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Compact differences of weighted composition operators on the weighted Bergman spaces

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Abstract
In this paper, we consider the compact differences of weighted composition operators on the standard weighted Bergman spaces. Some necessary and sufficient conditions for the differences of weighted composition operators to be compact are given, which extends Moorhouse's results in (J. Funct. Anal. 219:70-92, 2005).

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1 Introduction

Let **D** be the open unit disk in the complex plane **C** and **T** the boundary of **D**. Denote by $H(\mathbf{D})$ the space of all holomorphic functions on **D** and by $\mathbf{S}(\mathbf{D})$ the set of all holomorphic self-maps of **D**. Then, for $u \in H(\mathbf{D})$ and $\varphi \in \mathbf{S}(\mathbf{D})$, the weighted composition operator uC_φ induced by u and φ is given by

$$uC_\varphi(f) = u \cdot f \circ \varphi, \quad f \in H(\mathbf{D}).$$

When $u \equiv 1$, uC_φ is the composition operator C_φ, in other words, $C_\varphi(f) = f \circ \varphi, f \in H(\mathbf{D})$; when $\varphi(z) = z$, uC_φ is the multiplication operator M_u, *i.e.*, $M_u(f) = u \cdot f, f \in H(\mathbf{D})$. Broadly, one is interested in extracting properties of uC_φ acting on a given Banach space of holomorphic functions on **D** (boundedness, compactness, spectral properties, *etc.*) from function theoretic properties of u and φ and vice versa. In the past several decades, weighted composition operators on various spaces of holomorphic functions have been studied extensively, *e.g.*, [2–7].

As is well known, an early result of Shapiro and Taylor [8] in 1973 showed the nonexistence of the angular derivative of the inducing map at any point of the boundary of the unit disk is a necessary condition for the compactness of the composition operator on the Hardy space $H^2(\mathbf{D})$. Later, MacCluer and Shapiro [9] proved that this condition is a necessary and sufficient condition for the compactness of composition operators on the weighted Bergman spaces $A^p_\alpha(\mathbf{D})$ $(\alpha > -1)$. Using the Nevanlinna counting function, Shapiro [10] completely characterized those φ which induce compact composition operators on the Hardy space $H^2(\mathbf{D})$. With the basic questions such as compactness settled, it is natural to look at the topological structure of composition operators in the operator norm topology and this topic is of continuing interests in the theory of composition operators. Berkson [11] focused attention on the topological structure with his isolation result on $H^p(\mathbf{D})$ in 1981, which was refined later by Shapiro and Sundberg [12], and MacCluer [13]. In [12], Shapiro and Sundberg posed a question: Do the composition operators on $H^2(\mathbf{D})$ that differ from C_φ by a compact operator form the component of C_φ in the operator norm topology? While the same question was answered positively on the weighted Bergman spaces [13], this turned out to be not true on the Hardy space [14]. Some other results on differences of weighted composition operators on spaces of holomorphic functions can be found, for example, in [15–21]. In relation with the study of the topological structures, the difference or the linear sum of composition operators on various settings has been a very active topic [13, 17, 22–24]. Recently, Moorhouse [1] characterized completely the compactness for the difference of two composition operators on the Bergman space over the unit disk, and Al-Rawashdeh and Narayan [25] gave a sufficient condition for the same problem on the Hardy space. Here we continue this line to study compact differences of weighted composition operators acting on the standard weighted Bergman spaces.

The standard weighted Bergman space A^2_α $(\alpha > -1)$ is defined as follows:

$$A^2_\alpha := \left\{ f \in H(\mathbf{D}) : \|f\|^2_{A_\alpha} = \int_{\mathbf{D}} \left|f(z)\right|^2 d\lambda_\alpha(z) < \infty \right\},$$

where $d\lambda_\alpha(z) = \frac{1}{\pi}(\alpha+1)(1-|z|^2)^\alpha\, dA(z)$ and dA is the area measure on $\mathbf{D}$. As is well known, the Bergman space A^2_α is a reproducing kernel Hilbert space, the reproducing kernel at $z \in \mathbf{D}$ is $K_z(w) = \frac{1}{(1-\bar{z}w)^{\alpha+2}}$ and $\frac{1}{\|K_z\|_{A^2_\alpha}} K_z \to 0$ weakly as $|z| \to 1$.

In Section 2 we recall some related facts and results which are needed in the sequel, and then we prove our main results in Section 3. Section 4 deals with the compact perturbations of finite summations of a given weight composition operator.

Constants. Throughout the paper we use the letters C and c to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C and c will be often specified in the parentheses. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for non-negative quantities X and Y to mean $X \le CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2 Preliminaries

For $1 < t < \infty$ and $\xi \in \mathbf{T}$, let $\Delta_{t,\xi}$ be a non-tangential approach region at ξ defined by

$$\Delta_{t,\xi} := \left\{ z \in \mathbf{D} : |z - \xi| \le t\big(1 - |z|\big) \right\}$$

and $\Gamma_{t,\xi}$ the boundary curve of $\Delta_{t,\xi}$. Clearly $\Gamma_{t,\xi}$ has a corner at ξ with angle less than π. A function f is said to have a non-tangential limit at ξ, if $\lim_{z\to\xi} f(z)$ exists in each non-tangential region $\Delta_{t,\xi}$.

