K,W} = \int_{\partial K} p_W \, ds, \tag{6}$$

where ∂K is the boundary of K, and s is the arc length parameter of ∂K. The discriminant of $A_{K,W}(t) = 0$ is defined as the Wulff isoperimetric deficit (*cf.* [1]):

$$\Delta_W(K) = L_{K,W}^2 - 4A_K A_W. \tag{7}$$

When domain W is a unit disc, the Wulff isoperimetric deficit $\Delta_W(K)$ is the isoperimetric deficit of K.

Let

$$r_W = \max\{t | \text{some translate of } tW \text{ is contained in } K\}$$

and

$$R_W = \min\{t | \text{some translate of } tW \text{ contains } K\}$$

be the W-inradius and W-outradius of K, respectively. It is obvious that $0 < r_W \le R_W$.

If W is the unit disc, then r_W and R_W are, respectively, the radius of the maximum inscribed disc and the radius of the minimum circumscribed disc of K.

We first prove the following Wulff isoperimetric inequality:

$$L_{K,W}^2 - 4A_K A_W \ge 0, \tag{8}$$

where the equality holds if and only if K and W are homothetic. Then we consider the inequality of the form

$$L_{K,W}^2 - 4A_K A_W \ge B_W(K), \tag{9}$$

where $B_W(K)$ is an invariant of geometric significance of K and W and vanishes only when K and W are homothetic. The inequality of type (9) is called the Bonnesen-style Wulff isoperimetric inequality. Its reverse form, that is,

$$L_{K,W}^2 - 4A_K A_W \le U_W(K), \tag{10}$$

is called the reverse Bonnesen-style Wulff isoperimetric inequality. Here $U_W(K)$ is an invariant of K and W. We hope that $U_W(K)$ vanishes only when K and W are homothetic.

Our main results are the following.

Theorem 1 *Let K, W be two oval bodies in $\mathbb{R}^2$ with areas A_K and A_W, respectively. Let r_W, R_W be, respectively, the W-inradius and W-outradius of K. Then*

$$L_{K,W}^2 - 4A_K A_W \ge A_W^2 (R_W - r_W)^2 + (A_W R_W + A_W r_W - L_{K,W})^2, \tag{11}$$

$$L_{K,W}^2 - 4A_K A_W \ge \frac{A_K A_W (R_W - r_W)^2}{R_W r_W} + \frac{(R_W r_W A_W - A_K)^2}{R_W r_W}. \tag{12}$$

Each equality holds if and only if K and W are homothetic.

Theorem 2 *Let K, W be two oval bodies in $\mathbb{R}^2$ with areas A_K and A_W, respectively. Let r_W, R_W be, respectively, the W-inradius and W-outradius of K. Then*

$$L^2_{K,W} - 4A_K A_W \leq 2A_W L_{K,W}(R_W - r_W), \tag{13}$$

$$L^2_{K,W} - 4A_K A_W \leq 2A_K L_{K,W}\left(\frac{1}{r_W} - \frac{1}{R_W}\right), \tag{14}$$

$$L^2_{K,W} - 4A_K A_W \leq 2L_{K,W}\left(A_W R_W - \frac{A_K}{R_W}\right), \tag{15}$$

$$L^2_{K,W} - 4A_K A_W \leq 2L_{K,W}\left(\frac{A_K}{r_W} - A_W r_W\right). \tag{16}$$

Each equality holds if and only if K and W are homothetic.

2 Preliminaries

The support function $p_K(u)$ of the convex body K is defined by

$$p_K(u) = \max\{x \cdot u | x \in K\},$$

where $u \in S^1$. For simplicity, we replace $p_K(u)$ by p_K. For two convex bodies K, W, we have

$$p_W \leq p_K \quad \text{if and only if} \quad W \subseteq K. \tag{17}$$

If the support functions of the convex bodies K, W are denoted by p_K, p_W, respectively, and $t_1, t_2 \in \mathbb{R}$, then the support function of $t_1 K + t_2 W$ is $t_1 p_K + t_2 p_W$.

The image of the convex body K at time $t \geq 0$ under the normal flow having speed $p_W(u)$ (the Wulff flow associated to W) is $K + tW$.

