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Regularized gradient-projection methods for finding the minimum-norm solution of the constrained convex minimization problem

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Abstract

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Assume that g is a real-valued convex function and the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Let $0 < \lambda < \frac{2}{L+2}$, $0 < \beta_n < 1$. We prove that the sequence $\{x_n\}$ generated by the iterative algorithm $x_{n+1} = P_C(I - \lambda(\nabla g + \beta_n I))x_n$, $\forall n \geq 0$ converges strongly to $q \in U$, where $q = P_U(0)$ is the minimum-norm solution of the constrained convex minimization problem, which also solves the variational inequality $\langle -q, p - q\rangle \leq 0$, $\forall p \in U$. Under suitable conditions, we obtain some strong convergence theorems. As an application, we apply our algorithm to solving the split feasibility problem in Hilbert spaces.

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Keywords: regularized gradient-projection method; minimum-norm; the constrained convex minimization problem; variational inequality

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers. Suppose that f is a contraction on H with coefficient $0 < \alpha < 1$. A nonlinear operator $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We use $\mathrm{Fix}(T)$ to denote the fixed point of T.

Firstly, consider the constrained convex minimization problem:

$$\min_{x\in C} g(x), \tag{1.1}$$

where $g : C \to \mathbb{R}$ is a real-valued convex function. Assume that the constrained convex minimization problem (1.1) is solvable, let U denote its solution set. The gradient-projection algorithm (GPA) is an effective method for solving the constrained convex minimization problem (1.1). A sequence $\{x_n\}$ generated by the following recursive formula:

$$x_{n+1} = P_C(I - \lambda\nabla g)x_n, \quad \forall n \geq 0, \tag{1.2}$$

where the parameter λ is real positive number. In general, if the gradient ∇g is L-Lipschitz continuous and η-strongly monotone, $0 < \lambda < \frac{2\eta}{L^2}$, the sequence $\{x_n\}$ generated by (1.2) converges strongly to a minimizer of (1.1). However, if the gradient ∇g is only to be $\frac{1}{L}$-ism with $L > 0$, $0 < \lambda < \frac{2}{L}$, the sequence $\{x_n\}$ generated by (1.2) converges weakly to a minimizer of (1.1).

Recently, many authors combined the constrained convex minimization problem with a fixed point problem [1–3] and proposed composited iterative algorithms to find a solution of the constrained convex minimization problem [4–7].

In 2000, Moudafi [8] introduced the viscosity approximation method for nonexpansive mappings.

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0. \tag{1.3}$$

In 2001, Yamada [9] introduced the so-called hybrid steepest-descent algorithm:

$$x_{n+1} = Tx_n - \mu\lambda_n FTx_n, \quad \forall n \geq 0, \tag{1.4}$$

where F is Lipschitzian and strongly monotone operator. In 2006, Marino and Xu [10] considered a generative algorithm:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.5}$$

where A is a strongly positive operator. In 2010, Tian [11] combined the iterative algorithm of (1.4), (1.5), and proposed a new iterative algorithm:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad \forall n \geq 0. \tag{1.6}$$

In 2010, Tian [12] generalized (1.6), obtained the following iterative algorithm:

$$x_{n+1} = \alpha_n \gamma Vx_n + (I - \mu\alpha_n F)Tx_n, \quad \forall n \geq 0, \tag{1.7}$$

where V is Lipschitzian operator. Based on these iterative algorithms, some authors combined GPA with averaged operator to solve the constrained convex minimization problem [13, 14].

In 2011, Ceng *et al.* [1] proposed a sequence $\{x_n\}$ generated by the following iterative algorithm:

$$x_{n+1} = P_C\bigl[\theta_n rh(x_n) + (I - \theta_n \mu F)T_n(x_n)\bigr], \quad \forall n \geq 0, \tag{1.8}$$

where $h : C \to H$ is an l-Lipschitzian mapping with a constant $l > 0$, and $F : C \to H$ is a k-Lipschitzian and η-strongly monotone operator with constants $k, \eta > 0$. $\theta_n = \frac{2-\lambda_n L}{4}$, $P_C(I - \lambda_n \nabla g) = \theta_n I + (1 - \theta_n)T_n$, $\forall n \geq 0$. Then a sequence $\{x_n\}$ generated by (1.8) converges strongly to a minimizer of (1.1).