Let φ be a holomorphic self-map of $\mathbf{D}$. We say that φ has a finite angular derivative at $\xi \in \mathbf{T}$, if there exists a point $\eta \in \mathbf{T}$, such that the non-tangential limit as $z \to \xi$ of the difference quotient $\frac{\eta-\varphi(z)}{\xi-z}$ exists as a finite complex value. Write

$$\varphi'(\xi) := \angle\lim_{z\to\xi} \frac{\eta - \varphi(z)}{\xi - z}.$$

Denote $F(\varphi) := \{\xi \in \mathbf{T} : |\varphi'(\xi)| < \infty\}$. For $\xi \in F(\varphi)$, by the Julia-Carathéodory theorem in [26], we have

$$\left|\varphi'(\xi)\right| = \lim_{\substack{z\to\xi \\ z\in\Gamma_{t,\xi}}} \frac{1-|\varphi(z)|^2}{1-|z|^2} \tag{2.1}$$

for any $t > 1$.

For any $z \in \mathbf{D}$, let σ_z be the involutive automorphism of $\mathbf{D}$ which exchanges 0 to z, namely,

$$\sigma_z(w) = \frac{z-w}{1-\bar{z}w}, \quad w \in \mathbf{D}.$$

The pseudo-hyperbolic distance on $\mathbf{D}$ is defined by

$$\rho(z,w) = \left|\sigma_z(w)\right| = \left|\frac{z-w}{1-\bar{z}w}\right|, \quad z,w \in \mathbf{D}.$$

Then, for any $z, w \in \mathbf{D}$, it is easy to see that

$$1 - \left|\frac{z-w}{1-\bar{z}w}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}$$

and

$$1-\rho(z,w) \le \frac{1-|z|^2}{|1-\bar{z}w|} \le 1+\rho(z,w). \tag{2.2}$$

Moreover, for any $z \in \mathbf{D}$ and $0 < r < 1$, let

$$E_r(z) := \left\{w \in \mathbf{D} : \rho(z,w) < r\right\}$$

be the pseudo-hyperbolic disk with 'center' z and 'radius' r. It is well known that, for given $0 < r < 1$,

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \le \frac{1-|z|^2}{1-|w|^2} \le \frac{1+\rho(z,w)}{1-\rho(z,w)}, \quad w \in E_r(z), \tag{2.3}$$

and

$$\lambda_\alpha\left[E_r(z)\right] \approx \left(1-|z|^2\right)^{2+\alpha}, \quad z \in \mathbf{D}, \tag{2.4}$$

where the constants in the estimate above depend only on r and α. In the sequel, we set $\rho(z) := \rho(\varphi_1(z), \varphi_2(z))$ for the pseudo-hyperbolic distance of $\varphi_1(z)$ and $\varphi_2(z)$.

The following lemma is cited from [1].

Lemma 2.1 *For $\alpha > -1$, let φ be a holomorphic self-map of $\mathbf{D}$ and u a non-negative, bounded, and measurable function on $\mathbf{D}$. Define the measure $u\lambda_\alpha$ by $u\lambda_\alpha(E) := \int_E u(z)\,d\lambda_\alpha(z)$ on all Borel subsets $E \subseteq \mathbf{D}$. If*

$$\lim_{|z|\to 1} u(z)\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0,$$

then $u\lambda_\alpha \circ \varphi^{-1}$ is a compact α-Carleson measure and the inclusion map $I_\alpha : A^2_\alpha \to L^2(u\lambda_\alpha \circ \varphi^{-1})$ is compact.

For more details as regards Carleson measures, see Section 2.2 in [26].

3 Compact difference

Let $\varphi \in \mathbf{S}(\mathbf{D})$ and $u \in H(\mathbf{D})$. If the weighted composition operator uC_φ is bounded on A^2_α ($\alpha > -1$), then the adjoint $(uC_\varphi)^*$ of uC_φ satisfies

$$(uC_\varphi)^* K_z(w) = \overline{u(z)} K_{\varphi(z)}(w), \quad z, w \in \mathbf{D}.$$

For $\varphi \in \mathbf{S}(\mathbf{D})$, by the Schwarz-Pick theorem in [26],

$$\frac{1-|z|}{1-|\varphi(z)|} < \frac{1+|\varphi(0)|}{1-|\varphi(0)|} < \infty \tag{3.1}$$

for any $z \in \mathbf{D}$.

The following lemma can be obtained by modifying Lemma 5.1 in [27] (*e.g.*, at the third line on p.2929 in [27]). See also Proposition 3.2 in [28] in a different form for the unit ball case. Here, we give a more elementary proof for convenience.

Lemma 3.1 *Let φ_1 and φ_2 be holomorphic self-maps of $\mathbf{D}$. Then, for any $\xi \in F(\varphi_1)$, the following holds*:

$$\lim_{t\to\infty} \lim_{\substack{z\to\xi \\ z\in\Gamma_{t,\xi}}} \frac{1-|\varphi_1(z)|^2}{1-\varphi_1(z)\overline{\varphi_2(z)}} = \begin{cases} 1, & \text{if } \varphi_1(\xi) = \varphi_2(\xi) \text{ and } \varphi_1{}'(\xi) = \varphi_2{}'(\xi), \\ 0, & \text{otherwise.} \end{cases}$$