Let p_K be the support function of K, then

$$ds = \left(p_K + p_K''\right) d\theta; \tag{18}$$

$$A_K = \frac{1}{2}\int_{S^1} p_K \left(p_K + p_K''\right) d\theta; \tag{19}$$

$$L_K = \int_{S^1} \left(p_K + p_K''\right) d\theta = \int_{S^1} p_K \, d\theta. \tag{20}$$

Proposition 1 (Poincaré lemma [1]) *Let f be a function on $[0, a]$ whose first derivative is square integrable and such that*

$$f(0) = f(a) = 0.$$

Then

$$\int_0^a f'(x)^2 \, dx \geq \left(\frac{\pi}{a}\right)^2 \int_0^a f(x)^2 \, dx, \tag{21}$$

where equality holds if and only if $f = A\cos(\pi x/a) + B\sin(\pi x/a)$. In particular, if $a \leq \pi$, then

$$\int_0^a f'(x)^2 \, dx \geq \int_0^a f(x)^2 \, dx. \tag{22}$$

Inequality (22) *holds as an equality if and only if*

$$f = c \sin(\pi x / a)$$

for some constant c.

3 Bonnesen-style Wulff isoperimetric inequalities

To prove our main results, we need the following lemmas.

Lemma 1 *Let K, W be two convex bodies in $\mathbb{R}^2$. Let $p_K(\theta)$ and $p_W(\theta)$ be support functions of K and W, respectively. If $W \subseteq K$ such that $p_W(\theta) \neq p_K(\theta)$, $\theta \in [\theta_0, \theta_0 + \pi]$ for some θ_0, then there exist $\epsilon > 0$ and $v \in S^1$ such that $W + \epsilon \cdot v \subset K$.*

Proof Choose vectors $u, v \in S^1$ corresponding to angles $\theta, \theta_0 + \frac{\pi}{2}$, respectively. If $\theta \in [\theta_0, \theta_0 + \pi]^c$, the complement of $[\theta_0, \theta_0 + \pi]$, since the angle between u and v is strictly greater than $\frac{\pi}{2}$, hence $u \cdot v < 0$. By (17), we have

$$p_{W + \epsilon \cdot v}(u) = p_W(u) + \epsilon(u \cdot v) < p_K(u) \tag{23}$$

for $\epsilon > 0$.

If $\theta \in [\theta_0, \theta_0 + \pi]$, since $p_W(\theta) \neq p_K(\theta)$ when $\theta \in [\theta_0, \theta_0 + \pi]$ and $W \subseteq K$, we have

$$p_W(u) < p_K(u).$$

For $\epsilon > 0$ small enough, we have

$$p_{W + \epsilon \cdot v}(u) = p_W(u) + \epsilon(u \cdot v) < p_K(u). \tag{24}$$

Hence

$$p_{W + \epsilon \cdot v}(u) < p_K(u),$$

for all $u \in S^1$, that is, $W + \epsilon \cdot v \subset K$. □

Lemma 2 *Let K, W be two oval bodies in $\mathbb{R}^2$. Let r_W, R_W be, respectively, the W-inradius and W-outradius of K. Then the equation $A_{K,W}(t) = 0$ has two roots t_1, t_2 such that*

$$t_1 \leq -R_W \leq -r_W \leq t_2 < 0. \tag{25}$$

Each inequality in (25) *holds as an equality if and only if K and W are homothetic. In particular, when $r_W \leq t \leq R_W$,*

$$A_{K,W}(-t) \leq 0. \tag{26}$$

Inequality (26) *is strict whenever $r_W < t < R_W$. When $t = r_W$ or $t = R_W$, equality will occur in* (26) *if and only if K and W are homothetic.*

Proof There is at least one point where $\partial(r_W W)$ is tangent to ∂K for $\theta \in [\theta_0, \theta_0 + \pi]$ with all θ_0. If the conclusion fails, that is, there exists θ_0 such that $p_{r_W W}(\theta) \neq p_K(\theta)$ for $\theta \in [\theta_0, \theta_0 + \pi]$, choose the vector v corresponding to the angle $\theta_0 + \frac{\pi}{2}$. By Lemma 1, if we move $r_W W$ by v for $\epsilon > 0$ small enough, then $r_W W + \epsilon \cdot v$ continues to lie in the interior of K and has no points of tangency. This contradicts the maximality of r_W.