On the other hand, Xu [15] proposed that regularization can be used to find the minimum-norm solution of the minimization problem.

Consider the following regularized minimization problem:

$$\min_{x \in C} g_\beta(x) := g(x) + \frac{\beta}{2}\|x\|^2,$$

where the regularization parameter $\beta > 0$. g is a convex function and the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Then the sequence $\{x_n\}$ generated by the following formula:

$$x_{n+1} = P_C(I - \lambda \nabla g_{\beta_n})x_n = P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n, \quad \forall n \geq 0, \tag{1.9}$$

where the regularization parameters $0 < \beta_n < 1$, $0 < \lambda < \frac{2}{L}$ converges weakly. But, if a sequence $\{x_n\}$ defined by

$$x_{n+1} = P_C(I - \lambda_n \nabla g_{\beta_n})x_n = P_C\bigl(I - \lambda_n(\nabla g + \beta_n I)\bigr)x_n, \quad \forall n \geq 0, \tag{1.10}$$

where the initial guess $x_0 \in C$, $\{\lambda_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \lambda_n \leq \frac{\beta_n}{(L+\beta_n)^2}$, $\forall n \geq 0$,
(ii) $\beta_n \to 0$ (and $\lambda_n \to 0$) as $n \to \infty$,
(iii) $\sum_{n=1}^{\infty} \lambda_n \beta_n = \infty$,
(iv) $\frac{(|\lambda_n - \lambda_{n-1}| + |\lambda_n \beta_n - \lambda_{n-1}\beta_{n-1}|)}{(\lambda_n \beta_n)^2} \to 0$ as $n \to \infty$.

Then the sequence $\{x_n\}$ generated by (1.10) converges strongly to x^*, which is the minimum-norm solution of (1.1) [15].

Secondly, Yu et al. [16] proposed a strong convergence theorem with a regularized-like method to find an element of the set of solutions for a monotone inclusion problem in a Hilbert space.

Theorem 1.1 ([16]) *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $L > 0$, F is a $\frac{1}{L}$-ism mapping of C into H. Let B be a maximal monotone mapping on H and let G be a maximal monotone mapping on H such that the domains of B and G are included in C. Let $J_\rho = (I + \rho B)^{-1}$ and $T_r = (I + rG)^{-1}$ for each $\rho > 0$ and $r > 0$. Suppose that $(F + B)^{-1}(0) \cap G^{-1}(0) \neq \emptyset$. Let $\{x_n\} \subset H$ defined by*

$$x_{n+1} = J_\rho\bigl(I - \rho(F + \beta_n I)\bigr)T_r x_n, \quad \forall n > 0, \tag{1.11}$$

where $\rho \in (0, \infty)$, $\beta_n \in (0, 1)$, $r \in (0, \infty)$. Assume that

(i) $0 < a \leq \rho < \frac{2}{2+L}$,
(ii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$.

Then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $\overline{x}$, where $\overline{x} = P_{(F+B)^{-1}(0)\cap G^{-1}(0)}(0)$.

From the article of Yu et al. [16], we obtain a new condition of parameter ρ, $0 < \rho < \frac{2}{L+2}$, which is used widely in our article. Motivated and inspired by Lin, when $0 < \lambda < \frac{2}{L+2}$, $\{\beta_n\}$ satisfy certain conditions, a sequence $\{x_n\}$ generated by the iterative algorithm (1.9):

$$x_{n+1} = P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n, \quad \forall n \geq 0,$$

converges strongly to a point $q \in U$, where $q = P_U(0)$ is the minimum-norm solution of the constrained convex minimization problem.

Finally, we give concrete example and the numerical results to illustrate our algorithm is with fast convergence.

2 Preliminaries

In this part, we introduce some lemmas that will be used in the rest part. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We use '$\rightarrow$' to denote strong convergence of the sequence $\{x_n\}$ and use '$\rightharpoonup$' to denote weak convergence.