Proof First we notice that

$$\begin{aligned} \frac{1-|\varphi_1(z)|^2}{1-\varphi_1(z)\overline{\varphi_2(z)}} &= \frac{1-|\varphi_1(z)|^2}{1-|z|^2} \cdot \frac{1-|z|^2}{1-\varphi_1(z)\overline{\varphi_2(z)}} \\ &= \frac{1-|\varphi_1(z)|^2}{1-|z|^2} \cdot \frac{1}{\frac{1-|\varphi_1(z)|^2}{1-|z|^2} + \frac{\varphi_1(z)\overline{(\varphi_1(z)-\varphi_2(z))}}{1-|z|^2}} \\ &= \frac{1-|\varphi_1(z)|^2}{1-|z|^2} \cdot \frac{1}{\frac{1-|\varphi_1(z)|^2}{1-|z|^2} + \triangle(z)}, \end{aligned}$$

where $\triangle(z) = \frac{\varphi_1(z)\overline{(\varphi_1(z)-\varphi_2(z))}}{1-|z|^2}$, and

$$\liminf_{z\to\xi} \bigl|\triangle(z)\bigr| \ge \liminf_{z\to\xi} \frac{1}{2} \frac{1-|\varphi_2(z)|}{1-|z|}. \tag{3.2}$$

If φ_2 has no finite angular derivative at ξ, namely,

$$\liminf_{z\to\xi} \frac{1-|\varphi_2(z)|}{1-|z|} = \infty,$$

then, for any $t > 1$, by (3.2), we have

$$\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \left|\triangle(z)\right| = \infty.$$

If φ_2 has finite angular derivative at ξ and $\varphi_1(\xi) \neq \varphi_2(\xi)$, then it follows clearly that

$$\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \left|\triangle(z)\right| = \infty.$$

If φ_2 has finite angular derivative at ξ and $\varphi_1(\xi) = \varphi_2(\xi)$, then

$$\begin{aligned}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \left|\triangle(z)\right| &= \lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \left|\frac{\overline{\xi - z}}{1-|z|^2}\varphi_1(z)\overline{\left(\frac{\varphi_2(\xi)-\varphi_2(z)}{\xi - z} - \frac{\varphi_1(\xi)-\varphi_1(z)}{\xi - z}\right)}\right| \\ &= \frac{t}{2}\left|\varphi_2'(\xi) - \varphi_1'(\xi)\right|.\end{aligned}$$

Thus if $\varphi_1(\xi) = \varphi_2(\xi)$ and $\varphi_1'(\xi) = \varphi_2'(\xi)$, we have

$$\lim_{t\to\infty}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \triangle(z) = 0.$$

Otherwise if $\varphi_1(\xi) = \varphi_2(\xi)$ and $\varphi_1'(\xi) \neq \varphi_2'(\xi)$, then

$$\lim_{t\to\infty}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}} \left|\triangle(z)\right| = \infty.$$

Consequently, we get the desired result. □

To further study compact differences of weighted composition operators on A_α^2, we define $F_u(\varphi)$ as

$$F_u(\varphi) := \left\{\xi \in \mathbf{T} : \limsup_{z\to\xi}\left|u(z)\right|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} \neq 0\right\}.$$

It is easy to check that $F_u(\varphi) \subseteq F(\varphi)$ if u is bounded. To avoid the trivial case, in the sequel we assume $F_{u_i}(\varphi_i) \neq \emptyset$, $i = 1, 2$, *i.e.*, neither $u_1C_{\varphi_1}$ nor $u_2C_{\varphi_2}$ is compact on A_α^2.

In the following we take the test functions

$$g_w(z) := \frac{(1-|w|^2)^{\frac{1}{2}}}{(1-\overline{w}z)^{\frac{\alpha+3}{2}}}, \quad w, z \in \mathbf{D}.$$

First note that $\{g_w\}$ is bounded in A_α^2. Indeed, note that

$$I_c(w) = \int_{\mathbf{D}} \frac{(1-|z|^2)^\alpha}{|1-\overline{w}z|^{\alpha+2+c}}\,dA(z) \approx \frac{1}{(1-|w|^2)^c}, \quad |w| \to 1$$

for $c > 0$ by Lemma 4.2.2 in [29], and then

$$\|g_w\|_{A_\alpha^2}^2 = \frac{\alpha+1}{\pi}\int_{\mathbf{D}} \frac{(1-|w|^2)(1-|z|^2)^\alpha}{|1-\overline{w}z|^{\alpha+3}}\,dA(z) < \infty.$$

Again it is well known that $g_w(z) \to 0$ uniformly on any compact subset of $\mathbf{D}$, and hence that $g_w \to 0$ weakly as $|w| \to 1$.

We now give some necessary conditions for the difference of weighted composition operators to be compact.

Theorem 3.2 *Let φ_1, φ_2 be holomorphic self-maps of $\mathbf{D}$ and let u_1, u_2 be bounded holomorphic functions on $\mathbf{D}$ such that neither $u_1C_{\varphi_1}$ nor $u_2C_{\varphi_2}$ is compact on A^2_α. If $u_1C_{\varphi_1} - u_2C_{\varphi_2}$ is compact on A^2_α, then the following statements are true*:

(1) $F_{u_1}(\varphi_1) = F_{u_2}(\varphi_2)$.

(2) $\angle\lim_{z\to\xi} |u_1(z) - u_2(z)| = 0$ *for any* $\xi \in F_{u_1}(\varphi_1)$.

(3) $\lim_{|z|\to 1} \rho(z)(|u_1(z)|^2 \frac{1-|z|^2}{1-|\varphi_1(z)|^2} + |u_2(z)|^2 \frac{1-|z|^2}{1-|\varphi_2(z)|^2}) = 0$.