By integration by parts we have

$$
\begin{aligned}
A_{K,W}(-r_W) &= \frac{1}{2} \int_{S^1} (p_K - r_W p_W)\big(p_K + p_K'' - r_W(p_W + p_W'')\big)\, d\theta \\
&= \frac{1}{2} \int_{S^1} (p_K - r_W p_W)^2\, d\theta + \frac{1}{2} \int_{S^1} (p_K - r_W p_W)\big(p_K'' - r_W p_W''\big)\, d\theta \\
&= \frac{1}{2} \int_{S^1} (p_K - r_W p_W)^2\, d\theta - \frac{1}{2} \int_{S^1} \big(p_K' - r_W p_W'\big)^2\, d\theta.
\end{aligned}
\tag{27}
$$

Let $\theta_1, \theta_2, \dots, \theta_N$ be points where $\partial(r_W W)$ are tangent to ∂K. We can break up the right-hand side of (27) into integrals over the intervals $[\theta_i, \theta_{i+1}]$ ($1 \le i \le N-1$). Since every set $[\theta, \theta + \pi]$ contains a point where $r_W W$ is tangent to K, we have

$$
\theta_{i+1} - \theta_i \le \pi.
$$

Let

$$
f = p_K - r_W p_W,
$$

then

$$
f(\theta_i) = 0
$$

at each point of tangency. Applying inequality (22) in the Poincaré lemma, we have

$$
A_{K,W}(-r_W) \le 0,
\tag{28}
$$

where the equality holds if and only if

$$
p_K - r_W p_W = c \sin\left(\frac{\pi \theta}{\alpha}\right).
$$

Since the convex body K contains tW, then

$$
c \sin\left(\frac{\pi \theta}{\alpha}\right) = p_K - r_W p_W \ge 0
$$

for all θ. This leads to

$$
c = 0,
$$

that is, K and W are homothetic. In a similar way, we have

$$
A_{K,W}(-R_W) \le 0,
\tag{29}
$$

where the equality holds if and only if K and W are homothetic. Thus the equation $A_{K,W}(t) = 0$ has two roots t_1, t_2, and

$$t_1 + t_2 = -\frac{L_{K,W}}{A_W} < 0, \qquad t_1 t_2 = \frac{A_K}{A_W} > 0,$$

and therefore

$$t_1 < 0, \qquad t_2 < 0.$$

Therefore

$$t_1 \leq -R_W \leq -r_W \leq t_2.$$

In particular, according to (5), when $r_W \leq t \leq R_W$, we have

$$A_{K,W}(-t) \leq 0.$$

If $r_W < t < R_W$, then $t_1 < -t < t_2$. Inequality (26) is strict. Therefore, equality occurs in (26) only when $t = r_W$ or $t = R_W$, that is, K and W are homothetic. Lemma 2 is proved. □

Remark Inequality (26) has been mentioned in Green and Osher's work (cf. [1]) without proof. For general convex bodies, Luo, Xu and Zhou [17] have also obtained inequality (26) by the integral geometry method. However, it is difficult to obtain the equality condition of inequality (26) for general convex bodies. Via the method of convex geometric analysis, a complete proof of inequality (26) with equality condition is given in [9].

By (28) or (29), the sufficient condition for root existence of equation $A_{K,W}(t) = 0$ is that the discriminant of $A_{K,W}(t) = 0$ is non-negative. We obtain the following Wulff isoperimetric inequality.

Corollary 1 *Let K, W be two oval bodies in $\mathbb{R}^2$ with areas A_K, A_W, then*

$$L_{K,W}^2 - 4A_K A_W \geq 0,$$

the equality holds if and only if K and W are homothetic.

Proof of Theorem 1 By inequalities (28), (29), we have, respectively,

$$A_K - L_{K,W} r_W + A_W r_W^2 \leq 0, \tag{30}$$

$$A_K - L_{K,W} R_W + A_W R_W^2 \leq 0. \tag{31}$$

Then inequalities (30), (31) can be, respectively, rewritten as

$$-2A_K A_W \geq 2A_W^2 r_W^2 - 2L_{K,W} r_W A_W,$$

$$-2A_K A_W \geq 2A_W^2 R_W^2 - 2L_{K,W} R_W A_W.$$

Therefore, we have

$$L_{K,W}^2 - 4A_K A_W \geq L_{K,W}^2 + 2A_W^2 r_W^2 + 2A_W^2 R_W^2 - 2L_{K,W} r_W A_W - 2L_{K,W} R_W A_W$$

$$= A_W^2 r_W^2 + A_W^2 R_W^2 - 2A_W^2 r_W R_W + L_{K,W}^2 + A_W^2 r_W^2 + A_W^2 R_W^2$$

$$+ 2A_W^2 r_W R_W - 2L_{K,W} r_W A_W - 2L_{K,W} R_W A_W$$

$$= A_W^2 (R_W - r_W)^2 + (A_W R_W + A_W r_W - L_{K,W})^2,$$

where the equality holds if and only if the equalities of (28), (29) hold, that is, K and W are homothetic. This proves inequality (11).

