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Pointwise approximation by a Durrmeyer variant of Bernstein-Stancu operators

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Abstract

In the present paper, we introduce a kind of Durrmeyer variant of Bernstein-Stancu operators, and we obtain the direct and converse results of approximation by the operators.

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1 Introduction

For any $f \in C_{[0,1]}$, the corresponding Bernstein operators and Bernsetin-Durrmeyer operators are defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x) \tag{1.1}$$

and

$$D_n(f,x) := (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_0^1 f(t)p_{nk}(t)\,dt, \tag{1.2}$$

respectively, where $p_{nk}(x) := \binom{n}{k}x^k(1-x)^{n-k}$, $k = 0,1,\ldots,n$. Both $B_n(f,x)$ and $D_n(f,x)$ have played very important roles in approximation theory and computer science. There are many generalizations of the operators $B_n(f,x)$ and $D_n(f,x)$. Among them, Gadjiev and Ghorbanalizadeh [1] introduced the following new generalized Bernstein-Stancu type operators with shifted knots:

$$S_{n,\alpha,\beta}(f,x) := \left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) q_{nk}(x), \tag{1.3}$$

where $x \in A_n := [\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, and

$$q_{nk}(x) := \binom{n}{k}\left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-k}, \quad k = 0,1,\ldots,n,$$

with α_k, β_k, $k = 1,2$ positive numbers satisfying $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. Obviously, when $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $S_{n,\alpha,\beta}(f,x)$ reduces to the classical Bernstein operators in (1.1), when $\alpha_2 = \beta_2 = 0$, it reduces to the so-called Bernstein-Stancu operators which were introduced by Stancu [2]:

$$B_{n,\alpha,\beta}(f,x) := \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) p_{nk}(x). \tag{1.4}$$

Some approximation properties and generalizations of the operators $S_{n,\alpha,\beta}(f,x)$ can be found in [3–5].

Motivated by (1.3), we introduce the following generalization of the operators (1.2):

$$\widetilde{S}_{n,\alpha,\beta}(f,x) := \left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n} \lambda_{nk}^{-1} q_{nk}(x) \int_{A_n} q_{nk}(t) f\left(\frac{nt+\alpha_1}{n+\beta_1}\right) dt,$$

where

$$\lambda_{nk} = \int_{A_n} q_{nk}(t)\, dt, \quad k = 0,1,\ldots,n,$$

and α_k, β_k, $k = 1,2$ positive numbers satisfying $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$.

By Lemma 1 in Section 2, we observe that $\widetilde{S}_{n,\alpha,\beta}(f,x)$ can be rewritten as follows:

$$\widetilde{S}_{n,\alpha,\beta}(f,x) = \left(\frac{n+\beta_2}{n}\right)^{2n+1} \sum_{k=0}^{n} q_{nk}(x)(n+1) \int_{A_n} q_{nk}(t) f\left(\frac{nt+\alpha_1}{n+\beta_1}\right) dt.$$

Especially, when $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$, $\widetilde{S}_{n,\alpha,\beta}(f,x)$ reduces to the classical Bernstein-Durrmeyer operators in (1.2). Many authors have studied some special cases of the operators $\widetilde{S}_{n,\alpha,\beta}(f,x)$. For example, the case $\alpha_1 = \alpha_2 = \beta_1 = 0$ in [6] by Jung, Deo, and Dhamija, the case $\alpha_1 = \beta_1 = 0$ in [7] by Acar, Aral, and Gupta.

The main purpose of the present paper is to establish pointwise direct and converse approximation theorems of approximation by $\widetilde{S}_{n,\alpha,\beta}(f,x)$. To state our result, we need some notations:

$$\omega_{\varphi^\lambda}^2(f,t) = \sup_{0<h\leq t} \sup_{x\pm h\varphi^\lambda \in A_n} \left|\Delta_{h\varphi^\lambda}^2 f(x)\right|, \tag{1.5}$$

$$D_\lambda^2 = \left\{f \in C(A_n), f' \in A.C._{\text{loc}}, \left\|\varphi^{2\lambda} f''\right\| < +\infty\right\},$$

$$K_{\varphi^\lambda}\left(f,t^2\right) = \inf_{g\in D_\lambda^2} \left\{\|f-g\| + t^2 \left\|\varphi^{2\lambda} g''\right\|\right\}, \tag{1.6}$$

$$\overline{D}_\lambda^2 = \left\{f \in D_\lambda^2, \left\|f''\right\| < +\infty\right\},$$

$$\overline{K}_{\varphi^\lambda}\left(f,t^2\right) = \inf_{g\in \overline{D}_\lambda^2} \left\{\|f-g\| + t^2 \left\|\varphi^{2\lambda} g''\right\| + t^{4/(2-\lambda)} \left\|g''\right\|\right\}, \tag{1.7}$$

and $\varphi(x) = \sqrt{(x - \frac{\alpha_2}{n+\beta_2})(\frac{n+\alpha_2}{n+\beta_2} - x)}$, $0 \leq \lambda \leq 1$. It is well known (see [8], Theorem 3.1.2) that

$$\omega_{\varphi^\lambda}^2(f,t) \sim K_{\varphi^\lambda}\left(f,t^2\right) \sim \overline{K}_{\varphi^\lambda}\left(f,t^2\right), \tag{1.8}$$

where $x \sim y$ means that there exists a positive constant c such that $c^{-1}y \leq x \leq cy$.

