Xu et al. *Journal of Inequalities and Applications* (2017) 2017:7
DOI 10.1186/s13660-016-1275-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Second-order lower radial tangent derivatives and applications to set-valued optimization

Bihang Xu[1], Zhenhua Peng[2,3] and Yihong Xu[3*]

*Correspondence:
xuyihong@ncu.edu.cn
[3]Department of Mathematics, Nanchang University, Nanchang, 330031, China
Full list of author information is available at the end of the article

Abstract

We introduce the concepts of second-order radial composed tangent derivative, second-order radial tangent derivative, second-order lower radial composed tangent derivative, and second-order lower radial tangent derivative for set-valued maps by means of a radial tangent cone, second-order radial tangent set, lower radial tangent cone, and second-order lower radial tangent set, respectively. Some properties of second-order tangent derivatives are discussed, using which second-order necessary optimality conditions are established for a point pair to be a Henig efficient element of a set-valued optimization problem, and in the expressions the second-order tangent derivatives of the objective function and the constraint function are separated.

MSC: 46G05; 90C29; 90C46

Keywords: Henig efficiency; radial tangent derivative; set-valued optimization; optimality condition

1 Introduction

In recent years, first-order optimality conditions in the set-valued optimization have attracted a great deal of attention, and various derivative-like notions have been utilized to express these optimality conditions. For example, Gong *et al.* [1] introduced the concept of radial tangent cone and presented several kinds of necessary and sufficient conditions for some proper efficiencies, Kasimbeyli [2] gave necessary and sufficient optimality conditions based on the concept of the radial epiderivatives. At the same time, second-order optimality conditions and higher-order optimality conditions for vector optimization problems have been extensively studied in the literature (see [3–18]). Jahn *et al.* [3] proposed second-order epiderivatives for set-valued maps, and by using these concepts they gave second-order necessary optimality conditions and a sufficient optimality condition in set optimization. Khan and Isac [15] proposed the concept of a second-order composed contingent derivative for set-valued maps, using which they established second-order optimality conditions in set-valued optimization. With a second-order composed contingent derivative, Zhu *et al.* [4] established second-order Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a set-valued optimization problem. However, in [1, 4, 7–12, 14], in the expressions of first-order and higher-order optimality conditions, the tan-

gent derivatives of the objective function and the constraint function are not separated, and thus the properties of the derivatives of the objective function are not easily obtained from those of the constraint function.

On the other hand, some efficient points exhibit certain abnormal properties. To eliminate such anomalous efficient points, various concepts of proper efficiency have been introduced [19–22]. Henig [19] introduced the concept of Henig efficiency, which is very important for the study of set-valued optimization [13, 14, 20, 23].

In this paper, we introduce a new class of lower radial tangent cones and two new kinds of second-order tangent sets, using which we introduce four new kinds of second-order tangent derivatives. We discuss the properties of these second-order tangent derivatives, using which we establish second-order necessary optimality conditions for a point pair to be a Henig efficient element of a set-valued optimization problem.

2 Basic concepts

Throughout the paper, let X, Y, and Z be three real normed linear spaces, 0_X, 0_Y, and 0_Z denote the original points of X, Y, and Z, respectively. Let M be a nonempty subset of Y. As usual, we denote the interior, closure, and cone hull of M by $\operatorname{int} M$, $\operatorname{cl} M$, and $\operatorname{cone} M$, respectively. The cone hull of M is defined by

$$\operatorname{cone} M = \{\lambda m : \lambda \geq 0, m \in M\}.$$

Let C and D be closed convex pointed cones in Y and Z, respectively. A nonempty convex subset $B \subset C$ is called a base of C if $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$.

Denote the closed unit ball of Y by U. Suppose that C has a base B. Let $\delta := \inf\{\|b\| : b \in B\}$ and

$$C_\varepsilon(B) := \operatorname{cone}(\varepsilon U + B)$$

for all $0 < \varepsilon < \delta$. It is clear that $\delta > 0$ and $C_\varepsilon(B)$ is a pointed convex cone for all $0 < \varepsilon < \delta$ (see [21]).

Let $F : X \to 2^Y$ be a set-valued map. The domain, graph, and epigraph of F are defined respectively by

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\},$$
$$\operatorname{graph} F := \{(x, y) \in X \times Y : y \in F(x)\},$$
$$\operatorname{epi} F := \{(x, y) \in X \times Y : y \in F(x) + C\}.$$

Definition 2.1 (See [1]) Let A be a nonempty subset of X, and let $\hat{x} \in \operatorname{cl} A$. The radial tangent cone of A at $\hat{x}$, denoted by $R(A, \hat{x})$, is given by

$$R(A, \hat{x}) := \{u \in X : \exists t_n > 0 \text{ and } x_n \in A \text{ such that } t_n(x_n - \hat{x}) \to u\}. \tag{2.1}$$

Remark 2.1 Equation (2.1) is equivalent to

$$R(A, \hat{x}) = \{u \in X : \exists \lambda_n > 0 \text{ and } u_n \to u \text{ such that } \hat{x} + \lambda_n u_n \in A, \forall n \in N\},$$

where N denotes the set of positive integers.

Definition 2.2 (See [24]) Let A be a nonempty subset of X, and let $\hat{x} \in \operatorname{cl} A$. The contingent cone of A at $\hat{x}$, denoted by $T(A,\hat{x})$, is given by

$$T(A,\hat{x}) := \{u \in X : \exists t_n \to 0^+ \text{ and } u_n \to u \text{ such that } \hat{x} + t_n u_n \in A, \forall n \in N\}. \tag{2.2}$$

Remark 2.2 (See [24]) Equation (2.2) is equivalent to

$$T(A,\hat{x}) := \{u \in X : \exists \lambda_n \to +\infty \text{ and } x_n \in A \text{ such that } x_n \to \hat{x} \text{ and } \lambda_n(x_n - \hat{x}) \to u\}.$$

Definition 2.3 (See [3]) Let A be a nonempty subset of X, and let $\hat{x} \in \operatorname{cl} A$. The second-order contingent set of A at $\hat{x}$ in the direction w, denoted by $T^2(A,\hat{x},w)$, is given by

$$T^2(A,\hat{x},w) := \left\{v \in X : \exists t_n \to 0^+ \text{ and } v_n \to v \text{ such that } \hat{x} + t_n w + \frac{1}{2}t_n^2 v_n \in A\right\}.$$

Definition 2.4 (See [4, 15]) Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x},\hat{y}) \in \operatorname{graph} F$, and $(\hat{u},\hat{v}) \in X \times Y$. The second-order composed contingent derivative of F at $(\hat{x},\hat{y})$ in the direction $(\hat{u},\hat{v})$ is the set-valued map $D''F(\hat{x},\hat{y},\hat{u},\hat{v}) : X \to 2^Y$ defined by

$$\operatorname{graph} D''F(\hat{x},\hat{y},\hat{u},\hat{v}) = T\big(T\big(\operatorname{graph} F,(\hat{x},\hat{y})\big),(\hat{u},\hat{v})\big).$$

Definition 2.5 (See [3]) Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x},\hat{y}) \in \operatorname{graph} F$, and $(\hat{u},\hat{v}) \in X \times Y$. The second-order contingent derivative of F at $(\hat{x},\hat{y})$ in the direction $(\hat{u},\hat{v})$ is the set-valued map $D^2F(\hat{x},\hat{y},\hat{u},\hat{v}) : X \to 2^Y$ defined by

$$D^2F(\hat{x},\hat{y},\hat{u},\hat{v})(x) = \{y \in Y : (x,y) \in T^2\big(\operatorname{graph} F,(\hat{x},\hat{y}),(\hat{u},\hat{v})\big)\}.$$

In the following, we introduce a new class of lower radial tangent cones and two new kinds of second-order tangent sets.