Recall P_C is the metric projection from H into C, then to each point $x \in H$, the unique point $P_C \in C$ satisfy the property:

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

P_C has the following characteristics.

Lemma 2.1 ([17]) *For a given $x \in H$:*
(1) $z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0$, $\forall y \in C$;
(2) $z = P_C x \Longleftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
(3) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$.
From (3), *we can derive that P_C is nonexpansive and monotone.*

Lemma 2.2 (Demiclosed principle [18]) *Let $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

Lemma 2.3 ([19]) *Let $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $(0, 1)$ *and such that*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\rightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \alpha_n |\delta_n| < \infty$.
Then $\lim_{n\rightarrow\infty} a_n = 0$.

3 Main results

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Assume that $g : C \rightarrow \mathbb{R}$ is real-valued convex function and the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Suppose that the minimization problem (1.1) is consistent and let U denote its solution set. Let $0 < \lambda < \frac{2}{L+2}$, $0 < \beta_n < 1$. Consider the following mapping G_n on C defined by

$$G_n x = P_C\big(I - \lambda(\nabla g + \beta_n I)\big)x, \quad \forall x \in C, n \in \mathbb{N}.$$

We have

$$\begin{aligned} \|G_n x - G_n y\|^2 &= \big\|P_C\big(I - \lambda(\nabla g + \beta_n I)\big)x - P_C\big(I - \lambda(\nabla g + \beta_n I)\big)y\big\|^2 \\ &\leq \big\|\big(I - \lambda(\nabla g + \beta_n I)\big)x - \big(I - \lambda(\nabla g + \beta_n I)\big)y\big\|^2 \\ &= (1 - \lambda\beta_n)^2\|x - y\|^2 + \lambda^2\big\|\nabla g(x) - \nabla g(y)\big\|^2 \\ &\quad - 2\lambda(1 - \lambda\beta_n)\big\langle x - y, \nabla g(x) - \nabla g(y)\big\rangle \\ &\leq (1 - \lambda\beta_n)^2\|x - y\|^2 + \lambda^2\big\|\nabla g(x) - \nabla g(y)\big\|^2 \end{aligned}$$

$$\begin{aligned}
&\quad - \frac{2}{L}\lambda(1-\lambda\beta_n)\big\|\nabla g(x)-\nabla g(y)\big\|^2 \\
&\le (1-\lambda\beta_n)^2\|x-y\|^2 - \lambda\left(\frac{2}{L}(1-\lambda)-\lambda\right)\big\|\nabla g(x)-\nabla g(y)\big\|^2 \\
&\le (1-\lambda\beta_n)^2\|x-y\|^2.
\end{aligned}$$

That is,

$$\|G_n x - G_n y\| \le (1-\lambda\beta_n)\|x-y\|.$$

Since $0 < 1-\lambda\beta_n < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point x_n, such that

$$x_n = P_C\big(I - \lambda(\nabla g + \beta_n I)\big)x_n.$$

Next, we prove that the sequence $\{x_n\}$ converges strongly to $q \in U$, which also solves the variational inequality

$$\langle -q, p-q\rangle \le 0, \quad \forall p \in U. \tag{3.1}$$

Equivalently, $q = P_U(0)$, that is, q is the minimum-norm solution of the constrained convex minimization problem.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $g : C \to \mathbb{R}$ is real-valued convex function and assume that the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Assume that $U \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$x_n = P_C\big(I - \lambda(\nabla g + \beta_n I)\big)x_n, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Let λ, $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \lambda < \frac{2}{2+L}$,

(ii) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a point $q \in U$, where $q = P_U(0)$, which is the minimum-norm solution of the minimization problem (1.1) *and also solves the variational inequality* (3.1).