Our first result can be read as follows.

Theorem 1 *Let f be a continuous function on A_n, $\lambda \in [0,1]$ be a fixed positive number. Then there exists a positive constant C only depending on λ, α_1, α_2, β_1, and β_2 such that*

$$\left|\widetilde{S}_{n,\alpha,\beta}(f,x) - f(x)\right| \le C\left(\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \omega\left(f, \frac{1}{n}\right)\right), \tag{1.9}$$

where $\delta_n(x) = \varphi(x) + 1/\sqrt{n} \sim \max\{\varphi(x), 1/\sqrt{n}\}$, and $\omega(f,t)$ is the usual modulus of continuity of f on A_n.

Throughout the paper, C denotes either a positive absolute constant or a positive constant that may depend on some parameters but not on f, x, and n. Their values may be different at different locations.

For the converse result, we have the following.

Theorem 2 *Let f be a continuous function on A_n, $0 < \alpha < \frac{2}{2-\lambda}$, $0 \le \lambda \le 1$. Then*

$$\left|\widetilde{S}_{n,\alpha,\beta}(f,x) - f(x)\right| = O\left(\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^\alpha\right) \tag{1.10}$$

implies that

$$\text{(i)}\quad \omega^2_{\varphi^\lambda}(f,t) = O\left(t^\alpha\right); \qquad \text{(ii)}\quad \omega(f,t) = O\left(t^{\alpha(1-\lambda/2)}\right). \tag{1.11}$$

2 Auxiliary lemmas

Lemma 1 *We have*

$$\lambda_{kn} = \int_{A_n} q_{nk}(t)\,dt = \left(\frac{n}{n+\beta_2}\right)^{n+1} \frac{1}{n+1}, \quad k = 0,1,\dots,n. \tag{2.1}$$

Proof For $p, q = 1, 2, \dots$, set

$$\begin{aligned} B^*(p,q) &:= \int_{A_n} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^{p-1} \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{q-1} dx \\ &= \int_0^{\frac{n}{n+\beta_2}} x^{p-1} \left(\frac{n}{n+\beta_2} - x\right)^{q-1} dx. \end{aligned}$$

Then

$$\begin{aligned} B^*(p,q) &= \frac{q-1}{p} \int_0^{\frac{n}{n+\beta_2}} x^p \left(\frac{n}{n+\beta_2} - x\right)^{q-2} dx \\ &= \frac{q-1}{p} \int_0^{\frac{n}{n+\beta_2}} \left(\frac{n}{n+\beta_2}x^{p-1} - x^{p-1}\left(\frac{n}{n+\beta_2} - x\right)\right)\left(\frac{n}{n+\beta_2} - x\right)^{q-2} dx \\ &= \frac{q-1}{p} \cdot \frac{n}{n+\beta_2} B^*(p,q-1) - \frac{q-1}{p} B^*(p,q), \end{aligned}$$

which implies that

$$B^*(p,q) = \frac{q-1}{p+q-1} \cdot \frac{n}{n+\beta_2} B^*(p,q-1).$$

Therefore,

$$\begin{aligned}\lambda_{kn} &= \binom{n}{k} B^*(k+1, n-k+1)\\ &= \left(\frac{n}{n+\beta_2}\right)^{n-k} \binom{n}{k} \frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n+1)n\cdots(k+2)} B^*(k+1,1)\\ &= \left(\frac{n}{n+\beta_2}\right)^{n-k} \frac{k+1}{(n+1)} \int_0^{\frac{n}{n+\beta_2}} x^k\,dx\\ &= \left(\frac{n}{n+\beta_2}\right)^{n+1} \frac{1}{n+1}.\end{aligned}$$

□

Lemma 2 *For any $x \in A_n$, we have*

$$\widetilde{S}_{n,\alpha,\beta}\big((t-x)^2, x\big) \le \frac{C}{n}\delta_n^2(x). \tag{2.2}$$

Proof Write

$$\widetilde{D}_{n,\alpha,\beta}(f,x) := \left(\frac{n+\beta_2}{n}\right)^{2n+1} \sum_{k=0}^{n} q_{nk}(x)(n+1)\int_{A_n} q_{nk}(t)f(t)\,dt.$$

Then [7]

$$\begin{aligned}&\widetilde{D}_{n,\alpha,\beta}(1,x) = 1, \widetilde{D}_{n,\alpha,\beta}(t,x) = \frac{n}{n+2}x + \frac{n+2\alpha_2}{(n+2)(n+\beta_2)}, \qquad (2.3)\\ &\widetilde{D}_{n,\alpha,\beta}\big(t^2,x\big) = \left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 \frac{n(n-1)}{(n+2)(n+3)}\\ &\quad + \frac{n}{n+\beta_2}\left(x - \frac{\alpha_2}{n+\beta_2}\right)\frac{4n}{(n+2)(n+3)}\\ &\quad + \left(\frac{n}{n+\beta_2}\right)^2 \frac{2}{(n+2)(n+3)} + \frac{2n\alpha_2}{(n+2)(n+\beta_2)}\left(x - \frac{\alpha_2}{n+\beta_2}\right)\\ &\quad + \frac{2n\alpha_2}{(n+2)(n+\beta_2)^2} + \left(\frac{\alpha_2}{n+\beta_2}\right)^2,\end{aligned}$$

and

$$\widetilde{D}_{n,\alpha,\beta}\big((t-x)^2, x\big) \le \frac{C}{n}\delta_n^2(x).$$