Proof Denote $T := u_1C_{\varphi_1} - u_2C_{\varphi_2}$ for short, and assume that T is compact on A^2_α. For $\xi \in F_{u_1}(\varphi_1)$, it is easy to see that

$$\lim_{\substack{w\to\xi \\ w\in\Gamma_{t,\xi}}} \|Tg_{\varphi_1(w)}\|^2_{A^2_\alpha} = 0 \tag{3.3}$$

for any $t > 1$. Using the submean value type inequality in [30] and equation (2.4), then, for a given $0 < r < 1$,

$$\|Tg_{\varphi_1(w)}\|^2_{A^2_\alpha} \geq \int_{E_r(w)} \left|Tg_{\varphi_1(w)}(z)\right|^2 d\lambda_\alpha(z) \gtrsim \left(1-|w|^2\right)^{\alpha+2} \left|Tg_{\varphi_1(w)}(w)\right|^2.$$

So by (3.3)

$$\lim_{\substack{w\to\xi \\ w\in\Gamma_{t,\xi}}} \left(\frac{1-|w|^2}{1-|\varphi_1(w)|^2}\right)^{\alpha+2} \left|u_1(w) - u_2(w)\left(\frac{1-|\varphi_1(w)|^2}{1-\overline{\varphi_1(w)}\varphi_2(w)}\right)^{\frac{\alpha+3}{2}}\right|^2 = 0$$

for all $t > 1$. Since u_1, u_2 are bounded holomorphic functions on $\mathbf{D}$ and $\xi \in F(\varphi_1)$, then it follows from our assumption $\xi \in F_{u_1}(\varphi_1)$ that

$$\lim_{t\to\infty} \lim_{\substack{w\to\xi \\ w\in\Gamma_{t,\xi}}} \frac{1-|\varphi_1(w)|^2}{1-\overline{\varphi_2(w)}\varphi_1(w)} = 1,$$

and therefore

$$\lim_{t\to\infty} \lim_{\substack{w\to\xi \\ w\in\Gamma_{t,\xi}}} \left|u_1(w) - u_2(w)\right| = 0.$$

So (2) is obtained, and thus $F_{u_1}(\varphi_1) \subseteq F_{u_2}(\varphi_2)$ by Lemma 3.1. Similarly, we have $F_{u_2}(\varphi_2) \subseteq F_{u_1}(\varphi_1)$. Thus the proof for (1) is complete.

To prove (3), we assume that there exists a sequence $\{z_n\}$ with $|z_n| \to 1$ such that

$$\lim_{n\to\infty} \rho(z_n)\left(\left|u_1(z_n)\right|^2 \frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} + \left|u_2(z_n)\right|^2 \frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}\right) > 0.$$

Without loss of generality, we may further assume that

$$\left|u_1(z_n)\right|^2\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2}\geq\left|u_2(z_n)\right|^2\frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}$$

for all n. Then

$$\limsup_{n\to\infty}\rho(z_n)\left|u_1(z_n)\right|^2\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2}>0. \tag{3.4}$$

Due to (3.1) and the boundedness of u_1 on $\mathbf{D}$, by passing to a subsequence if necessary, we can suppose that

$$\lim_{n\to\infty}\rho(z_n)=a_0,\qquad \lim_{n\to\infty}\left|u_1(z_n)\right|^2=a_1\quad\text{and}\quad\lim_{n\to\infty}\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2}=a_2$$

for some constants $a_0\in(0,1]$, $a_1>0$, $a_2>0$. Then $\lim_{n\to\infty}|u_2(z_n)|^2=a_1$ by the obtained facts (1) and (2). Actually, we may further assume that

$$\lim_{n\to\infty}\frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}=a_3$$

for some $a_3>0$. We put $f_n:=K_{z_n}/\|K_{z_n}\|_{A^2_\alpha}$, where K_{z_n} is the reproducing kernel function at $z_n\in\mathbf{D}$ in A^2_α for each $n\geq1$. So $f_n\to0$ weakly as $n\to\infty$. We will arrive at a contradiction to the compactness of $u_1C_{\varphi_1}-u_2C_{\varphi_2}$ by showing $(u_1C_{\varphi_1}-u_2C_{\varphi_2})^*f_n\nrightarrow0$ $(n\to\infty)$ in A^2_α. In fact, notice that

$$\begin{aligned}
&\left\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})^*f_n\right\|^2_{A^2_\alpha}\\
&\quad=\frac{1}{\|K_{z_n}\|^2_{A^2_\alpha}}\int_{\mathbf{D}}\left|\overline{u_1(z_n)}K_{\varphi_1(z_n)}-\overline{u_2(z_n)}K_{\varphi_2(z_n)}\right|^2d\lambda_\alpha\\
&\quad\gtrsim\left|u_1(z_n)\right|^2\left(\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2}\right)^{\alpha+2}+\left|u_2(z_n)\right|^2\left(\frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}\right)^{\alpha+2}\\
&\qquad-2\left|u_1(z_n)u_2(z_n)\right|\left(1-\rho^2(z_n)\right)^{\frac{\alpha+2}{2}}\frac{(1-|z_n|^2)^{\alpha+2}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+2}{2}}(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+2}{2}}}\\
&\quad\geq2\left(1-\left(1-\rho^2(z_n)\right)^{\frac{\alpha+2}{2}}\right)\left|u_1(z_n)u_2(z_n)\right|\frac{(1-|z_n|^2)^{\alpha+2}}{(1-|\varphi_1(z_n)|^2)^{\frac{\alpha+2}{2}}(1-|\varphi_2(z_n)|^2)^{\frac{\alpha+2}{2}}}
\end{aligned}$$

for all n. Then

$$\liminf_{n\to\infty}\left\|(u_1C_{\varphi_1}-u_2C_{\varphi_2})^*f_n\right\|^2_{A^2_\alpha}\geq2\left(1-\left(1-a_0^2\right)^{\frac{\alpha+2}{2}}\right)a_1^2(a_2a_3)^{\frac{\alpha+2}{2}}>0.$$

The contradiction implies (3), which completes the proof. □

To give a sufficient condition for the compact difference of weighted composition operators, we need the following fact from [28], pp.95-97: for any $\varepsilon>0$ small enough and $f\in A^2_\alpha$,

$$\int_{\{z\in\mathbf{D}:\rho(z)\leq\varepsilon\}}\left|f(\varphi_1)-f(\varphi_2)\right|^2d\lambda_\alpha\lesssim\varepsilon^2\|f\|^2_{A^2_\alpha}. \tag{3.5}$$

To simplify our sufficient condition, we use the following simple lemma.