Proof First, we claim that $\{x_n\}$ is bounded. Indeed, pick any $p \in U$, then we have

$$\begin{aligned}
\|x_n - p\| &= \big\|P_C\big(I-\lambda(\nabla g+\beta_n I)\big)x_n - P_C(I-\lambda\nabla g)p\big\| \\
&\le \big\|\big(I-\lambda(\nabla g+\beta_n I)\big)x_n - \big(I-\lambda(\nabla g+\beta_n I)\big)p\big\| \\
&\quad + \big\|\big(I-\lambda(\nabla g+\beta_n I)\big)p - (I-\lambda\nabla g)p\big\| \\
&\le (1-\lambda\beta_n)\|x_n-p\| + \lambda\beta_n\|p\|.
\end{aligned}$$

Then we derive that

$$\|x_n - p\| \le \|p\|,$$

and hence $\{x_n\}$ is bounded.

Next, we claim that $\|x_n - P_C(I - \lambda\nabla g)x_n\| \to 0$. Indeed

$$\begin{aligned}\|x_n - P_C(I - \lambda\nabla g)x_n\| &= \|P_C(I - \lambda(\nabla g + \beta_n I))x_n - P_C(I - \lambda\nabla g)x_n\| \\ &\leq \|(I - \lambda(\nabla g + \beta_n I))x_n - (I - \lambda\nabla g)x_n\| \\ &\leq \lambda\beta_n\|x_n\|.\end{aligned}$$

Since $\{x_n\}$ is bounded, $\beta_n \to 0$ $(n \to \infty)$, we obtain

$$\|x_n - P_C(I - \lambda\nabla g)x_n\| \to 0.$$

∇g is $\frac{1}{L}$-ism. Consequently, $P_C(I - \lambda\nabla g)$ is a nonexpansive self-mapping on C. As a matter of fact, we have for each $x, y \in C$

$$\begin{aligned}&\|P_C(I - \lambda\nabla g)x - P_C(I - \lambda\nabla g)y\|^2 \\ &\quad\leq \|(I - \lambda\nabla g)x - (I - \lambda\nabla g)y\|^2 \\ &\quad= \|x - y - \lambda(\nabla g(x) - \nabla g(y))\|^2 \\ &\quad= \|x - y\|^2 - 2\lambda\langle x - y, \nabla g(x) - \nabla g(y)\rangle + \lambda^2\|\nabla g(x) - \nabla g(y)\|^2 \\ &\quad\leq \|x - y\|^2 - \lambda\left(\frac{2}{L} - \lambda\right)\|\nabla g(x) - \nabla g(y)\|^2 \\ &\quad\leq \|x - y\|^2.\end{aligned}$$

$\{x_n\}$ is bounded, consider a subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$. Then by Lemma 2.2, we obtain $z \in U$.

On the other hand

$$\begin{aligned}\|x_n - z\|^2 &= \|P_C(I - \lambda(\nabla g + \beta_n I))x_n - P_C(I - \lambda\nabla g)z\|^2 \\ &\leq \langle (I - \lambda(\nabla g + \beta_n I))x_n - (I - \lambda\nabla g)z, x_n - z\rangle \\ &= \langle (I - \lambda(\nabla g + \beta_n I))x_n - (I - \lambda(\nabla g + \beta_n I))z, x_n - z\rangle \\ &\quad+ \langle -\lambda\beta_n z, x_n - z\rangle \\ &\leq (1 - \lambda\beta_n)\|x_n - z\|^2 + \lambda\beta_n\langle -z, x_n - z\rangle.\end{aligned}$$

Thus

$$\|x_n - z\|^2 \leq \langle -z, x_n - z\rangle.$$

In particular

$$\|x_{n_i} - z\|^2 \leq \langle -z, x_{n_i} - z\rangle.$$

Since $x_{n_i} \rightharpoonup z$. Then we derive that $x_{n_i} \to z$ as $i \to \infty$.

Let q be the minimum-norm solution of U, that is, $q = P_U(0)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$. As the above proof, we know that $x_{n_i} \to z$, $z \in U$.