By the facts that

$$\begin{aligned}&\widetilde{S}_{n,\alpha,\beta}(1,x) = \widetilde{D}_{n,\alpha,\beta}(1,x) = 1,\\ &\widetilde{S}_{n,\alpha,\beta}(t,x) = \frac{n}{n+\beta_1}\widetilde{D}_{n,\alpha,\beta}(t,x) + \frac{\alpha_1}{n+\beta_1}, \qquad (2.4)\end{aligned}$$

and

$$\widetilde{S}_{n,\alpha,\beta}\big(t^2,x\big) = \frac{n^2}{(n+\beta_1)^2}\widetilde{D}_{n,\alpha,\beta}\big(t^2,x\big) + \frac{2n\alpha_1}{(n+\beta_1)^2}\widetilde{D}_{n,\alpha,\beta}(t,x) + \frac{\alpha_1^2}{(n+\beta_1)^2},$$

we get

$$\begin{aligned}
\widetilde{S}_{n,\alpha,\beta}\big((t-x)^2,x\big) &= \frac{n^2}{(n+\beta_1)^2}\widetilde{D}_{n,\alpha,\beta}\big((t-x)^2,x\big) \\
&\quad + \left(\frac{2n^2x}{(n+\beta_1)^2} + \frac{2n\alpha_1}{(n+\beta_1)^2} - \frac{2nx}{n+\beta_1}\right)\widetilde{D}_{n,\alpha,\beta}(t,x) \\
&\quad + \frac{\alpha_1^2}{(n+\beta_1)^2} - \frac{2\alpha_1 x}{n+\beta_1} + x^2 - \frac{n^2}{(n+\beta_1)^2}x^2 \\
&= \frac{n^2}{(n+\beta_1)^2}\widetilde{D}_{n,\alpha,\beta}\big((t-x)^2,x\big) + \frac{(\beta_1^2+4\beta_1)n+2\beta_1^2}{(n+\beta_1)^2(n+2)}x^2 \\
&\quad + \frac{2\alpha_1(\beta_1+\beta_2+2)n^2+2n\alpha_1(\beta_1\beta_2+2\beta_1+2\beta_2)+4\alpha_1\beta_1\beta_2}{(n+\beta_1)^2(n+2)(n+\beta_2)}x \\
&\quad + \frac{\alpha_1^2}{(n+\beta_1)^2} \\
&\leq \widetilde{D}_{n,\alpha,\beta}\big((t-x)^2,x\big) + \frac{C}{n^2} \\
&\leq \frac{C}{n}\delta_n^2(x).
\end{aligned}$$

□

Lemma 3 *For any given $\gamma \geq 0$, we have*

$$\sum_{k=0}^{n}\left|\frac{k+\alpha_2}{n+\beta_2}-x\right|^{\gamma}\left|q_{nk}(x)\right| \leq C\frac{\delta_n^{\gamma}(x)}{n^{\gamma/2}}, \quad x\in[0,1]. \tag{2.5}$$

Proof It was showed in [3] that

$$\sum_{k=0}^{n}\left|\frac{k+\alpha_1}{n+\beta_1}-x\right|^{\gamma}\left|q_{nk}(x)\right| \leq C\frac{(\delta_n^{*}(x))^{\gamma}}{n^{\gamma/2}}, \quad x\in[0,1], \tag{2.6}$$

where $\delta_n^*(x) := \psi(x) + \frac{1}{\sqrt{n}}$ and $\psi(x) = \sqrt{x(1-x)}$. We verify that

$$\delta_n^*(x) \sim \delta_n(x), \quad x\in[0,1]. \tag{2.7}$$

In fact, when $x \in [\frac{2\alpha_2+1}{n+\beta_2}, \frac{n-\beta_2+2\alpha_2}{n+\beta_2}]$, we have

$$\begin{aligned}
&\frac{1}{2}x \leq x - \frac{\alpha_2}{n+\beta_2} \leq x, \\
&\frac{1}{2}(1-x) \leq \frac{n+\alpha_2}{n+\beta_2} - x \leq 1-x.
\end{aligned}$$

Thus,

$$\psi(x) \sim \varphi(x),$$

which implies (2.7) for $x \in [\frac{2\alpha_2+1}{n+\beta_2}, \frac{n-\beta_2+2\alpha_2}{n+\beta_2}]$. When $x \in [0, \frac{2\alpha_2+1}{n+\beta_2}) \cup (\frac{n-\beta_2+2\alpha_2}{n+\beta_2}, 1]$, we have

$$\delta_n^*(x) \sim \delta_n(x) \sim \frac{1}{\sqrt{n}}, \tag{2.8}$$

and thus (2.7) also holds.