Lemma 3.3 *Let φ_1, φ_2 be holomorphic self-maps of* $\mathbf{D}$ *and let u_1, u_2 be bounded holomorphic functions on* $\mathbf{D}$. *If*

$$\lim_{z\to\xi}\bigl|u_1(z)-u_2(z)\bigr| = 0 \quad \textit{for any } \xi \in F_{u_1}(\varphi_1)\cup F_{u_2}(\varphi_2)$$

and

$$\lim_{|z|\to 1}\rho(z)\left(\bigl|u_1(z)\bigr|^2\frac{1-|z|^2}{1-|\varphi_1(z)|^2}+\bigl|u_2(z)\bigr|^2\frac{1-|z|^2}{1-|\varphi_2(z)|^2}\right) = 0,$$

then $F_{u_1}(\varphi_1) = F_{u_2}(\varphi_2)$.

Proof If $F_{u_1}(\varphi_1) \neq F_{u_2}(\varphi_2)$, we may assume that $\xi \in F_{u_1}(\varphi_1)$ but $\xi \notin F_{u_2}(\varphi_2)$. Then $\xi \in F(\varphi_1)$, and $\xi \notin F(\varphi_2)$ by the assumption $\lim_{z\to\xi}|u_1(z)-u_2(z)| = 0$ and $\xi \in F_{u_1}(\varphi_1)$. Hence by Lemma 3.1,

$$\lim_{t\to\infty}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}}\frac{1-|\varphi_1(z)|^2}{1-\varphi_1(z)\overline{\varphi_2(z)}} = 0.$$

Note that

$$1-\rho^2(z) = \frac{(1-|\varphi_1(z)|^2)(1-|\varphi_2(z)|^2)}{|1-\overline{\varphi_1(z)}\varphi_2(z)|^2},$$

and (2.2) implies

$$\frac{1-|\varphi_2(z)|^2}{|1-\overline{\varphi_1(z)}\varphi_2(z)|} \leq 2.$$

So

$$\lim_{t\to\infty}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}}\rho(z) = 1,$$

and then

$$\lim_{t\to\infty}\lim_{\substack{z\to\xi\\ z\in\Gamma_{t,\xi}}}\rho(z)\left(\bigl|u_1(z)\bigr|^2\frac{1-|z|^2}{1-|\varphi_1(z)|^2}+\bigl|u_2(z)\bigr|^2\frac{1-|z|^2}{1-|\varphi_2(z)|^2}\right) \neq 0.$$

This leads to a contradiction to the assumption. Thus $F_{u_1}(\varphi_1) = F_{u_2}(\varphi_2)$. □

We are now ready to give our sufficiency theorem.

Theorem 3.4 *Let φ_1, φ_2 be holomorphic self-maps of* $\mathbf{D}$ *and let u_1, u_2 be bounded holomorphic functions on* $\mathbf{D}$. *If the following hold*:

(1) $\lim_{z\to\xi}|u_1(z)-u_2(z)| = 0$ *for any* $\xi \in F_{u_1}(\varphi_1)\cup F_{u_2}(\varphi_2)$ *and*

(2) $\lim_{|z|\to 1}\rho(z)(|u_1(z)|^2\frac{1-|z|^2}{1-|\varphi_1(z)|^2}+|u_2(z)|^2\frac{1-|z|^2}{1-|\varphi_2(z)|^2}) = 0$,

then $u_1C_{\varphi_1}-u_2C_{\varphi_2}$ *is compact on* A^2_α.

Proof Assume that $\{f_n\}$ is any bounded sequence in A^2_α such that $f_n \to 0$ $(n \to \infty)$ uniformly on each compact subsets of $\mathbf{D}$. Given $\varepsilon > 0$, we put

$$Q := \{z \in \mathbf{D} : \rho(z) \le \varepsilon\}, \qquad Q' := \mathbf{D} \backslash Q.$$

Now we can write

$$\|(u_1C_{\varphi_1} - u_2C_{\varphi_2})f_n\|^2_{A^2_\alpha} = \int_{\mathbf{D}} |u_1C_{\varphi_1}f_n - u_2C_{\varphi_2}f_n|^2\,d\lambda_\alpha = \int_Q + \int_{Q'} \tag{3.6}$$

for each n.