Then we derive that

$$\begin{aligned}
\|x_n - q\|^2 &= \bigl\|P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - q\bigr\|^2 \\
&\le \bigl\langle \bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - (I - \lambda\nabla g)q, x_n - q\bigr\rangle \\
&= \bigl\langle \bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - \bigl(I - \lambda(\nabla g + \beta_n I)\bigr)q, x_n - q\bigr\rangle \\
&\quad + \langle -\lambda\beta_n q, x_n - q\rangle \\
&\le (1 - \lambda\beta_n)\|x_n - q\|^2 + \lambda\beta_n\langle -q, x_n - q\rangle.
\end{aligned}$$

Thus

$$\|x_n - q\|^2 \le \langle -q, x_n - q\rangle.$$

In particular

$$\|x_{n_i} - q\|^2 \le \langle -q, x_{n_i} - q\rangle.$$

Since $x_{n_i} \to z$, $z \in U$,

$$\|z - q\|^2 \le \langle -q, z - q\rangle \le 0.$$

So, we have $z = q$. From the arbitrariness of $z \in U$, it follows that $q \in U$ is a solution of the variational inequality (3.1). By the uniqueness of solution of the variational inequality (3.1), we conclude that $x_n \to q$ as $n \to \infty$, where $q = P_U(0)$. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ is real-valued convex function and assume that the gradient ∇g is $\frac{1}{L}$-ism with $L > 0$. Assume that $U \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

where λ and $\{\beta_n\}$ satisfy the following conditions:

(i) $0 < \lambda < \frac{2}{L+2}$;

(ii) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point $q \in U$, where $q = P_U(0)$, which is the minimum-norm solution of the minimization problem (1.1) *and also solves the variational inequality* (3.1).

Proof First, we claim that $\{x_n\}$ is bounded. Indeed, pick any $p \in U$, then we know that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - p\| &\le \bigl\|P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)p\bigr\| \\
&\quad + \bigl\|P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)p - P_C(I - \lambda\nabla g)p\bigr\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\lambda\beta_n)\|x_n - p\| + \lambda\beta_n\|p\| \\
&\leq \max\bigl\{\|x_n - p\|, \|p\|\bigr\}.
\end{aligned}$$

By the introduction

$$\|x_n - p\| \leq \max\bigl\{\|x_1 - p\|, \|p\|\bigr\},$$

and hence $\{x_n\}$ is bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$.

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \bigl\|P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - P_C\bigl(I - \lambda(\nabla g + \beta_{n-1} I)\bigr)x_{n-1}\bigr\| \\
&\leq \bigl\|\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - \bigl(I - \lambda(\nabla g + \beta_{n-1} I)\bigr)x_{n-1}\bigr\| \\
&= \bigl\|\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - \bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_{n-1} \\
&\quad - \lambda\beta_n x_{n-1} + \lambda\beta_{n-1} x_{n-1}\bigr\| \\
&\leq (1-\lambda\beta_n)\|x_n - x_{n-1}\| + \lambda|\beta_n - \beta_{n-1}| \cdot \|x_{n-1}\| \\
&\leq (1-\lambda\beta_n)\|x_n - x_{n-1}\| + \lambda|\beta_n - \beta_{n-1}| \cdot M,
\end{aligned}$$

where $M = \sup\{\|x_n\| : n \in \mathbb{N}\}$. Hence, by Lemma 2.3, we have

$$\|x_{n+1} - x_n\| \to 0.$$

Then we claim that $\|x_n - P_C(I - \lambda\nabla g)x_n\| \to 0$.

$$\begin{aligned}
\bigl\|x_n - P_C(I - \lambda\nabla g)x_n\bigr\| &= \bigl\|x_n - x_{n+1} + x_{n+1} - P_C(I - \lambda\nabla g)x_n\bigr\| \\
&\leq \|x_n - x_{n+1}\| + \bigl\|P_C\bigl(I - \lambda(\nabla g + \beta_n I)\bigr)x_n - P_C(I - \lambda\nabla g)x_n\bigr\| \\
&\leq \|x_n - x_{n+1}\| + \lambda\beta_n \cdot \|x_n\| \\
&\leq \|x_n - x_{n+1}\| + \lambda\beta_n \cdot M,
\end{aligned}$$

since $\beta_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, we have

$$\bigl\|x_n - P_C(I - \lambda\nabla g)x_n\bigr\| \to 0.$$

Next, we show that

$$\limsup_{n\to\infty}\langle -q, x_n - q\rangle \leq 0. \tag{3.4}$$

Let q be the minimum-norm solution of U, that is, $q = P_U(0)$. Since $\{x_n\}$ is bounded, without loss of generality, we assume that $x_{n_j} \rightharpoonup z$. By the same argument as in the proof of Theorem 3.1, we have $z \in U$.