Now, by (2.6) and (2.7), we have

$$\begin{aligned}\sum_{k=0}^{n}\left|\frac{k+\alpha_2}{n+\beta_2}-x\right|^{\gamma}\left|q_{nk}(x)\right| &\le \sum_{k=0}^{n}\left|\frac{k+\alpha_2}{n+\beta_2}-\frac{k+\alpha_1}{n+\beta_1}\right|^{\gamma}\left|q_{nk}(x)\right|+\sum_{k=0}^{n}\left|\frac{k+\alpha_1}{n+\beta_1}-x\right|^{\gamma}\left|q_{nk}(x)\right| \\ &\le \frac{C}{n^{\gamma}}\sum_{k=0}^{n}\left|q_{nk}(x)\right|+C\frac{\delta_n^{\gamma}(x)}{n^{\gamma/2}} \\ &\le C\frac{\delta_n^{\gamma}(x)}{n^{\gamma/2}}.\end{aligned}$$

□

Lemma 4 *For any $x \in A_n$, we have*

$$\sum_{k=0}^{n} q_{nk}(x)(n+1)\int_{A_n}\delta_n^2(t)q_{nk}(t)\,dt \le C\delta_n^2(x) \tag{2.9}$$

and

$$\sum_{k=0}^{n-1} q_{n-1,k}(x)n\int_{A_n}\delta_n^{-2}(t)q_{n+1,k+1}(t)\,dt \le C\delta_n^{-2}(x). \tag{2.10}$$

Proof By a similar calculation to that of Lemma 1, we have

$$\int_{A_n}\varphi^2(t)q_{nk}(t)\,dt = \left(\frac{n}{n+\beta_2}\right)^{n+3}\frac{(n-k+1)(k+1)}{(n+3)(n+2)(n+1)}. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{k}{n}-\frac{k^2}{n^2}\right)q_{nk}(x) &= \left(\frac{n}{n+\beta_2}\right)^{n-1}\left(x-\frac{\alpha_2}{n+\beta_2}\right)-\left(\frac{n}{n+\beta_2}\right)^{n-1}\frac{(x-\frac{\alpha_2}{n+\beta_2})}{n} \\ &\quad -\frac{n-1}{n}\left(\frac{n}{n+\beta_2}\right)^{n-2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2 \\ &= \frac{n-1}{n}\left(\frac{n}{n+\beta_2}\right)^{n-2}\varphi^2(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n} q_{nk}(x)(n+1)\int_{A_n}\delta_n^2(t)q_{nk}(t)\,dt &\le 2\sum_{k=0}^{n} q_{nk}(x)(n+1)\int_{A_n}\left(\varphi^2(t)+\frac{1}{n}\right)q_{nk}(t)\,dt \\ &\le 2\sum_{k=0}^{n} q_{nk}(x)\left(\frac{n}{n+\beta_2}\right)^{n+3}\frac{(n-k+1)(k+1)}{(n+3)(n+2)} \\ &\quad +\frac{C}{n}\sum_{k=0}^{n} q_{nk}(x) \\ &\le C\sum_{k=0}^{n} q_{nk}(x)\left(\frac{(n-k)k}{n^2}+\frac{1}{n}\right)+\frac{C}{n} \\ &\le C\delta_n^2(x),\end{aligned}$$

which proves (2.9).

By Lemma 1, we have

$$\begin{aligned} n\int_{A_n}\delta_n^{-2}(t)q_{n+1,k+1}(t)\,dt &\le Cn\int_{A_n}\big(\varphi^{-2}(t)+n\big)q_{n+1,k+1}(t)\,dt \\ &\le Cn\bigg(\int_{A_n}\varphi^{-2}(t)q_{n+1,k+1}(t)\,dt+1\bigg) \\ &= Cn\bigg(\frac{(n+1)n}{(k+1)(n-k)}\int_{A_n}q_{n-1,k}(t)\,dt+1\bigg) \\ &\le Cn\bigg(\frac{(n+1)}{(k+1)(n-k)}+1\bigg) \\ &\le Cn. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=0}^{n-1}q_{n-1,k}(x)n\int_{A_n}\delta_n^{-2}(t)q_{n+1,k+1}(t)\,dt &\le Cn\sum_{k=0}^{n}q_{n-1,k}(x) \\ &= Cn\le C\delta_n^{-2}(x). \end{aligned}$$

Hence, (2.10) is proved. □

Lemma 5 *If f is r times differentiable on* $[0,1]$, *then*

$$\begin{aligned} \widetilde{S}^{(r)}_{n,\alpha,\beta}(f,x) &= \left(\frac{n+\beta_2}{n}\right)^{2n+1}\left(\frac{n}{n+\beta_1}\right)^{r}\frac{(n+1)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}q_{n-r,k}(x) \\ &\quad\times\int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}}q_{n+r,k+r}(t)f^{(r)}\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt. \end{aligned} \tag{2.12}$$