Definition 2.6 Let Q be a nonempty subset of $X \times Y$, and let $(\hat{x},\hat{y}) \in \operatorname{cl} Q$. The lower radial tangent cone of Q at $(\hat{x},\hat{y})$ is defined by

$$R_l\big(Q,(\hat{x},\hat{y})\big) := \{(u,v) \in X \times Y : \forall t_n > 0, \forall u_n \to u, \exists v_n \to v \text{ such that } (\hat{x} + t_n u_n, \hat{y} + t_n v_n) \in Q\}.$$

Definition 2.7 Let Q be a nonempty subset of $X \times Y$, and let $(\hat{x},\hat{y}) \in \operatorname{cl} Q$. The second-order lower radial tangent set of Q at $(\hat{x},\hat{y})$ in the direction $(\hat{u},\hat{v})$, denoted by $R_l^2(Q,(\hat{x},\hat{y}),(\hat{u},\hat{v}))$, is given by

$$R_l^2\big(Q,(\hat{x},\hat{y}),(\hat{u},\hat{v})\big) := \left\{(u,v) \in X \times Y : \forall t_n > 0, \forall u_n \to u, \exists v_n \to v \text{ such that } \left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2 u_n, \hat{y} + t_n\hat{v} + \frac{1}{2}t_n^2 v_n\right) \in Q\right\}.$$

Definition 2.8 Let A be a nonempty subset of X, and let $\hat{x} \in \operatorname{cl} A$. The second-order radial tangent set of A at $\hat{x}$ in the direction w, denoted by $R^2(A, \hat{x}, w)$, is given by

$$R^2(A, \hat{x}, w) := \left\{ v \in X : \exists t_n > 0 \text{ and } v_n \to v \text{ such that } \hat{x} + t_n w + \frac{1}{2} t_n^2 v_n \in A \right\}.$$

Remark 2.3 Let $\emptyset \neq Q \subset X \times Y$, $(\hat{x}, \hat{y}) \in \operatorname{cl} Q$. Then

(i) $R_l(Q, (\hat{x}, \hat{y})) \subset T(Q, (\hat{x}, \hat{y})) \subset R(Q, (\hat{x}, \hat{y}))$;

(ii) $R_l^2(Q, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})) \subset T^2(Q, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})) \subset R^2(Q, (\hat{x}, \hat{y}), (\hat{u}, \hat{v}))$.

However, none of the inverse inclusions is necessarily true, as is shown in the following example.

Example 2.1 Let R be the set of real numbers, $X = Y = R$, $Q = \{(-\frac{1}{n}, \frac{1}{n^2}) : n = 1, 2, \ldots\} \cup \{(x, y) : x \geq 0, y \geq 0\} \cup \{(-1, -1)\}$, and $(\hat{x}, \hat{y}) = (\hat{u}, \hat{v}) = (0, 0)$. A direct calculation gives $R_l^2(Q, (0,0), (0,0)) = \{(x, y) : x > 0, y \geq 0\}$, $T^2(Q, (0,0), (0,0)) = \{(x, y) : x \geq 0, y \geq 0\} \cup \{(x, 0) : x < 0\}$, and $R^2(Q, (0,0), (0,0)) = \{(x, y) : x \geq 0, y \geq 0\} \cup \{(x, 0) : x < 0\} \cup \{(x, x) : x < 0\} \cup \bigcup_{n=1}^{\infty} \{\lambda(-\frac{1}{n}, \frac{1}{n^2}) : \lambda > 0\}$.

3 The second-order lower radial tangent derivative

In this section, by virtue of the radial tangent cone, the second-order radial tangent set, the lower radial tangent cone, and the second-order lower radial tangent set, we introduce the concepts of the second-order radial composed tangent derivative, the second-order radial tangent derivative, the second-order lower radial composed tangent derivative, and the second-order lower radial tangent derivative for a set-valued map. Furthermore, we discuss some important properties of the second-order lower radial composed tangent derivative and the second-order lower radial tangent derivative.

Definition 3.1 Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \operatorname{graph} F$, and $(\hat{u}, \hat{v}) \in X \times Y$. The second-order radial composed tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ is the set-valued map $R''F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) : X \to 2^Y$ defined by

$$\operatorname{graph} R''F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R\big(R\big(\operatorname{epi} F, (\hat{x}, \hat{y})\big), (\hat{u}, \hat{v})\big).$$

If $R(R(\operatorname{epi} F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})) \neq \emptyset$, then F is said to be second-order radial composed derivable at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ or that the second-order radial composed tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ exists.

Definition 3.2 Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \operatorname{graph} F$, and $(\hat{u}, \hat{v}) \in X \times Y$. The second-order radial tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ is the set-valued map $R^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) : X \to 2^Y$ defined by

$$\operatorname{graph} R^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R^2\big(\operatorname{epi} F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})\big).$$

If $R^2(\operatorname{epi} F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})) \neq \emptyset$, then F is called second-order radial derivable at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ or that the second-order radial tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ exists.

Definition 3.3 Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \operatorname{graph} F$, and $(\hat{u}, \hat{v}) \in X \times Y$. The second-order lower radial composed tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ is the set-valued map $R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) : X \to 2^Y$ defined by

$$\operatorname{graph} R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R_l\big(R_l\big(\operatorname{epi} F, (\hat{x}, \hat{y})\big), (\hat{u}, \hat{v})\big).$$

If $R_l(R_l(\operatorname{epi} F, (\hat{x}, \hat{y})), (\hat{u}, \hat{v})) \neq \emptyset$, then F is said to be second-order lower radial composed derivable at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ or that the second-order lower radial composed tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ exists.

Definition 3.4 Let $F : X \to 2^Y$ be a set-valued map, $(\hat{x}, \hat{y}) \in \operatorname{graph} F$, and $(\hat{u}, \hat{v}) \in X \times Y$. The second-order lower radial tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ is the set-valued map $R_l^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) : X \to 2^Y$ defined by

$$\operatorname{graph} R_l^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R_l^2\big(\operatorname{epi} F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})\big).$$

If $R_l^2(\operatorname{epi} F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})) \neq \emptyset$, then F is called second-order lower radial derivable at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ or that the second-order lower radial tangent derivative of F at $(\hat{x}, \hat{y})$ in the direction $(\hat{u}, \hat{v})$ exists.