$$\limsup_{n\to\infty}\langle -q, x_n - q\rangle = \lim_{j\to\infty}\langle -q, x_{n_j} - q\rangle = \langle -q, z - q\rangle \leq 0.$$

Then

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \big\|P_C\big(I-\lambda(\nabla g+\beta_n I)\big)x_n - P_C(I-\lambda\nabla g)q\big\|^2\\ &= \big\langle P_C\big(I-\lambda(\nabla g+\beta_n I)\big)x_n - P_C\big(I-\lambda(\nabla g+\beta_n I)\big)q, x_{n+1}-q\big\rangle\\ &\quad + \big\langle P_C\big(I-\lambda(\nabla g+\beta_n I)\big)q - P_C(I-\lambda\nabla g)q, x_{n+1}-q\big\rangle\\ &\le (1-\lambda\beta_n)\|x_n-q\|\cdot\|x_{n+1}-q\| + \lambda\beta_n\langle -q, x_{n+1}-q\rangle\\ &\le \frac{1-\lambda\beta_n}{2}\|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 + \lambda\beta_n\langle -q, x_{n+1}-q\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1-\lambda\beta_n)\|x_n-q\|^2 + 2\lambda\beta_n\langle -q, x_{n+1}-q\rangle\\ &= (1-\lambda\beta_n)\|x_n-q\|^2 + 2\lambda\beta_n\delta_n,\end{aligned}$$

where $\delta_n = \langle -q, x_{n+1}-q\rangle$.

It is easy to see that $\lim_{n\to\infty}\lambda\beta_n = 0$, $\sum_{n=1}^{\infty}\lambda\beta_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, by Lemma 2.3, the sequence $\{x_n\}$ converges strongly to q, where $q = P_U(0)$. This completes the proof. □

4 Application

In this part, we will illustrate the practical value of our algorithm in the split feasibility problem. In 1994, Censor and Elfving [20] came up with the split feasibility problem. The SFP is formulated as finding a point x with the property:

$$x\in C \quad \text{and} \quad Ax\in Q, \tag{4.1}$$

where C and Q are nonempty closed and convex subset of real Hilbert spaces H_1 and H_2, $A: H_1\to H_2$ is bounded linear operator.

Next, we consider the constrained convex minimization problem:

$$\min_{x\in C} g(x) = \min_{x\in C}\frac{1}{2}\|Ax-P_QAx\|^2. \tag{4.2}$$

If x^* is a solution of SFP, then $Ax^*\in Q$ and $Ax^*-P_QAx^*=0$, x^* is the solution of the minimization problem (4.2). The gradient of g is ∇g, where $\nabla g = A^*(I-P_Q)A$. Applying Theorem 3.2, we obtain the following theorem.

Theorem 4.1 *Assume that the SFP* (4.1) *is consistent. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $A: H_1\to H_2$ is bounded linear operator, $W\ne\emptyset$, where W denotes the solution set of SFP* (4.1). *Let $\{x_n\}$ be a sequence generated by $x_1\in C$ and*

$$x_{n+1} = P_C\big(I-\lambda\big(A^*(I-P_Q)A+\beta_n I\big)\big)x_n, \quad \forall n\in\mathbb{N}. \tag{4.3}$$

Let λ and $\{\beta_n\}$ satisfy the following conditions:

(i) $0<\lambda<\frac{2}{2+\|A\|^2}$;

(ii) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.
Then $\{x_n\}$ *converges strongly to a point* $q \in W$, *where* $q = P_W(0)$.