Proof By using Leibniz's theorem, we have

$$\begin{aligned} \widetilde{S}^{(r)}_{n,\alpha,\beta}(f,x) &= \left(\frac{n+\beta_2}{n}\right)^{2n+1}\sum_{i=0}^{r}\sum_{k=i}^{n-r+i}\binom{r}{i}\frac{(-1)^{r-i}(n+1)!}{(k-i)!(n-k-r+i)!} \\ &\quad\times\left(x-\frac{\alpha_2}{n+\beta_2}\right)^{k-i}\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k-r+i}\int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}}q_{nk}(t)f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt \\ &= \left(\frac{n+\beta_2}{n}\right)^{2n+1}\sum_{k=i}^{n-r+i}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{r-i}(n+1)!}{(n-r)!}q_{n-r,k-i}(x) \\ &\quad\times\int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}}q_{nk}(t)f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt \\ &= \left(\frac{n+\beta_2}{n}\right)^{2n+1}\frac{(n+1)!}{(n-r)!}\sum_{k=0}^{n-r}(-1)^r q_{n-r,k}(x) \\ &\quad\times\int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}}\sum_{i=0}^{r}\binom{r}{i}(-1)^i q_{n,k+i}(t)f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)dt. \end{aligned}$$

Since

$$\frac{d^r}{dt^r} q_{n+r,k+r}(t) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \frac{(n+r)!}{n!} q_{n,k+i}(t),$$

we have

$$\widetilde{S}^{(r)}_{n,\alpha,\beta}(f,x) = \left(\frac{n+\beta_2}{n}\right)^{2n+1} \frac{(n+1)!n!}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} q_{n-r,k}(x) \\ \times \int_{\frac{\alpha_2}{n+\beta_2}}^{\frac{n+\alpha_2}{n+\beta_2}} (-1)^r q^{(r)}_{n+r,k+r}(t) f\left(\frac{nt+\alpha_1}{n+\beta_1}\right) dt.$$

We obtain the required result by integrating by parts r times. □

Set

$$\|f\|_0 = \sup_{x\in A_n} \left\{ \left|\delta_n^{\alpha(\lambda-1)}(x) f(x)\right| \right\};$$

$$C_{\alpha,\lambda} = \left\{ f \in C(A_n), \|f\|_0 < +\infty \right\};$$

$$\|f\|_1 = \sup_{x\in A_n} \left\{ \left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x) f'(x)\right| \right\};$$

$$C^1_{\alpha,\lambda} = \left\{ f \in C_{\alpha,\lambda}, \|f\|_1 < +\infty \right\};$$

$$\|f\|_2 = \sup_{x\in A_n} \left\{ \left|\delta_n^{2+\alpha(\lambda-1)}(x) f''(x)\right| \right\};$$

$$C^2_{\alpha,\lambda} = \left\{ f \in C_{\alpha,\lambda}, f' \in A.C._{\mathrm{loc}}, \|f\|_2 < +\infty \right\};$$

$$K^1_{\alpha,\lambda}(f,t) = \inf_{g\in C^1_{\alpha,\lambda}} \left\{ \|f-g\|_0 + t\|g\|_1 \right\};$$

$$K^2_{\alpha,\lambda}(f,t) = \inf_{g\in C^2_{\alpha,\lambda}} \left\{ \|f-g\|_0 + t\|g\|_2 \right\}.$$

Lemma 6 *If* $0 \le \lambda \le 1$, $0 < \alpha < 2$, *then*

$$\left\|\widetilde{S}_{n,\alpha,\beta}(f)\right\|_1 \le Cn^{1/(2-\lambda)} \|f\|_0, \quad f \in C_{\alpha,\lambda}, \tag{2.13}$$

$$\left\|\widetilde{S}_{n,\alpha,\beta}(f)\right\|_1 \le C\|f\|_1, \quad f \in C^1_{\alpha,\lambda}. \tag{2.14}$$

Proof Firstly, we prove (2.13) by considering the following two cases.

Case 1. $x \in B_n := [\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}]$. In this case, we have

$$\varphi(x) \ge \min\left(\varphi\left(\frac{\alpha_2+1}{n+\beta_2}\right), \varphi\left(\frac{n+\alpha_2-1}{n+\beta_2}\right)\right) \ge \frac{C}{\sqrt{n}},$$

which means that

$$\delta_n(x) \sim \varphi(x) \quad \text{for } x \in B_n. \tag{2.15}$$

By simple calculations, we have

$$q'_{nk}(x) = n\varphi^{-2}(x)\left(\frac{k+\alpha_2}{n+\beta_2} - x\right)q_{nk}(x) \tag{2.16}$$

and

$$\begin{aligned}\delta_n\left(\frac{nt+\alpha_1}{n+\beta_1}\right) &= \sqrt{\left(t - \frac{\alpha_2}{n+\beta_2} + \frac{\alpha_1-\beta_1 t}{n+\beta_1}\right)\left(\frac{n+\alpha_2}{n+\beta_2} - t + \frac{\beta_1 t - \alpha_1}{n+\beta_1}\right)} + \frac{1}{\sqrt{n}} \\ &= \sqrt{\varphi^2(t) + O\left(\frac{1}{n}\right)} + \frac{1}{\sqrt{n}} \sim \varphi(t) + \frac{1}{\sqrt{n}} = \delta_n(t).\end{aligned} \tag{2.17}$$