Inequalities (30), (31) can also be rewritten, respectively, as follows:

$$L_{K,W} r_W \geq A_W r_W^2 + A_K,$$

$$L_{K,W} R_W \geq A_W R_W^2 + A_K.$$

Therefore

$$L_{K,W}^2 r_W R_W - 4A_K A_W r_W R_W \geq A_W^2 r_W^2 R_W^2 + A_K A_W r_W^2 + A_K A_W R_W^2 + A_K^2$$

$$- 4A_K A_W r_W R_W$$

$$= A_K A_W (R_W - r_W)^2 + (R_W r_W A_W - A_K)^2.$$

Hence, we have

$$L_{K,W}^2 - 4A_K A_W \geq \frac{A_K A_W (R_W - r_W)^2}{R_W r_W} + \frac{(R_W r_W A_W - A_K)^2}{R_W r_W},$$

where the equality holds if and only if K and W are homothetic. Inequality (12) is proved. □

Let W be the unit disc, then $L_{K,W}^2 = L_K^2$, $A_W = \pi$. Therefore we have the following.

Corollary 2 *Let K be an oval body in $\mathbb{R}^2$ with area A_K and perimeter L_K. Let r and R be, respectively, the radius of the maximum inscribed disc and the radius of the minimum circumscribed disc of K. Then*

$$L_K^2 - 4\pi A_K \geq \pi^2 (R - r)^2 + (\pi R + \pi r - L_K)^2, \tag{32}$$

$$L_K^2 - 4\pi A_K \geq \frac{\pi A_K (R - r)^2}{Rr} + \frac{(\pi Rr - A_K)^2}{Rr}. \tag{33}$$

Each equality holds if and only if K is a disc.

It should be noted that (32) is obtained in [24], which is stronger than the Bonnesen isoperimetric inequality (4).

4 Reverse Bonnesen-style Wulff isoperimetric inequalities

To prove reverse Bonnesen-style Wulff isoperimetric inequalities in Theorem 2, we need the following Wulff isoperimetric inequalities.

Lemma 3 *Let K, W be two oval bodies in $\mathbb{R}^2$ with areas A_K and A_W. Let r_W, R_W be, respectively, the W-inradius and W-outradius of K. Then*

$$r_W \leq \frac{2A_K}{L_{K,W}} \leq \sqrt{\frac{A_K}{A_W}} \leq \frac{L_{K,W}}{2A_W} \leq R_W, \tag{34}$$

$$\frac{A_K}{A_W R_W} \leq \frac{2A_K}{L_{K,W}} \leq \sqrt{\frac{A_K}{A_W}} \leq \frac{L_{K,W}}{2A_W} \leq \frac{A_K}{A_W r_W}. \tag{35}$$

Each equality holds if and only if K and W are homothetic.

Proof The Wulff isoperimetric inequality

$$L_{K,W}^2 - 4A_K A_W \geq 0$$

can be rewritten as

$$\frac{2A_K}{L_{K,W}} \leq \sqrt{\frac{A_K}{A_W}} \leq \frac{L_{K,W}}{2A_W}. \tag{36}$$

Each inequality holds as an equality if and only if K and W are homothetic. Recalling (6), (18) and (19), we have

$$2A_K = \int_{S^1} p_K \left(p_K + p_K'' \right) d\theta,$$

$$r_W L_{K,W} = \int_{S^1} r_W p_W \left(p_K + p_K'' \right) d\theta.$$

By the definition of r_W, we have

$$r_W p_W \leq p_K$$

for all θ, which leads to

$$r_W L_{K,W} \leq 2A_K,$$

that is,

$$r_W \leq \frac{2A_K}{L_{K,W}}, \tag{37}$$

where the equality holds if and only if $r_W p_W = p_K$ for all θ, that is, K and W are homothetic. By the definition of $L_{K,W}$ in (6), we have

$$L_{K,W} = \int_{S^1} p_W \left(p_K + p_K'' \right) d\theta = \int_{S^1} p_K \left(p_W + p_W'' \right) d\theta.$$

Via the area formula (19), we have

$$2A_W R_W = \int_{S^1} R_W p_W \left(p_W + p_W''\right) d\theta \geq \int_{S^1} p_K \left(p_W + p_W''\right) d\theta = L_{K,W}.$$

Hence, we have

$$\frac{L_{K,W}}{2A_W} \leq R_W, \tag{38}$$

where the equality holds if and only if $R_W p_W = p_K$ for all θ, that is, K and W are homothetic.