Proposition 3.1 *Suppose that $E \subset X$ and the second-order lower radial composed tangent derivative of $F : X \to 2^Y$ at $(\hat{x}, \hat{y}) \in \operatorname{graph} F$ in the direction $(\hat{u}, \hat{v})$ exists. Then*

$$R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R\big(R(E, \hat{x}), \hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

Proof Let $v \in R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(R(R(E, \hat{x}), \hat{u}))$. Then there exists $u \in R(R(E, \hat{x}), \hat{u})$ such that

$$v \in R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(u).$$

Thus,

$$(u, v) \in \operatorname{graph} R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R_l\big(R_l\big(\operatorname{epi} F, (\hat{x}, \hat{y})\big), (\hat{u}, \hat{v})\big). \tag{3.1}$$

From $u \in R(R(E, \hat{x}), \hat{u})$ it follows that there exist sequences $t_n > 0$ and $u_n \to u$ such that

$$\hat{u} + t_n u_n \in R(E, \hat{x}).$$

Therefore, there exist sequences $t_n^k > 0$ and $u_n^k \to \hat{u} + t_n u_n$ such that

$$\hat{x} + t_n^k u_n^k \in E.$$

For such t_n and u_n, it follows from (3.1) that there exists a sequence $v_n \to v$ such that

$$(\hat{u} + t_n u_n, \hat{v} + t_n v_n) \in R_l\big(\operatorname{epi} F, (\hat{x}, \hat{y})\big).$$

Then, for the same t_n^k and u_n^k, there exists a sequence $v_n^k \to \hat{v} + t_n v_n$ such that

$$\big(\hat{x} + t_n^k u_n^k, \hat{y} + t_n^k v_n^k\big) \in \operatorname{epi} F,$$

and, consequently,

$$\hat{y} + t_n^k v_n^k \in F\big(\hat{x} + t_n^k u_n^k\big) + C.$$

Thus,

$$v_n^k \in \frac{1}{t_n^k}\big(F\big(\hat{x} + t_n^k u_n^k\big) + C - \hat{y}\big),$$

and, consequently,

$$v_n^k \in \operatorname{cone}\big(F(E) + C - \hat{y}\big).$$

Since $v_n^k \to \hat{v} + t_n v_n$ as $k \to \infty$, we obtain

$$\hat{v} + t_n v_n \in \operatorname{clcone}\big(F(E) + C - \hat{y}\big).$$

Thus,

$$v_n \in \frac{1}{t_n}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big),$$

and, consequently,

$$v_n \in \operatorname{cone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

Taking $n \to \infty$, we get

$$v \in \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

So,

$$R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R\big(R(E, \hat{x}), \hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big). \qquad \square$$

Proposition 3.2 *Suppose that $E \subset X$ and the second-order lower radial tangent derivative of $F : X \to 2^Y$ at $(\hat{x}, \hat{y}) \in \operatorname{graph} F$ in the direction $(\hat{u}, \hat{v})$ exists. Then*

$$R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R^2(E, \hat{x}, \hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

Proof Let $v \in R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(R^2(E, \hat{x}, \hat{u}))$. Then there exists $u \in R^2(E, \hat{x}, \hat{u})$ such that

$$v \in R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(u).$$

Thus,

$$(u, v) \in \operatorname{graph} R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) = R_l^2\big(\operatorname{epi} F, (\hat{x}, \hat{y}), (\hat{u}, \hat{v})\big). \tag{3.2}$$

From $u \in R^2(E,\hat{x},\hat{u})$ it follows that there exist sequences $t_n > 0$ and $u_n \to u$ such that

$$\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2 u_n \in E.$$

For such t_n and u_n, it follows from (3.2) that there exists a sequence $v_n \to v$ such that

$$\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2 u_n, \hat{y} + t_n\hat{v} + \frac{1}{2}t_n^2 v_n\right) \in \operatorname{epi} F.$$

Then

$$\hat{y} + t_n\hat{v} + \frac{1}{2}t_n^2 v_n \in F\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2 u_n\right) + C,$$

and, consequently,

$$\hat{v} + \frac{1}{2}t_n v_n \in \frac{1}{t_n}\left(F\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2 u_n\right) + C - \hat{y}\right).$$

Thus,

$$\hat{v} + \frac{1}{2}t_n v_n \in \operatorname{cone}\big(F(E) + C - \hat{y}\big).$$

Hence,

$$v_n \in \frac{2}{t_n}\big(\operatorname{cone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

Therefore,

$$v_n \in \operatorname{cone}\big(\operatorname{cone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

Taking $n \to \infty$, we get

$$v \in \operatorname{clcone}\big(\operatorname{cone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big).$$

So,

$$R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big). \qquad \square$$

Remark 3.1 If we substitute $D''F(\hat{x},\hat{y},\hat{u},\hat{v})$ or $R''F(\hat{x},\hat{y},\hat{u},\hat{v})$ for $R_l''F(\hat{x},\hat{y},\hat{u},\hat{v})$ in Proposition 3.1, then none of the inclusions

$$D''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big)$$

and

$$R''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big)$$

is necessarily true. If we substitute $D^2F(\hat{x},\hat{y},\hat{u},\hat{v})$ or $R^2F(\hat{x},\hat{y},\hat{u},\hat{v})$ for $R_l^2F(\hat{x},\hat{y},\hat{u},\hat{v})$ in Proposition 3.2, then none of the inclusions

$$D^2F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big)$$

and

$$R^2F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big)$$

is necessarily true, as is shown in the following example.

Example 3.1 Let R be the set of real numbers, $X = Y = R$, $C = \{t : t \geq 0\}$, and $E = \{x : x \geq 0\}$. Define the set-valued map $F : X \to 2^Y$ by

$$F(x) = \begin{cases} \{y : y \geq 0\} & \text{if } x \geq 0, \\ \{y : y \geq \sqrt[3]{x}\} & \text{otherwise.} \end{cases}$$

(i) Let $(\hat{x},\hat{y}) = (0,0)$, $(\hat{u},\hat{v}) = (0,-1)$. A direct calculation gives

$$R(E,0) = R\big(R(E,0),0\big) = [0,+\infty),$$
$$T\big(\operatorname{epi}F,(0,0)\big) = R\big(\operatorname{epi}F,(0,0)\big) = \big\{(x,y) : x > 0, y \geq 0\big\} \cup \big\{(x,y) : x \leq 0, y \in R\big\},$$
$$T\big(T\big(\operatorname{epi}F,(0,0)\big),(0,-1)\big) = \big\{(x,y) : x \leq 0, y \in R\big\},$$
$$R\big(R\big(\operatorname{epi}F,(0,0)\big),(0,-1)\big) = \big\{(x,y) : x \leq 0, y \in R\big\} \cup \big\{(x,y) : x > 0, y \geq 0\big\},$$
$$D''F(0,0,0,-1)(x) = \begin{cases} R, & x \leq 0, \\ \emptyset, & x > 0, \end{cases}$$
$$R''F(0,0,0,-1)(x) = \begin{cases} R, & x \leq 0, \\ \{y : y \geq 0\}, & x > 0, \end{cases}$$
$$R_l\big(\operatorname{epi}F,(0,0)\big) = \big\{(x,y) : x \in R, y \geq 0\big\},$$
$$R_l\big(R_l\big(\operatorname{epi}F,(0,0)\big),(0,-1)\big) = \emptyset,$$
$$R_l''F(0,0,0,-1)(x) = \emptyset, \quad x \in R.$$

Consequently,

$$D''F(0,0,0,-1)\big(R\big(R(E,0),0\big)\big) = R''F(0,0,0,-1)\big(R\big(R(E,0),0\big)\big) = R,$$
$$R_l''F(0,0,0,-1)\big(R\big(R(E,0),0\big)\big) = \emptyset,$$
$$\operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big) = [0,+\infty).$$

Then, the inclusion of Proposition 3.1 is true. However,

$$D''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big) \not\subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big)$$

and

$$R''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big) \not\subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big).$$