By (2.1), (2.15)-(2.17), and Hölder's inequality, we have

$$\begin{aligned}&\left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\widetilde{S}'_{n,\alpha,\beta}(f,x)\right| \\ &\quad\le Cn\varphi^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)-2}(x)\left(\frac{n+\beta_2}{n}\right)^{2n+1} \\ &\qquad\times\sum_{k=0}^{n} q_{nk}(x)\left|\frac{k+\alpha_2}{n+\beta_2} - x\right|(n+1)\left|\int_{A_n} f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)q_{nk}(t)\,dt\right| \\ &\quad\le Cn\|f\|_0\varphi^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)-2}(x)\sum_{k=0}^{n} q_{nk}(x)\left|\frac{k+\alpha_2}{n+\beta_2} - x\right|(n+1)\left|\int_{A_n}\delta_n^{\alpha(1-\lambda)}(t)q_{nk}(t)\,dt\right| \\ &\quad\le Cn\|f\|_0\varphi^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)-2}(x)\sum_{k=0}^{n} q_{nk}(x)\left|\frac{k+\alpha_2}{n+\beta_2} - x\right|\left((n+1)\int_{A_n}\delta_n^2(t)q_{nk}(t)\,dt\right)^{\alpha(1-\lambda)/2} \\ &\qquad\times\left((n+1)\int_{A_n} q_{nk}(t)\,dt\right)^{1-\alpha(1-\lambda)/2} \\ &\quad\le Cn\|f\|_0\varphi^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)-2}(x)\sum_{k=0}^{n} q_{nk}(x)\left|\frac{k+\alpha_2}{n+\beta_2} - x\right|\left((n+1)\int_{A_n}\delta_n^2(t)q_{nk}(t)\,dt\right)^{\alpha(1-\lambda)/2}.\end{aligned}$$

By (2.9), (2.15) (2.5), and Hölder's inequality again, we have

$$\begin{aligned}&\left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\widetilde{S}'_{n,\alpha,\beta}(f,x)\right| \\ &\quad\le Cn\|f\|_0\varphi^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)-2}(x)\left(\sum_{k=0}^{n} q_{nk}(x)\left|\frac{k+\alpha_2}{n+\beta_2} - x\right|^{\frac{1}{1-\alpha(1-\lambda)/2}}\right)^{1-\alpha(1-\lambda)/2} \\ &\qquad\times\left(\sum_{k=0}^{n} q_{nk}(x)(n+1)\int_{A_n}\delta_n^2(t)q_{nk}(t)\,dt\right)^{\alpha(1-\lambda)/2} \\ &\quad\le Cn^{1/2}\|f\|_0\varphi^{\frac{2(1-\lambda)}{2-\lambda}-1}(x) \le Cn^{1/(2-\lambda)}\|f\|_0.\end{aligned} \tag{2.18}$$

Case 2. $x \in B_n^c = [\frac{\alpha_2}{n+\beta_2}, \frac{\alpha_2+1}{n+\beta_2}) \cup (\frac{n+\alpha_2-1}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$. In this case, we have

$$\delta_n(x) \sim \frac{1}{\sqrt{n}}, \quad x \in B_n^c. \tag{2.19}$$

Noting that

$$q'_{nk}(x) = n\big(q_{n-1,k-1}(x) - q_{n-1,k}(x)\big)$$

with $q_{n-1,-1}(x) = q_{n-1,n}(x) = 0$, we get

$$\widetilde{S}'_{n,\alpha,\beta}(f,x) = n\sum_{k=0}^{n-1} q_{n-1,k}(x)(x)(n+1)\int_{A_n} f\left(\frac{nt+\alpha_1}{n+\beta_1}\right)\big(q_{n,k+1}(t) - q_{n,k}(t)\big)\,dt.$$

Then, by using (2.17) and Hölder's inequality twice,

$$\begin{aligned}
&\big|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\widetilde{S}'_{n,\alpha,\beta}(f,x)\big| \\
&\quad\le Cn\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\|f\|_0\left|\sum_{k=0}^{n-1} q_{n-1,k}(x)(n+1)\int_{A_n}\delta_n^{\alpha(1-\lambda)}(t)\big(q_{n,k+1}(t)+q_{n,k}(t)\big)\,dt\right| \\
&\quad\le Cn\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\|f\|_0\sum_{k=0}^{n-1} q_{n-1,k}(x)\left((n+1)\int_{A_n}\delta_n^{2}(t)\big(q_{n,k+1}(t)+q_{n,k}(t)\big)\,dt\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\le Cn\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\|f\|_0\left(\sum_{k=0}^{n-1} q_{n-1,k}(x)(n+1)\int_{A_n}\delta_n^{2}(t)\big(q_{n,k+1}(t)+q_{n,k}(t)\big)\,dt\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\le Cn\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\|f\|_0\delta_n^{\alpha(1-\lambda)} \\
&\quad\le Cn^{\frac{1}{2-\lambda}}\|f\|_0,
\end{aligned}\tag{2.20}$$

where in the fourth inequality, we used the following fact, which can be deduced exactly in the same way as (2.10):

$$\sum_{k=0}^{n-1} q_{n-1,k}(x)(n+1)\int_{A_n}\delta_n^2(t)q_{nk^*}(t)\,dt \le C\delta_n^2(x).$$

We obtain (2.13) by combining (2.18) and (2.20).