Proof We only need to show that ∇g is $\frac{1}{\|A\|^2}$-ism, then Theorem 4.1 can be obtained by Theorem 3.2.

$$\nabla g = A^*(I - P_Q)A.$$

Since P_Q is firmly nonexpansive, so P_Q is $\frac{1}{2}$-averaged mapping, then $I - P_Q$ is 1-ism, for any $x, y \in C$, we derive that

$$\begin{aligned}
\bigl\langle \nabla g(x) - \nabla g(y), x - y \bigr\rangle &= \bigl\langle A^*(I - P_Q)Ax - A^*(I - P_Q)Ay, x - y \bigr\rangle \\
&= \bigl\langle (I - P_Q)Ax - (I - P_Q)Ay, Ax - Ay \bigr\rangle \\
&\geq \bigl\| (I - P_Q)Ax - (I - P_Q)Ay \bigr\|^2 \\
&= \frac{1}{\|A\|^2} \cdot \bigl\| A^*\bigl((I - P_Q)Ax - (I - P_Q)Ay\bigr) \bigr\|^2 \\
&= \frac{1}{\|A\|^2} \cdot \bigl\| \nabla g(x) - \nabla g(y) \bigr\|^2.
\end{aligned}$$

So, ∇g is $\frac{1}{\|A\|^2}$-ism. □

5 Numerical result

In this part, we use the algorithm in Theorem 4.1 to solve a system of linear equations. Then we calculate the 4×4 system of linear equations.

Example 1 Let $H_1 = H_2 = \mathbb{R}^4$. Take

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & -2 & 3 & -3 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 4 & 3 \end{pmatrix}, \tag{5.1}$$

$$b = \begin{pmatrix} -2 \\ -10 \\ 6 \\ 18 \end{pmatrix}. \tag{5.2}$$

Then the SFP can be formulated as the problem of finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q,$$

where $C = \mathbb{R}^4$, $Q = \{b\}$. That is, x^* is the solution of the system of linear equations $Ax = b$, and

$$x^* = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}. \tag{5.3}$$

Table 1 Numerical results as regards Example 1

n	x_n^1	x_n^2	x_n^3	x_n^4	E_n
0	1.0000	1.0000	1.0000	1.0000	5.74E+00
100	1.2292	2.8506	1.8424	4.0887	3.28E–01
1,000	1.2208	2.9107	1.8691	4.0722	2.81E–01
5,000	1.1128	2.9543	1.9331	4.0369	1.42E–01
10,000	1.0298	2.9880	1.9824	4.0097	3.79E–02

Table 2 Numerical results as regards Example 1

n	x_n^1	x_n^2	x_n^3	x_n^4	E_n
0	1.0000	1.0000	1.0000	1.0000	3.74E+00
100	0.6070	2.0706	1.7816	3.9672	1.03E+00
1,000	1.0094	2.8884	1.9496	4.0123	1.23E–01
5,000	1.0353	2.9643	1.9702	4.0133	5.99E–02
10,000	1.0307	2.9769	1.9774	4.0109	4.59E–02

Take $P_C = I$, where I denotes the 4×4 identity matrix. Given the parameters $\beta_n = \frac{1}{(n+2)^2}$ for $n \geq 0$, $\lambda = \frac{3}{200}$. Then by Theorem 4.1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = x_n - \frac{3}{200}A^*Ax_n + \frac{3}{200}A^*b - \frac{3}{200(n+2)^2}x_n.$$

As $n \to \infty$, we have $\{x_n\} \to x^* = (1,3,2,4)^T$.

From Table 1, we can easily see that with iterative number increasing x_n approaches to the exact solution x^* and the errors gradually approach zero.

In Tian and Jiao [21], they use another iterative algorithm to calculate the same example.

Compare Table 1 with Table 2, we find that if the parameters β_n are the same, when $\lambda \to \frac{2}{L+2}$, our algorithm is with fast convergence.

6 Conclusion

In a real Hilbert space, there are many methods to solve the constrained convex minimization problem. However, most of them cannot find the minimum-norm solution. In this article, we use the regularized gradient-projection algorithm to find the minimum-norm solution of the constrained convex minimization problem, where $0 < \lambda < \frac{2}{L+2}$. Then under some suitable conditions, new strong convergence theorems are obtained. Finally, we apply this algorithm to the split feasibility problem and use a concrete example and numerical results to illustrate that our algorithm has fast convergence.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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