Let $\chi_{Q'}$ be the characteristic function of Q', then by the assumption (2),

$$\lim_{|z|\to 1} \chi_{Q'}\left(|u_1(z)|^2 \frac{1-|z|^2}{1-|\varphi_1(z)|^2} + |u_2(z)|^2 \frac{1-|z|^2}{1-|\varphi_2(z)|^2}\right) = 0.$$

So by Lemma 2.1, for the second term of the right-hand side of (3.6),

$$\begin{aligned}
&\int_{Q'} |u_1f_n(\varphi_1) - u_2f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\quad\lesssim \int_{\mathbf{D}} |\chi_{Q'}u_1C_{\varphi_1}(f_n)|^2\,d\lambda_\alpha + \int_{\mathbf{D}} |\chi_{Q'}u_2C_{\varphi_2}(f_n)|^2\,d\lambda_\alpha \to 0
\end{aligned}$$

as $n \to \infty$. For any $\xi \in F_{u_1}(\varphi_1)$, by the assumption (1), there exists $\delta(\xi) > 0$ such that $|u_1(z) - u_2(z)| < \varepsilon$ whenever $|z - \xi| < \delta(\xi)$. We decompose Q into two parts, $Q := H_1 + H_2$, where $H_1 := Q \cap (\bigcup_{\xi \in F_{u_1}(\varphi_1)} \{z \in \mathbf{D} : |z - \xi| < \delta(\xi)\})$ and $H_2 := Q \backslash H_1$. Also, for the first term of the right-hand side of (3.6), we have

$$\begin{aligned}
&\int_Q |u_1f_n(\varphi_1) - u_2f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\quad\lesssim \int_Q |u_1|^2 |f_n(\varphi_1) - f_n(\varphi_2)|^2\,d\lambda_\alpha + \int_Q |u_1 - u_2|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\quad\lesssim \int_Q |f_n(\varphi_1) - f_n(\varphi_2)|^2\,d\lambda_\alpha + \int_Q |u_1 - u_2|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\quad\lesssim \int_Q |f_n(\varphi_1) - f_n(\varphi_2)|^2\,d\lambda_\alpha + \sum_{i=1}^{i=2} \int_{H_i} |u_1 - u_2|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\quad\lesssim \int_Q |f_n(\varphi_1) - f_n(\varphi_2)|^2\,d\lambda_\alpha + \int_{H_1} |u_1 - u_2|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha \\
&\qquad + \sum_{i=1}^{i=2} \int_{H_2} |u_i|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha.
\end{aligned} \tag{3.7}$$

Note that by (3.5)

$$\int_Q |f_n(\varphi_1) - f_n(\varphi_2)|^2\,d\lambda_\alpha \lesssim \varepsilon^2$$

for all n. Also, by the definition of H_1, we can easily get

$$\int_{H_1} |u_1 - u_2|^2 |f_n(\varphi_2)|^2\,d\lambda_\alpha \le \varepsilon^2 \|C_{\varphi_2}f_n\|^2_{A^2_\alpha} \lesssim \varepsilon^2$$

for all n and

$$\lim_{|z|\to 1} \chi_{H_2} \big|u_1(z)\big|^2 \frac{1-|z|^2}{1-|\varphi_1(z)|^2} = 0. \tag{3.8}$$

We now claim that

$$\lim_{|z|\to 1} \chi_{H_2} \big|u_2(z)\big|^2 \frac{1-|z|^2}{1-|\varphi_2(z)|^2} = 0 \tag{3.9}$$

and

$$\lim_{|z|\to 1} \chi_{H_2} \big|u_1(z)\big|^2 \frac{1-|z|^2}{1-|\varphi_2(z)|^2} = 0. \tag{3.10}$$

Indeed, if either (3.9) or (3.10) fails, then we will arrive at a contradiction to (3.8), and thus the desired is obtained. To this end, we assume that there exist some $\eta \in \mathbf{T}$ and a sequence $z_n \in H_2$ satisfying $z_n \to \eta$ such that

$$\lim_{n\to\infty} \big|u_2(z_n)\big|^2 \frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2} > 0, \tag{3.11}$$

or

$$\lim_{n\to\infty} \big|u_1(z_n)\big|^2 \frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2} > 0. \tag{3.12}$$

If (3.11) holds, then $\eta \in F_{u_2}(\varphi_2)$. Thus $\eta \in F_{u_1}(\varphi_1)$ due to the fact that $F_{u_2}(\varphi_2) = F_{u_1}(\varphi_1)$ by Lemma 3.3. If (3.12) holds, then

$$\lim_{n\to\infty} \big|u_1(z_n)\big|^2 \frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} > 0,$$

because

$$\frac{1-|z_n|^2}{1-|\varphi_1(z_n)|^2} \geq \frac{1-\varepsilon}{1+\varepsilon} \frac{1-|z_n|^2}{1-|\varphi_2(z_n)|^2}$$

by $z_n \in H_2$ and (2.3). Thus we also have $\eta \in F_{u_1}(\varphi_1)$. This leads to a contradiction to (3.8). So our claim holds. Thus by Lemma 2.1, we have

$$\sum_{i=1}^{i=2} \int_{H_2} |u_i|^2 \big|f_n(\varphi_2)\big|^2 \, d\lambda_\alpha = \sum_{i=1}^{i=2} \int_{\mathbf{D}} |\chi_{H_2} u_i|^2 \big|f_n(\varphi_2)\big|^2 \, d\lambda_\alpha \to 0$$

as $n \to \infty$. Therefore the proof is complete. □

The following, given in [27] and [1], respectively, are immediate consequences of Theorems 3.2 and 3.4.