Now, we begin to prove (2.14). If $(\frac{2}{2-\lambda}-\alpha)(\lambda-1)<0$, by (2.12) and using Hölder's inequality twice, we get

$$\begin{aligned}
&\big|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\widetilde{S}'_{n,\alpha,\beta}(f,x)\big| \\
&\quad\le C\|f\|_1\left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)n\sum_{k=0}^{n-1} q_{n-1,k}(x)\int_{A_n} q_{n+1,k+1}(t)\delta_n^{(\frac{2}{2-\lambda}-\alpha)(\lambda-1)}(t)\,dt\right| \\
&\quad\le C\|f\|_1\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\sum_{k=0}^{n-1} q_{n-1,k}(x)\left(n\int_{A_n} q_{n+1,k+1}(t)\delta_n^{-2}(t)\,dt\right)^{\frac12(\frac{2}{2-\lambda}-\alpha)(1-\lambda)} \\
&\qquad\times\left(n\int_{A_n} q_{n+1,k+1}(t)\,dt\right)^{1-\frac12(\frac{2}{2-\lambda}-\alpha)(1-\lambda)} \\
&\quad\le C\|f\|_1\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\sum_{k=0}^{n-1} q_{n-1,k}(x)\left(n\int_{A_n} q_{n+1,k+1}(t)\delta_n^{-2}(t)\,dt\right)^{\frac12(\frac{2}{2-\lambda}-\alpha)(1-\lambda)} \\
&\qquad\text{(by (2.1))}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|f\|_1 \delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\left(\sum_{k=0}^{n-1} q_{n-1,k}(x) n \int_{A_n} q_{n+1,k+1}(t)\delta_n^{-2}(t)\,dt\right)^{\frac{1}{2}(\frac{2}{2-\lambda}-\alpha)(1-\lambda)} \\
&\leq C\|f\|_1,
\end{aligned}$$

where, in the last inequality, (2.10) is applied.

If $(\frac{2}{2-\lambda}-\alpha)(\lambda-1) > 0$, by using (2.9) instead of (2.10), we also can deduce that

$$\left|\delta_n^{(\frac{2}{2-\lambda}-\alpha)(1-\lambda)}(x)\widetilde{S}'_{n,\alpha,\beta}(f,x)\right| \leq C\|f\|_1. \qquad \square$$

Lemma 7 *If* $0 \leq \lambda \leq 1$, $0 < \alpha < 2$, *then*

$$\left\|\widetilde{S}_{n,\alpha,\beta}(f)\right\|_2 \leq Cn\|f\|_0, \quad f \in C_{\alpha,\lambda}, \tag{2.21}$$

$$\left\|\widetilde{S}_{n,\alpha,\beta}(f)\right\|_2 \leq C\|f\|_2, \quad f \in C^2_{\alpha,\lambda}. \tag{2.22}$$

Proof It can be proved in a way similar to Lemma 6. □

Lemma 8 *For* $0 < t < \frac{1}{8}$, $\frac{t}{2} \leq x \leq 1-\frac{t}{2}$, $x \in [0,1]$, $\beta < 2$, *we have*

$$\int_{-t/2}^{t/2} \delta_n^{-\beta}(x+u)\,du \leq C(\beta)t\delta_n^{-\beta}(x). \tag{2.23}$$

Lemma 9 *For* $0 < t < \frac{1}{4}$, $t \leq x \leq 1-t$, $x \in [0,1]$, $0 \leq \beta \leq 2$, *we have*

$$\int_{-t/2}^{t/2}\int_{-t/2}^{t/2} \delta_n^{-\beta}(x+u+v)\,du\,dv \leq Ct^2\delta_n^{-\beta}(x). \tag{2.24}$$

It has been shown in [9] that Lemma 8 and Lemma 9 are valid when $\delta_n(t)$ is replaced by $\delta_n^*(t)$, which combining with (2.8) proves Lemma 8 and Lemma 9.

3 Proofs of theorems

3.1 Proof of Theorem 1

Define the auxiliary operators $\mathbf{S}_{n,\alpha,\beta}(f,x)$ as follows:

$$\mathbf{S}_{n,\alpha,\beta}(f,x) = \widetilde{S}_{n,\alpha,\beta}(f,x) + L_{n,\alpha,\beta}(f,x), \tag{3.1}$$

where

$$L_{n,\alpha,\beta}(f,x) = f(x) - f\big(\widetilde{S}_{n,\alpha,\beta}(t,x)\big).$$

By (2.3) and (2.4), we have

$$\left|\widetilde{S}_{n,\alpha,\beta}(t,x) - x\right| \leq \frac{C}{n}, \tag{3.2}$$

$$\mathbf{S}_{n,\alpha,\beta}(1,x) = 1, \qquad \mathbf{S}_{n,\alpha,\beta}(t-x,x) = 0, \tag{3.3}$$

and

$$\|\mathbf{S}_{n,\alpha,\beta}\| \leq 3. \tag{3.4}$$

It follows from (3.2) that

$$\left|L_{n,\alpha,\beta}(f,x)\right| \le \omega\left(f, \left|\widetilde{S}_{n,\alpha,\beta}(t,x)-x\right|\right) \le C\omega\left(f, \frac{1}{n}\right). \tag{3.5}$$