(ii) Let $(\hat{x},\hat{y})=(0,0)$, $(\hat{u},\hat{v})=(0,0)$. A direct calculation gives

$$\begin{aligned}
&R\big(R(E,0),0\big)=R^2(E,0,0)=R(E,0)=[0,+\infty),\\
&T\big(T\big(\operatorname{epi}F,(0,0)\big),(0,0)\big)=R\big(R\big(\operatorname{epi}F,(0,0)\big),(0,0)\big)=T^2\big(\operatorname{epi}F,(0,0),(0,0)\big)\\
&\qquad\qquad=R^2\big(\operatorname{epi}F,(0,0),(0,0)\big)\\
&\qquad\qquad=T\big(\operatorname{epi}F,(0,0)\big)=R\big(\operatorname{epi}F,(0,0)\big)\\
&\qquad\qquad=\big\{(x,y):x>0,y\geq 0\big\}\cup\big\{(x,y):x\leq 0,y\in R\big\},\\
&D''F(0,0,0,0)(x)=R''F(0,0,0,0)(x)=\begin{cases}R, & x\leq 0,\\ \{y:y\geq 0\}, & x>0,\end{cases}\\
&D^2F(0,0,0,0)(x)=R^2F(0,0,0,0)(x)=\begin{cases}R, & x\leq 0,\\ \{y:y\geq 0\}, & x>0,\end{cases}\\
&R_l\big(R_l\big(\operatorname{epi}F,(0,0)\big),(0,0)\big)=R_l^2\big(\operatorname{epi}F,(0,0),(0,0)\big)\\
&\qquad\qquad=R_l\big(\operatorname{epi}F,(0,0)\big)=\big\{(x,y):x\in R,y\geq 0\big\},\\
&R_l''F(0,0,0,0)(x)=R_l^2F(0,0,0,0)(x)=[0,+\infty),\quad x\in R.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&D''F(0,0,0,0)\big(R\big(R(E,0),0\big)\big)=R''F(0,0,0,0)\big(R\big(R(E,0),0\big)\big)\\
&\qquad\qquad=D^2F(0,0,0,0)\big(R^2(E,0,0)\big)\\
&\qquad\qquad=R^2F(0,0,0,0)\big(R^2(E,0,0)\big)=R,\\
&R_l''F(0,0,0,0)\big(R\big(R(E,0),0\big)\big)=R_l^2F(0,0,0,0)\big(R^2(E,0,0)\big)=[0,+\infty),\\
&\operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big)=\operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big)=[0,+\infty).
\end{aligned}$$

Then, the inclusions of Propositions 3.1 and 3.2 are true. However,

$$\begin{aligned}
&D''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big)\not\subset\operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big),\\
&R''F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R\big(R(E,\hat{x}),\hat{u}\big)\big)\not\subset\operatorname{clcone}\big(\operatorname{clcone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big),\\
&D^2F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big)\not\subset\operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big),
\end{aligned}$$

and

$$R^2F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big)\not\subset\operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big).$$

4 Second-order necessary optimality conditions

Let $F:X\to 2^Y$, $G:X\to 2^Z$, and $(F,G):X\to 2^{Y\times Z}$ be defined by $(F,G)(x)=F(x)\times G(x)$.

Consider the following optimization problem with set-valued maps:

$$\begin{aligned}&(\mathrm{VP}) \quad \min F(x),\\ &\qquad \text{s.t. } G(x)\cap(-D)\neq\emptyset, \quad x\in X.\end{aligned}$$

The feasible set of (VP) is denoted by $\hat{E}$, that is, $\hat{E}=\{x\in X : G(x)\cap(-D)\neq\emptyset\}$.

Definition 4.1 (See [13, 19, 21]) Let $\hat{x}\in\hat{E}$, $\hat{y}\in F(\hat{x})$. A pair $(\hat{x},\hat{y})$ is called a Henig efficient element of (VP) if there exists $\varepsilon\in(0,\delta)$ such that

$$\big(F(\hat{E})-\hat{y}\big)\cap\big(-\operatorname{intcone}(\varepsilon U+B)\big)=\emptyset,$$

where $\delta:=\inf\{\|b\| : b\in B\}$, $F(\hat{E})=\bigcup_{x\in\hat{E}}F(x)$, and U is the closed unit ball of Y.

Definition 4.2 (See [10]) The interior tangent cone $\mathrm{IT}(S,\bar{y})$ of S at $\bar{y}$ is the set of all $y\in Y$ such that for any $t_n\to 0^+$ and $y_n\to y$, we have $\bar{y}+t_ny_n\in S$.

Remark 4.1 (See [10]) If $S\subset Y$ is convex, $\bar{y}\in S$, and $\operatorname{int}S\neq\emptyset$, then

$$\mathrm{IT}(S,\bar{y})=\mathrm{IT}(\operatorname{int}S,\bar{y})=\operatorname{intcone}(S-\bar{y}).$$

Theorem 4.1 *Suppose that* $(\hat{x},\hat{y})$ *is a Henig efficient element of* (VP), $\hat{z}\in G(\hat{x})\cap(-D)$, $(\hat{u},\hat{v},\hat{w})\in X\times(-C)\times(-D)$, F *is second-order lower radial composed derivable at* $(\hat{x},\hat{y})$ *in the direction* $(\hat{u},\hat{v})$, *and* G *is second-order radial composed derivable at* $(\hat{x},\hat{z})$ *in the direction* $(\hat{u},\hat{w})$. *Then there exists* $\hat{\varepsilon}\in(0,\delta)$ *such that*

$$\big(R_l''F(\hat{x},\hat{y},\hat{u},\hat{v})(x),R''G(\hat{x},\hat{z},\hat{u},\hat{w})(x)\big)\cap\big(\big(-\operatorname{intcone}(\hat{\varepsilon}U+B)\big)\times(-\operatorname{int}D)\big)=\emptyset \tag{4.1}$$

for all $x\in\operatorname{dom}R_l''F(\hat{x},\hat{y},\hat{u},\hat{v})\cap\operatorname{dom}R''G(\hat{x},\hat{z},\hat{u},\hat{w})$.

Proof Since $(\hat{x},\hat{y})$ is a Henig efficient element of (VP), there exists a number $\varepsilon_0\in(0,\delta)$ such that

$$\big(F(\hat{E})-\hat{y}\big)\cap\big(-\operatorname{intcone}(\varepsilon_0U+B)\big)=\emptyset. \tag{4.2}$$

On the contrary, suppose that (4.1) does not hold. Then there exist $\bar{x}\in\operatorname{dom}R_l''F(\hat{x},\hat{y},\hat{u},\hat{v})\cap\operatorname{dom}R''G(\hat{x},\hat{z},\hat{u},\hat{w})$, $\bar{y}\in R_l''F(\hat{x},\hat{y},\hat{u},\hat{v})(\bar{x})$, and $\bar{z}\in R''G(\hat{x},\hat{z},\hat{u},\hat{w})(\bar{x})$ such that

$$\bar{y}\in-\operatorname{intcone}(\varepsilon_0U+B) \tag{4.3}$$

and

$$\bar{z}\in-\operatorname{int}D. \tag{4.4}$$

From $\bar{z}\in R''G(\hat{x},\hat{z},\hat{u},\hat{w})(\bar{x})$ it follows that

$$(\bar{x},\bar{z})\in\operatorname{graph}R''G(\hat{x},\hat{z},\hat{u},\hat{w})=R\big(R\big(\operatorname{epi}G,(\hat{x},\hat{z})\big),(\hat{u},\hat{w})\big).$$