By (36), (37) and (38), we have

$$r_W \leq \frac{2A_K}{L_{K,W}} \leq \sqrt{\frac{A_K}{A_W}} \leq \frac{L_{K,W}}{2A_W} \leq R_W.$$

Inequalities (34) are proved.

Inequalities (38), (37) can, respectively, be rewritten as

$$\frac{A_K}{A_W R_W} \leq \frac{2A_K}{L_{K,W}}, \qquad \frac{L_{K,W}}{2A_W} \leq \frac{A_K}{A_W r_W}.$$

Together with (36) and the above inequalities, inequalities (35) follow. □

Proof of Theorem 2 By inequalities (34), we have

$$2A_W L_{K,W} r_W \leq 4A_K A_W \leq L_{K,W}^2 \leq 2A_W L_{K,W} R_W, \tag{39}$$

then

$$L_{K,W}^2 - 4A_K A_W \leq 2A_W L_{K,W} R_W - 2A_W L_{K,W} r_W = 2A_W L_{K,W}(R_W - r_W),$$

where the equality holds if and only if each equality of (34) holds, that is, K and W are homothetic. This is inequality (13).

By inequalities (35), we have

$$2\frac{A_K L_{K,W}}{R_W} \leq 4A_K A_W \leq L_{K,W}^2 \leq 2\frac{A_K L_{K,W}}{r_W}, \tag{40}$$

then

$$L_{K,W}^2 - 4A_K A_W \leq \frac{2A_K L_{K,W}}{r_W} - \frac{2A_K L_{K,W}}{R_W} = 2A_K L_{K,W}\left(\frac{1}{r_W} - \frac{1}{R_W}\right),$$

where the equality holds if and only if each equality of (35) holds, then K and W are homothetic. This is inequality (14).

By (39) and (40), we obtain

$$2\frac{A_K L_{K,W}}{R_W} \leq 4A_K A_W \leq L_{K,W}^2 \leq 2A_W L_{K,W} R_W,$$

then

$$L_{K,W}^2 - 4A_K A_W \leq 2A_W L_{K,W} R_W - \frac{2A_K L_{K,W}}{R_W}.$$

According to the equality conditions of (39) and (40), the equality holds for (15) if and only if K and W are homothetic. This gives inequality (15).

By (39) and (40) again, we get

$$2A_W L_{K,W} r_W \leq 4A_K A_W \leq L_{K,W}^2 \leq 2\frac{A_K L_{K,W}}{r_W}, \tag{41}$$

then

$$L_{K,W}^2 - 4A_K A_W \leq \frac{2A_K L_{K,W}}{r_W} - 2A_W L_{K,W} r_W.$$

From the equality conditions of (39) and (40) again, the equality of (16) holds if and only if K and W are homothetic. This gives inequality (16). Theorem 2 is proved. □

Let W be a unit disc. Direct consequences of Theorem 2 are as follows.

Corollary 3 *Let K be an oval body in $\mathbb{R}^2$ with area A_K and perimeter L_K. Let r and R be, respectively, the radius of the maximum inscribed disc and the radius of the minimum circumscribed disc of K. Then*

$$L_K^2 - 4\pi A_K \leq 2\pi L_K (R - r), \tag{42}$$

$$L_K^2 - 4\pi A_K \leq 2A_K L_K \left(\frac{1}{r} - \frac{1}{R} \right), \tag{43}$$

$$L_K^2 - 4\pi A_K \leq 2L_K \left(\pi R - \frac{A_K}{R} \right), \tag{44}$$

$$L_K^2 - 4\pi A_K \leq 2L_K \left(\frac{A_K}{r} - \pi r \right). \tag{45}$$

Each equality holds if and only if K is a disc.

The reverse Bonnesen-style inequality (42) is obtained by Bokowski, Heil, Zhou, Ma and Xu (*cf.* [4, 27]).

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