From (1.7) and (1.8), for any fixed x, λ, and n, we may choose a $g_{n,x,\lambda}(t) \in \overline{D}^2_\lambda$ such that

$$\|f-g\| \le C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right), \tag{3.6}$$

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^2 \left\|\varphi^{2\lambda}g''\right\| \le C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right), \tag{3.7}$$

$$\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{4/(2-\lambda)} \left\|g''\right\| \le C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right). \tag{3.8}$$

By (3.4) and (3.6), we have

$$\begin{aligned}\left|\mathbf{S}_{n,\alpha,\beta}(f,x)-f(x)\right| &\le \left|\mathbf{S}_{n,\alpha,\beta}\left((f-g),x\right)\right| + \left|f(x)-g(x)\right| + \left|\mathbf{S}_{n,\alpha,\beta}(g,x)-g(x)\right| \\ &\le 4\|f-g\| + \left|\mathbf{S}_{n,\alpha,\beta}(g,x)-g(x)\right| \\ &\le C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right) + \left|\mathbf{S}_{n,\alpha,\beta}(g,x)-g(x)\right|. \end{aligned} \tag{3.9}$$

Noting that $\varphi^{2\lambda}(x)$ and $\delta_n^{2\lambda}(x)$ are concave functions on $[0,1]$, for any $t,x \in [0,1]$, and u between x and t, say $u = \theta x + (1-\theta)t$, $0 \le \theta \le 1$, we have

$$\frac{|t-u|}{\varphi^{2\lambda}(u)} = \frac{\theta|t-x|}{\varphi^{2\lambda}(\theta x+(1-\theta)t)} \le \frac{\theta|t-x|}{\theta\varphi^{2\lambda}(x)+(1-\theta)\varphi^{2\lambda}(t)} \le \frac{|t-x|}{\varphi^{2\lambda}(x)}, \tag{3.10}$$

$$\frac{|t-u|}{\delta_n^{2\lambda}(u)} \le \frac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{3.11}$$

By using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du,$$

(3.3), and (3.11),

$$\begin{aligned}\left|\mathbf{S}_{n,\alpha,\beta}(g,x)-g(x)\right| &= \left|\mathbf{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du, x\right)\right| \\ &\le \left|\widetilde{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du, x\right)\right| \\ &\quad + \left|\int_x^{\widetilde{S}_{n,\alpha,\beta}(t,x)} \left(\widetilde{S}_{n,\alpha,\beta}(t,x)-u\right)g''(u)\,du\right|. \end{aligned}$$

When $x \in B_n$, by (2.15), (3.10), (3.2), and (2.2), we have

$$\begin{aligned}\left|\mathbf{S}_{n,\alpha,\beta}(g,x)-g(x)\right| &\le C\left\|\varphi^{2\lambda}g''\right\|\widetilde{S}_{n,\alpha,\beta}\left(\frac{(t-x)^2}{\varphi^{2\lambda}(x)}, x\right) + \varphi^{-2\lambda}(x)\left\|\varphi^{2\lambda}g''\right\|\left(\widetilde{S}_{n,\alpha,\beta}(t,x)-x\right)^2 \\ &\le Cn^{-1}\delta_n^{2-2\lambda}(x)\left\|\varphi^{2\lambda}g''\right\| \\ &\le C\omega^2_{\varphi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right), \end{aligned} \tag{3.12}$$

where in the last inequality, (3.7) is applied.

When $x \in B_n^c$, by (2.19), (3.10), (3.2), and (2.2), we have

$$\begin{aligned}
\left|\mathbf{S}_{n,\alpha,\beta}(g,x) - g(x)\right| &\leq C\left\|\delta_n^{2\lambda} g''\right\| \widetilde{S}_{n,\alpha,\beta}\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)}, x\right) + \delta_n^{-2\lambda}(x)\left\|\delta_n^{2\lambda} g''\right\| \left(\widetilde{S}_{n,\alpha,\beta}(t,x) - x\right)^2 \\
&\leq Cn^{-1}\delta_n^{2-2\lambda}(x)\left(\left\|\varphi^{2\lambda} g''\right\| + \frac{1}{n^\lambda}\left\|g''\right\|\right) \\
&\leq Cn^{-1}\delta_n^{2-2\lambda}(x)\left\|\varphi^{2\lambda} g''\right\| + C\left(n^{-1/2}\delta_n^{1-\lambda}(x)\right)^{4/(2-\lambda)}\left\|g''\right\| \\
&\leq C\omega_{\varphi^\lambda}^2\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right), \qquad (3.13)
\end{aligned}$$

where in the last inequality, we used (3.7) and (3.8).

We complete the proof of Theorem 1 by combining (3.1), (3.5), (3.9), (3.12), and (3.13).

3.2 Proof of Theorem 2

With Lemma 6-Lemma 9, the proof of Theorem 2 can be found exactly in the same way as that of [9]. We omit the details here.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this work. All authors read and approved the final manuscript

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