Hence, there exist $t_n > 0$ and $(u_n, w_n) \in R(\operatorname{epi} G, (\hat{x}, \hat{z}))$ such that

$$t_n\big((u_n, w_n) - (\hat{u}, \hat{w})\big) \to (\bar{x}, \bar{z}). \tag{4.5}$$

From (4.4) it follows that there exists $N_1 \in N$ such that

$$t_n(w_n - \hat{w}) \in -\operatorname{int} D, \quad \forall n > N_1.$$

Since $-\operatorname{int} D$ is a cone, we obtain

$$w_n - \hat{w} \in -\operatorname{int} D, \quad \forall n > N_1.$$

Since $\hat{w} \in -D$ and $-D$ is a convex cone, it follows that

$$w_n \in -\operatorname{int} D - D = -\operatorname{int} D, \quad \forall n > N_1. \tag{4.6}$$

Since $(u_n, w_n) \in R(\operatorname{epi} G, (\hat{x}, \hat{z}))$, there exist sequences $t_n^k > 0$ and $(x_n^k, z_n^k) \in \operatorname{epi} G$ such that

$$t_n^k\big(\big(x_n^k, z_n^k\big) - (\hat{x}, \hat{z})\big) \to (u_n, w_n), \quad k \to +\infty. \tag{4.7}$$

It follows from (4.6) that there exists $K_1(n) \in N$ such that

$$t_n^k\big(z_n^k - \hat{z}\big) \in -\operatorname{int} D, \quad \forall n > N_1, \forall k > K_1(n).$$

Since $-\operatorname{int} D$ is a cone, we obtain

$$z_n^k - \hat{z} \in -\operatorname{int} D, \quad \forall n > N_1, \forall k > K_1(n).$$

Since $\hat{z} \in -D$ and $-D$ is a convex cone, it follows that

$$z_n^k \in -\operatorname{int} D - D = -\operatorname{int} D, \quad \forall n > N_1, \forall k > K_1(n).$$

Since $(x_n^k, z_n^k) \in \operatorname{epi} G$, we obtain $z_n^k \in G(x_n^k) + D$. Hence, there exists $\bar{z}_n^k \in G(x_n^k)$ such that $z_n^k \in \bar{z}_n^k + D$. Consequently,

$$\bar{z}_n^k \in z_n^k - D \subset -\operatorname{int} D - D = -\operatorname{int} D.$$

Thus, $G(x_n^k) \cap (-D) \neq \emptyset$, that is, $x_n^k \in \hat{E}$. It follows from (4.7) that $t_n^k(x_n^k - \hat{x}) \to u_n$ as $k \to \infty$, and hence, $u_n \in R(\hat{E}, \hat{x})$. It follows from (4.5) that $t_n(u_n - \hat{u}) \to \bar{x}$, and hence, $\bar{x} \in R(R(\hat{E}, \hat{x}), \hat{u})$. By Proposition 3.1, since $\bar{y} \in R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(\bar{x})$, we conclude that

$$\bar{y} \in R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R\big(R(\hat{E}, \hat{x}), \hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big).$$

From (4.3) it follows that

$$\operatorname{clcone}\big(\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

Since $-\operatorname{intcone}(\varepsilon_0 U + B)$ is open, we obtain

$$\operatorname{cone}\big(\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

Since $\operatorname{cone}(\varepsilon_0 U + B)$ is a pointed cone, it follows that

$$\big(\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset,$$

and thus,

$$\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(\hat{v} - \operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

It follows from $\hat{v} \in -C \subset -\operatorname{cone}(\varepsilon_0 U + B)$ that

$$\begin{aligned}\hat{v} - \operatorname{int}\big(\operatorname{cone}(\varepsilon_0 U + B)\big) &\subset -\operatorname{cone}(\varepsilon_0 U + B) - \operatorname{int}\big(\operatorname{cone}(\varepsilon_0 U + B)\big)\\ &\subset -\operatorname{intcone}(\varepsilon_0 U + B).\end{aligned}$$

Consequently,

$$\operatorname{clcone}\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

In the similar way, we conclude that

$$\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

Since $C \subset \operatorname{cone}(\varepsilon_0 U + B)$ and $\operatorname{cone}(\varepsilon_0 U + B)$ is a point cone, we obtain

$$\big(F(\hat{E}) - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon_0 U + B)\big) \neq \emptyset.$$

This is a contradiction to (4.2). The proof is completed. □

Corollary 4.1 *Suppose that* $(\hat{x}, \hat{y})$ *is a Henig efficient element of* (VP), $\hat{z} \in G(\hat{x}) \cap (-D)$, $(\hat{u}, \hat{v}, \hat{w}) \in X \times (-C) \times (-D)$, *F is second-order lower radial composed derivable at* $(\hat{x}, \hat{y})$ *in the direction* $(\hat{u}, \hat{v})$, *and G is second-order lower radial composed derivable at* $(\hat{x}, \hat{z})$ *in the direction* $(\hat{u}, \hat{w})$. *Then there exists a number* $\hat{\varepsilon} \in (0, \delta)$ *such that*

$$\big(R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), R_l'' G(\hat{x}, \hat{z}, \hat{u}, \hat{w})(x)\big) \cap \big(\big(-\operatorname{intcone}(\hat{\varepsilon} U + B)\big) \times (-\operatorname{int} D)\big) = \emptyset$$

for all $x \in \operatorname{dom} R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \cap \operatorname{dom} R_l'' G(\hat{x}, \hat{z}, \hat{u}, \hat{w})$.

Proof The proof follows directly from Theorem 4.1 and Remark 2.3(ii). □

Corollary 4.2 *Suppose that* $(\hat{x}, \hat{y})$ *is a Henig efficient element of* (VP), $\hat{z} \in G(\hat{x}) \cap (-D)$, $(\hat{u}, \hat{v}, \hat{w}) \in X \times (-C) \times (-D)$, *C has a convex base B, F is second-order lower radial composed derivable at* $(\hat{x}, \hat{y})$ *in the direction* $(\hat{u}, \hat{v})$, *and G is second-order lower radial composed derivable at* $(\hat{x}, \hat{z})$ *in the direction* $(\hat{u}, \hat{w})$. *Then there exists a number* $\hat{\varepsilon} \in (0, \delta)$ *such that*

$$\big(R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), R_l'' G(\hat{x}, \hat{z}, \hat{u}, \hat{w})(x)\big) \cap \big(\operatorname{IT}\big(-\operatorname{intcone}(\hat{\varepsilon} U + B), -\hat{v}\big) \times (-\operatorname{int} D)\big) = \emptyset$$

for all $x \in \operatorname{dom} R_l'' F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \cap \operatorname{dom} R_l'' G(\hat{x}, \hat{z}, \hat{u}, \hat{w})$.

Proof

$$\begin{aligned}\mathrm{IT}\big(-\operatorname{intcone}(\hat{\varepsilon}U+B),-\hat{v}\big) &= \operatorname{intcone}\big(-\operatorname{intcone}(\hat{\varepsilon}U+B)+\hat{v}\big)\\ &\subset -C-\operatorname{intcone}(\hat{\varepsilon}U+B)\\ &\subset -\operatorname{cone}(\hat{\varepsilon}U+B)-\operatorname{intcone}(\hat{\varepsilon}U+B)\\ &\subset -\operatorname{intcone}(\hat{\varepsilon}U+B).\end{aligned}$$

□

We provide the following example to explain Theorem 4.1 and Corollaries 4.1 and 4.2.