Corollary 3.5 *Let φ_1, φ_2 be holomorphic self-maps of $\mathbf{D}$ and a, b non-zero constants. If $F(\varphi_1) \neq \emptyset$ and $F(\varphi_2) \neq \emptyset$, then $aC_{\varphi_1} + bC_{\varphi_2}$ is compact on A^2_α if and only if $a + b = 0$ and $\lim_{|z|\to 1} \rho(z)(\frac{1-|z|^2}{1-|\varphi_1(z)|^2} + \frac{1-|z|^2}{1-|\varphi_2(z)|^2}) = 0$.*

Corollary 3.6 *Let φ_1, φ_2 be holomorphic self-maps of* $\mathbf{D}$, *then $C_{\varphi_1} - C_{\varphi_2}$ is compact on A^2_α if and only if* $\lim_{|z|\to 1}\rho(z)(\frac{1-|z|^2}{1-|\varphi_1(z)|^2} + \frac{1-|z|^2}{1-|\varphi_2(z)|^2}) = 0$.

Corollary 3.7 *Let u_1, u_2 be bounded holomorphic functions on* $\mathbf{D}$, *then $M_{u_1} - M_{u_2}$ is compact on A^2_α if and only if $u_1 = u_2$.*

4 Compact perturbation

In the final section, we consider the compact perturbation of finite summations of a given weighted composition operator.

Theorem 4.1 *For $i = 1, 2, \dots, N$, let $\varphi, \varphi_i \in \mathbf{S}(\mathbf{D})$ and u, u_i bounded holomorphic functions on* $\mathbf{D}$. *Suppose that $F_{u_i}(\varphi_i) \neq \emptyset$ for each i, $F_{u_i}(\varphi_i) \cap F_{u_j}(\varphi_j) = \emptyset$ $(i \neq j)$, $F_u(\varphi) = \bigcup_{i=1}^N F_{u_i}(\varphi_i)$. Define $\rho_i(z) := |\frac{\varphi(z)-\varphi_i(z)}{1-\overline{\varphi_i(z)}\varphi(z)}|$. If*

(1) $\lim_{z\to\xi}\rho_i(z)(|u(z)|^2\frac{1-|z|^2}{1-|\varphi(z)|^2} + |u_i(z)|^2\frac{1-|z|^2}{1-|\varphi_i(z)|^2}) = 0$, *and*

(2) $\lim_{z\to\xi}|u(z) - u_i(z)| = 0$ *for any* $\xi \in F_{u_i}(\varphi_i)$

for every $i = 1, 2, \dots, N$, then $uC_\varphi - \sum_{i=1}^N u_iC_{\varphi_i}$ is compact on A^2_α.

Proof Define $\mathbf{D}_i := \{z \in \mathbf{D} : |u_i(z)|^2\frac{1-|z|^2}{1-|\varphi_i(z)|^2} \geq |u_j(z)|^2\frac{1-|z|^2}{1-|\varphi_j(z)|^2}, \text{for all } j \neq i\}$ for each $i = 1, 2, \dots, N$. Fix $\varepsilon > 0$ and denote $E_i := \{z \in \mathbf{D}_i, \rho_i(z) \leq \varepsilon\}$ and $E_i' := \mathbf{D}_i \backslash E_i$. To end the proof, we assume that $\{f_n\}$ is any bounded sequence in A^2_α such that $f_n \to 0$ $(n \to \infty)$ uniformly on each compact subset of $\mathbf{D}$. For $1 \leq i \leq N$,

$$\begin{aligned}
&\int_{\mathbf{D}_i}\left|uf_n(\varphi) - \sum_{k=1}^N u_kf_n(\varphi_k)\right|^2 d\lambda_\alpha \\
&\quad\lesssim \int_{\mathbf{D}_i}\left|uf_n(\varphi) - u_if_n(\varphi_i)\right|^2 d\lambda_\alpha + \sum_{k\neq i}\int_{\mathbf{D}_i}\left|u_kf_n(\varphi_k)\right|^2 d\lambda_\alpha \\
&\quad\lesssim \int_{E_i}\left|uf_n(\varphi) - u_if_n(\varphi_i)\right|^2 d\lambda_\alpha + \int_{E_i'}\left|uf_n(\varphi)\right|^2 d\lambda_\alpha \\
&\qquad + \int_{E_i'}\left|u_if_n(\varphi_i)\right|^2 d\lambda_\alpha + \sum_{k\neq i}\int_{\mathbf{D}_i}\left|u_kf_n(\varphi_k)\right|^2 d\lambda_\alpha.
\end{aligned} \tag{4.1}$$

Let $\chi_{\mathbf{D}_i}$ and $\chi_{E_i'}$ be the characteristic functions of $\mathbf{D}_i$ and E_i', respectively, then it is obvious from the assumption (1) that

$$\lim_{|z|\to 1}\chi_{E_i'}\left(\left|u_i(z)\right|^2\frac{1-|z|^2}{1-|\varphi_i(z)|^2} + \left|u(z)\right|^2\frac{1-|z|^2}{1-|\varphi(z)|^2}\right) = 0. \tag{4.2}$$

Moreover,

$$\lim_{|z|\to 1}\chi_{\mathbf{D}_i}\left|u_k(z)\right|^2\frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0 \tag{4.3}$$

for fixed i and each $k \neq i$. Indeed, if (4.3) fails for some $k \neq i$, then there exist $\xi \in \mathbf{T}$ and $z_n \in \mathbf{D}_i$ satisfying $z_n \to \xi$ such that

$$\lim_{n\to\infty}\left|u_k(z_n)\right|^2\frac{1-|z_n|^2}{1-|\varphi_k(z_n)|^2} > 0.$$