Example 4.1 Let R be the set of real numbers, $X = Y = Z = R$, $C = D = \{t : t \geq 0\}$, $B = \{1\}$. Define the set-valued maps $F : X \to 2^Y$ and $G : X \to 2^Z$ by

$$F(x) = G(x) = \begin{cases} \{y : y \geq 0\} & \text{if } x \geq 0,\\ \{y : y \geq x^2\} & \text{otherwise.}\end{cases}$$

Let $(\hat{x},\hat{y}) = (0,0)$, $(\hat{u},\hat{v},\hat{w}) = (1,0,0) \in X \times (-C) \times (-D)$, $\varepsilon = \frac{1}{2}$. A direct calculation gives

$$\hat{z} \in G(0) \cap (-D) = \{0\},$$

$$R_l\big(\operatorname{epi} F,(0,0)\big) = R_l\big(\operatorname{epi} G,(0,0)\big) = \big\{(x,y) : x > 0, y \geq 0\big\},$$

$$R\big(\operatorname{epi} G,(0,0)\big) = \big\{(x,y) : x \in R, y \geq 0\big\},$$

$$R_l\big(R_l\big(\operatorname{epi} F,(0,0)\big),(1,0)\big) = R_l\big(R_l\big(\operatorname{epi} G,(0,0)\big),(1,0)\big) = \big\{(x,y) : x > 0, y \geq 0\big\},$$

$$R\big(R\big(\operatorname{epi} G,(0,0)\big),(1,0)\big) = \big\{(x,y) : x \in R, y \geq 0\big\},$$

$$R_l'' F(0,0,1,0)(x) = R_l'' G(0,0,1,0)(x) = \begin{cases} \{y : y \geq 0\} & \text{if } x > 0,\\ \emptyset & \text{otherwise,}\end{cases}$$

$$R'' G(0,0,1,0)(x) = [0,+\infty), \quad x \in R,$$

$$\mathrm{IT}\big(-\operatorname{intcone}(\varepsilon U+B),-\hat{v}\big) = -\operatorname{intcone}(\varepsilon U+B) = (-\infty,0).$$

Then, the inclusions of Theorem 4.1 and Corollaries 4.1 and 4.2 are true.

Theorem 4.2 *Suppose that* $(\hat{x},\hat{y})$ *is a Henig efficient element of* (VP), $\hat{z} \in G(\hat{x}) \cap (-D)$, $(\hat{u},\hat{v},\hat{w}) \in X \times (-C) \times (-D)$, F *is second-order lower radial derivable at* $(\hat{x},\hat{y})$ *in the direction* $(\hat{u},\hat{v})$, *and* G *is second-order radial derivable at* $(\hat{x},\hat{z})$ *in the direction* $(\hat{u},\hat{w})$. *Then there exists a number* $\hat{\varepsilon} \in (0,\delta)$ *such that*

$$\big(R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v})(x), R^2 G(\hat{x},\hat{z},\hat{u},\hat{w})(x)\big) \cap \big(\big(-\operatorname{intcone}(\hat{\varepsilon}U+B)\big) \times (-\operatorname{int} D)\big) = \emptyset \tag{4.8}$$

for all $x \in \operatorname{dom} R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v}) \cap \operatorname{dom} R^2 G(\hat{x},\hat{z},\hat{u},\hat{w})$.

Proof On the contrary, suppose that (4.8) does not hold. Then, for any $\varepsilon \in (0,\delta)$, there exist $\bar{x} \in \operatorname{dom} R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v}) \cap \operatorname{dom} R^2 G(\hat{x},\hat{z},\hat{u},\hat{w})$, $\bar{y} \in R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v})(\bar{x})$, and $\bar{z} \in R^2 G(\hat{x},\hat{z},\hat{u},\hat{w})(\bar{x})$ such that

$$\bar{y} \in -\operatorname{intcone}(\varepsilon U+B) \tag{4.9}$$

and

$$\bar{z} \in -\operatorname{int} D. \tag{4.10}$$

From $\bar{z} \in R^2G(\hat{x}, \hat{z}, \hat{u}, \hat{w})(\bar{x})$ it follows that

$$(\bar{x}, \bar{z}) \in \operatorname{graph} R^2G(\hat{x}, \hat{z}, \hat{u}, \hat{w}) = R^2\big(\operatorname{epi} G, (\hat{x}, \hat{z}), (\hat{u}, \hat{w})\big).$$

Hence, there exist $t_n > 0$, $x_n \to \bar{x}$, and $z_n \to \bar{z}$ such that

$$\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2x_n, \hat{z} + t_n\hat{w} + \frac{1}{2}t_n^2z_n\right) \in \operatorname{epi} G.$$

Thus,

$$\hat{z} + t_n\hat{w} + \frac{1}{2}t_n^2z_n \in G\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2x_n\right) + D. \tag{4.11}$$

The set of positive integers is denoted by N. From (4.10) and $z_n \to \bar{z}$ it follows that there exists $N_1 \in N$ such that

$$z_n \in -\operatorname{int} D, \quad \forall n > N_1.$$

Since $-\operatorname{int} D$ and $-D$ are convex cones, we obtain

$$\hat{z} + t_n\hat{w} + \frac{1}{2}t_n^2z_n \in -D - D - \operatorname{int} D = -\operatorname{int} D, \quad \forall n > N_1. \tag{4.12}$$

It follows from (4.11) that there exists $\tilde{z}_n \in G(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2x_n)$ such that

$$\hat{z} + t_n\hat{w} + \frac{1}{2}t_n^2z_n \in \{\tilde{z}_n\} + D.$$

Since (4.12) and D is a convex cone, we obtain

$$\tilde{z}_n \in \left\{\hat{z} + t_n\hat{w} + \frac{1}{2}t_n^2z_n\right\} - D \subset -\operatorname{int} D - D = -\operatorname{int} D \subset -D.$$

Thus,

$$G\left(\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2x_n\right) \cap (-D) \neq \emptyset,$$

that is,

$$\hat{x} + t_n\hat{u} + \frac{1}{2}t_n^2x_n \in \hat{E}.$$

From $t_n > 0$ and $x_n \to \bar{x}$ it follows that $\bar{x} \in R^2(\hat{E}, \hat{x}, \hat{u})$. By Proposition 3.2 and $\bar{y} \in R_l^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(\bar{x})$ we obtain

$$\bar{y} \in R_l^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R^2(\hat{E}, \hat{x}, \hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big).$$

It follows from (4.9) that

$$\operatorname{clcone}\big(\operatorname{cone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

Since $-\operatorname{intcone}(\varepsilon U + B)$ is open, we obtain

$$\operatorname{cone}\big(\operatorname{cone}\big(F(\hat{\hat{E}}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

Since $\operatorname{cone}(\varepsilon U + B)$ is a pointed cone, it follows that

$$\big(\operatorname{cone}\big(F(\hat{E}) + C - \hat{y}\big) - \hat{v}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset,$$

and thus,

$$\operatorname{cone}\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(\hat{v} - \operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

It follows from $\hat{v} \in -C \subset -\operatorname{cone}(\varepsilon U + B)$ that

$$\begin{aligned}\hat{v} - \operatorname{int}\big(\operatorname{cone}(\varepsilon U + B)\big) &\subset -\operatorname{cone}(\varepsilon U + B) - \operatorname{int}\big(\operatorname{cone}(\varepsilon U + B)\big)\\ &\subset -\operatorname{intcone}(\varepsilon U + B).\end{aligned}$$

Consequently,

$$\operatorname{cone}\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

In a similar way, we conclude that

$$\big(F(\hat{E}) + C - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

Since $C \subset \operatorname{cone}(\varepsilon U + B)$ and $\operatorname{cone}(\varepsilon U + B)$ is a pointed cone, we obtain

$$\big(F(\hat{E}) - \hat{y}\big) \cap \big(-\operatorname{intcone}(\varepsilon U + B)\big) \neq \emptyset.$$

This is a contradiction to the assumption that $(\hat{x}, \hat{y})$ is a Henig minimizer of (VP). □

Corollary 4.3 *Suppose that* $(\hat{x}, \hat{y})$ *is a Henig efficient element of* (VP), $\hat{z} \in G(\hat{x}) \cap (-D)$, $(\hat{u}, \hat{v}, \hat{w}) \in X \times (-C) \times (-D)$, *F is second-order lower radial derivable at* $(\hat{x}, \hat{y})$ *in the direction* $(\hat{u}, \hat{v})$, *and G is second-order lower radial derivable at* $(\hat{x}, \hat{z})$ *in the direction* $(\hat{u}, \hat{w})$. *Then there exists a number* $\hat{\varepsilon} \in (0, \delta)$ *such that*

$$\big(R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x), R_l^2 G(\hat{x}, \hat{z}, \hat{u}, \hat{w})(x)\big) \cap \big(\big(-\operatorname{intcone}(\hat{\varepsilon} U + B)\big) \times (-\operatorname{int} D)\big) = \emptyset$$

for all $x \in \operatorname{dom} R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \cap \operatorname{dom} R_l^2 G(\hat{x}, \hat{z}, \hat{u}, \hat{w})$.

Proof The proof follows immediately from Theorem 4.2 and Remark 2.3(ii). □

Corollary 4.4 *Suppose that* $(\hat{x},\hat{y})$ *is a Henig efficient element of* (VP), $\hat{z}\in G(\hat{x})\cap(-D)$, $(\hat{u},\hat{v},\hat{w})\in X\times(-C)\times(-D)$, B *is a base of* C, F *is second-order lower radial derivable at* $(\hat{x},\hat{y})$ *in the direction* $(\hat{u},\hat{v})$, *and* G *is second-order lower radial derivable at* $(\hat{x},\hat{z})$ *in the direction* $(\hat{u},\hat{w})$. *Then there exists a number* $\hat{\varepsilon}\in(0,\delta)$ *such that*

$$\big(R_l^2F(\hat{x},\hat{y},\hat{u},\hat{v})(x),R_l^2G(\hat{x},\hat{z},\hat{u},\hat{w})(x)\big)\cap\big(\mathrm{IT}\big(-\operatorname{intcone}(\hat{\varepsilon}U+B),-\hat{v}\big)\times(-\operatorname{int}D)\big)=\emptyset$$

for all $x\in\operatorname{dom}R_l^2F(\hat{x},\hat{y},\hat{u},\hat{v})\cap\operatorname{dom}R_l^2G(\hat{x},\hat{z},\hat{u},\hat{w})$.

Proof It is similar to the proof of Corollary 4.2. □

We give the following example to illustrate Theorem 4.2 and Corollaries 4.3 and 4.4.

Example 4.2 Let R be the set of real numbers, $X=Y=Z=R$, $C=D=\{t:t\geq0\}$, and $B=\{1\}$. Define the set-valued maps $F:X\to2^Y$ and $G:X\to2^Z$ by

$$F(x)=\{y:y\geq0\},\quad x\in R,$$
$$G(x)=\{y:y\geq x\},\quad x\in R.$$

Let $(\hat{x},\hat{y})=(0,0)$, $(\hat{u},\hat{v},\hat{w})=(-1,0,-1)$, and $\varepsilon=\frac{1}{2}$. A direct calculation gives

$$\hat{z}\in G(0)\cap(-D)=\{0\},$$
$$R_l^2\big(\operatorname{epi}F,(0,0),(-1,0)\big)=\big\{(x,y):x\in R,y\geq0\big\},$$
$$R^2\big(\operatorname{epi}G,(0,0),(-1,-1)\big)=\big\{(x,y):x\in R,y\geq x\big\},$$
$$R_l^2\big(\operatorname{epi}G,(0,0),(-1,-1)\big)=\big\{(x,y):x\in R,y\geq x\big\},$$
$$R_l^2F(0,0,-1,0)(x)=\{y:y\geq0\},\quad x\in R,$$
$$R_l^2G(0,0,-1,-1)(x)=R^2G(0,0,-1,-1)(x)=\{y:y\geq x\},\quad x\in R,$$
$$\mathrm{IT}\big(-\operatorname{intcone}(\varepsilon U+B),-\hat{v}\big)=-\operatorname{intcone}(\varepsilon U+B)=(-\infty,0).$$

Then, the inclusions of Theorem 4.2 and Corollaries 4.3 and 4.4 are true.

Let us recall that the upper (inferior) limit in the sense of Painlevé-Kuratowski of a set-valued map $\Phi:X\to2^Y$ is defined as $\limsup_{u\to\bar{u}}\Phi(u):=\{y\in Y:\exists u_n\to\bar{u},\exists y_n\in\Phi(u_n)$ such that $y_n\to y\}$ and $\liminf_{u\to\bar{u}}\Phi(u):=\{y\in Y:\forall u_n\to\bar{u},\exists y_n\in\Phi(u_n)$ such that $y_n\to y\}$. If $f:X\to Y$ is Fréchet differentiable at $\hat{x}\in X$, its Fréchet derivative is denoted by $f'(\hat{x})$.

The profile map of F is the set-valued map $F_+:X\to2^Y$ defined by $F_+(x)=F(x)+C$, $x\in\operatorname{dom}F$.

In what follows, we consider vector optimization.

Let $f:X\to Y$, $g:X\to Z$.

Consider the following vector optimization:

$$\text{(P)}\quad\min f(x),$$
$$\text{s.t. } g(x)\in-D,\quad x\in X.$$

Similarly to Definition 4.3 in [18], we introduce the following second-order generalized lower (upper) directional derivative for vector-valued functions.

Definition 4.3 Let $f : X \to Y$ be Fréchet differentiable at $\hat{x}$, and $\hat{u}, x \in X$. The parabolic second-order generalized lower directional derivative of $\hat{x}$ in the direction $(\hat{u}, x)$ is

$$\tilde{D}_l^2 f(\hat{x}, \hat{u})(x) := \liminf_{t>0, x' \to x} \frac{f(\hat{x} + t\hat{u} + \frac{1}{2}t^2 x') - f(\hat{x}) - tf'(\hat{x})\hat{u}}{\frac{1}{2}t^2}.$$

Remark 4.2 When the set-valued map F becomes to a vector-valued function f, which is Fréchet differentiable at $\hat{x}$, letting $\hat{v} := f'(\hat{x})\hat{u}$, we have

$$R_l^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \tilde{D}_l^2 f_+(\hat{x}, \hat{u})(x) = \liminf_{t>0, x' \to x} \frac{f_+(\hat{x} + t\hat{u} + \frac{1}{2}t^2 x') - f_+(\hat{x}) - tf'(\hat{x})\hat{u}}{\frac{1}{2}t^2}.$$