Then $\xi \in F_{u_k}(\varphi_k)$ and

$$\lim_{n\to\infty} \left|u_i(z_n)\right|^2 \frac{1-|z_n|^2}{1-|\varphi_i(z_n)|^2} > 0$$

by the definition of $\mathbf{D}_i$, which implies $\xi \in F_{u_i}(\varphi_i)$. So $F_{u_i}(\varphi_i) \cap F_{u_k}(\varphi_k) \neq \emptyset$ when $i \neq k$, which contradicts our assumption. Then the last three terms of (4.1) tend to 0 as $n \to \infty$ by Lemma 2.1. In the following, we consider the first term of (4.1) by a similar argument to the proof of Theorem 3.4. For any $\xi \in F_{u_i}(\varphi_i)$, there exists $\delta(\xi) > 0$ such that

$$\left|u(z) - u_i(z)\right| < \varepsilon,$$

whenever $|z-\xi| < \delta(\xi)$. We decompose E_i into two parts as $E_i = H_{i1} + H_{i2}$, where

$$H_{i1} := E_i \cap \left(\bigcup_{\xi \in F_u(\varphi)} \left\{ z \in \mathbf{D} : |z-\xi| < \delta(\xi) \right\} \right)$$

and $H_{i2} := E_i \backslash H_{i1}$. Note that

$$\begin{aligned}
&\int_{E_i} \left|uf_n(\varphi) - u_i f_n(\varphi_i)\right|^2 d\lambda_\alpha \\
&\quad \lesssim \int_{E_i} \left|f_n(\varphi) - f_n(\varphi_i)\right|^2 d\lambda_\alpha + \int_{E_i} |u-u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha \\
&\quad \lesssim \int_{E_i} \left|f_n(\varphi) - f_n(\varphi_i)\right|^2 d\lambda_\alpha + \sum_{j=1}^{j=2} \int_{H_{ij}} |u-u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha \\
&\quad \lesssim \int_{E_i} \left|f_n(\varphi) - f_n(\varphi_i)\right|^2 d\lambda_\alpha + \int_{H_{i1}} |u-u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha \\
&\qquad + \int_{H_{i2}} |u|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha + \int_{H_{i2}} |u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha.
\end{aligned} \tag{4.4}$$

Clearly, by (3.5)

$$\int_{E_i} \left|f_n(\varphi) - f_n(\varphi_i)\right|^2 d\lambda_\alpha \le \int_{\{z\in\mathbf{D}:\rho_i(z)\le\varepsilon\}} \left|f_n(\varphi) - f_n(\varphi_i)\right|^2 d\lambda_\alpha \lesssim \varepsilon^2$$

for all n. Also, by the definition of H_{i1}, we can easily get

$$\int_{H_{i1}} |u-u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha \le \varepsilon^2 \|C_{\varphi_i} f_n\|^2_{A^2_\alpha} \lesssim \varepsilon^2$$

for all n. Moreover,

$$\lim_{|z|\to 1} \chi_{H_{i2}} \left|u(z)\right|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0. \tag{4.5}$$

Indeed, if (4.5) fails, there exist some $\zeta \in \mathbf{T}$ and a sequence $\{z_n\} \subseteq H_{i2}$ satisfying $z_n \to \zeta$ such that

$$\lim_{|z|\to 1} \left|u(z)\right|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} > 0,$$

then $\zeta \in F_u(\varphi) = \bigcup_{k=1}^N F_{u_k}(\varphi_k)$. Since this $\zeta \notin F_{u_k}(\varphi_k)$ when $k \neq i$ by (4.3), then $\zeta \in F_{u_i}(\varphi_i)$, which contradicts the definition of H_{i2}. So (4.5) holds. We now claim that

$$\lim_{|z|\to 1} \chi_{H_{i2}} \left|u_i(z)\right|^2 \frac{1-|z|^2}{1-|\varphi_i(z)|^2} = 0 \tag{4.6}$$

and

$$\lim_{|z|\to 1} \chi_{H_{i2}} \left|u(z)\right|^2 \frac{1-|z|^2}{1-|\varphi_i(z)|^2} = 0. \tag{4.7}$$

Indeed, the argument for (4.6) is similar to (3.9) and we omit it. To prove that (4.7) holds, we assume that there exist some $\eta \in \mathbf{T}$ and a sequence $\{z_n\} \subseteq H_{i2}$ such that $z_n \to \eta$ and

$$\lim_{n\to\infty} \left|u(z_n)\right|^2 \frac{1-|z_n|^2}{1-|\varphi_i(z_n)|^2} > 0.$$

Note that

$$\frac{1-|z_n|^2}{1-|\varphi(z_n)|^2} \geq \frac{1-\varepsilon}{1+\varepsilon} \frac{1-|z_n|^2}{1-|\varphi_i(z_n)|^2},$$

because of $\{z_n\} \subseteq H_{i2}$ and (2.3). Then $\eta \in F_u(\varphi)$, which contradicts (4.5). Thus again by Lemma 2.1, we have

$$\int_{H_{i2}} |u_i|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha + \int_{H_{i2}} |u|^2 \left|f_n(\varphi_i)\right|^2 d\lambda_\alpha \to 0$$

as $n \to \infty$. So

$$\int_{\mathbf{D}_i} \left| uf_n(\varphi) - \sum_{k=1}^N u_k f_n(\varphi_k) \right|^2 d\lambda_\alpha \to 0,$$

and then

$$\left\| \left(uC_\varphi - \sum_{i=1}^N u_i C_{\varphi_i} \right) f_n \right\|_{A_\alpha^2}^2 \lesssim \sum_{i=1}^N \int_{\mathbf{D}_i} \left| uf_n(\varphi) - \sum_{k=1}^N u_k f_n(\varphi_k) \right|^2 d\lambda_\alpha \to 0$$

as $n \to \infty$, which completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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