Definition 4.4 Let $f : X \to Y$ be Fréchet differentiable at $\hat{x}$, and $\hat{u}, x \in X$. The parabolic second-order generalized upper directional derivative of $\hat{x}$ in the direction $(\hat{u}, x)$ is

$$\tilde{D}^2 f(\hat{x}, \hat{u})(x) := \limsup_{t>0, x' \to x} \frac{f(\hat{x} + t\hat{u} + \frac{1}{2}t^2 x') - f(\hat{x}) - tf'(\hat{x})\hat{u}}{\frac{1}{2}t^2}.$$

Remark 4.3 When the set-valued map F becomes to a vector-valued function f, which is Fréchet differentiable at $\hat{x}$, letting $\hat{v} := f'(\hat{x})\hat{u}$, we have

$$R^2 F(\hat{x}, \hat{y}, \hat{u}, \hat{v})(x) = \tilde{D}^2 f_+(\hat{x}, \hat{u})(x) = \limsup_{t>0, x' \to x} \frac{f_+(\hat{x} + t\hat{u} + \frac{1}{2}t^2 x') - f_+(\hat{x}) - tf'(\hat{x})\hat{u}}{\frac{1}{2}t^2}.$$

Corollary 4.5 *Suppose that* $(\hat{x}, \hat{y})$ *is a Henig efficient element of* (P) *and* $g(\hat{x}) \in -D$. *Then there exists a number* $\hat{\varepsilon} \in (0, \delta)$ *such that*

$$\big(\tilde{D}_l^2 f_+(\hat{x}, \hat{u})(x), \tilde{D}^2 g_+(\hat{x}, \hat{u})(x)\big) \cap \big(\big(-\operatorname{intcone}(\hat{\varepsilon}U + B)\big) \times (-\operatorname{int} D)\big) = \emptyset$$

for any $x \in \operatorname{dom} \tilde{D}_l^2 f_+(\hat{x}, \hat{u}) \cap \operatorname{dom} \tilde{D}^2 g_+(\hat{x}, \hat{u})$.

Proof The proof follows immediately from Theorem 4.2 and Remarks 4.2 and 4.3. □

5 Conclusions

In this paper, we introduced some new kinds of lower radial tangent cone, second-order lower radial tangent set, and second-order radial tangent set. By virtue of these concepts, second-order radial composed tangent derivative, second-order radial tangent derivative, second-order lower radial composed tangent derivative, and second-order lower radial tangent derivative for a set-valued map are introduced. Compared with the second-order composed contingent derivative $D''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ introduced in [4, 15], the second-order contingent derivative $D^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$, second-order radial composed tangent derivative $R''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$, and second-order radial tangent derivative $R^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$, second-order lower radial composed tangent derivative $R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$, and second-order lower radial tangent derivative $R_l^2F(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ have nice properties:

$$R_l''F(\hat{x}, \hat{y}, \hat{u}, \hat{v})\big(R\big(R(E, \hat{x}), \hat{u}\big)\big) \subset \operatorname{clcone}\big(\operatorname{clcone}\big(F(E) + C - \hat{y}\big) - \hat{v}\big)$$

and

$$R_l^2 F(\hat{x},\hat{y},\hat{u},\hat{v})\big(R^2(E,\hat{x},\hat{u})\big) \subset \operatorname{clcone}\big(\operatorname{cone}\big(F(E)+C-\hat{y}\big)-\hat{v}\big),$$

which are demonstrated in Propositions 3.1 and 3.2. Just applying these properties, we established second-order necessary optimality conditions for a point pair to be a Henig efficient element of a set-valued optimization problem where the second-order tangent derivatives of the objective function and constraint function are separated.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

Author details
[1]School of Information Engineering, Nanchang University, Nanchang, 330031, China. [2]School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China. [3]Department of Mathematics, Nanchang University, Nanchang, 330031, China.

Authors' information
Yihong Xu (1969-), Professor, Doctor, the major field of interest is the set-valued optimization.

Acknowledgements
This research was supported by the National Natural Science Foundation of China Grant 11461044 and the Natural Science Foundation of Jiangxi Province (20151BAB201027).

Received: 14 October 2016 Accepted: 9 December 2016 Published online: 04 January 2017

References

1. Gong, XH, Dong, HB, Wang, SY: Optimality conditions for proper efficient solutions of vector set-valued optimization. J. Math. Anal. Appl. **284**, 332-350 (2003)
2. Kasimbeyli, R: Radial epiderivatives and set-valued optimization. Optimization **58**, 521-534 (2009)
3. Jahn, J, Khan, AA, Zeilinger, P: Second-order optimality conditions in set optimization. J. Optim. Theory Appl. **125**, 331-347 (2005)
4. Zhu, SK, Li, SJ, Teo, KL: Second-order Karush-Kuhn-Tucker optimality conditions for set-valued optimization. J. Glob. Optim. **58**, 673-692 (2014)
5. Ning, E, Song, W, Zhang, Y: Second order sufficient optimality conditions in vector optimization. J. Glob. Optim. **54**, 537-549 (2012)
6. Xu, YH, Li, M, Peng, ZH: A note on 'Higher-order optimality conditions in set-valued optimization using Studniarski derivatives and applications to duality' [Positivity 18, 449-473(2014)]. Positivity **20**, 295-298 (2016)
7. Khan, AA, Tammer, C: Second-order optimality conditions in set-valued optimization via asymptotic derivatives. Optimization **62**, 743-758 (2013)
8. Li, SJ, Teo, KL, Yang, XQ: Higher-order optimality conditions for set-valued optimization. J. Optim. Theory Appl. **137**, 533-553 (2008)
9. Li, SJ, Zhu, SK, Teo, KL: New generalized second-order contingent epiderivatives and set-valued optimization problems. J. Optim. Theory Appl. **152**, 587-604 (2012)
10. Jiménez, B, Novo, V: Second-order necessary conditions in set constrained differentiable vector optimization. Math. Methods Oper. Res. **58**, 299-317 (2003)
11. Anh, NLH, Khanh, PQ: Higher-order optimality conditions in set-valued optimization using radial sets and radial derivatives. J. Glob. Optim. **56**, 519-536 (2013)
12. Anh, NLH, Khanh, PQ, Tung, LT: Higher-order radial derivatives and optimality conditions in nonsmooth vector optimization. Nonlinear Anal. **74**, 7365-7379 (2011)
13. Li, SJ, Chen, CR: Higher order optimality conditions for Henig efficient solutions in set-valued optimization. J. Math. Anal. Appl. **323**, 1184-1200 (2006)
14. Wang, QL, Li, SJ: Generalized higher-order optimality conditions for set-valued optimization under Henig efficiency. Numer. Funct. Anal. Optim. **30**, 849-869 (2009)
15. Khan, AA, Isac, G: Second-order optimality conditions in set-valued optimization by a new tangential derivative. Acta Math. Vietnam. **34**, 81-90 (2009)
16. Aghezzaf, B, Hachimi, M: Second-order optimality conditions in multiobjective optimization problems. J. Optim. Theory Appl. **102**, 37-50 (1999)
17. Xu, YH, Peng, ZH: Higher-order sensitivity analysis in set-valued optimization under Henig efficiency. J. Ind. Manag. Optim. (2016). doi:10.3934/jimo.2016019
18. Peng, ZH, Xu, YH: New second-order tangent epiderivatives and applications to set-valued optimization. J. Optim. Theory Appl. (2016). doi:10.1007/s10957-016-1011-1
19. Henig, MI: Proper efficiency with respect to cones. J. Optim. Theory Appl. **36**(3), 387-407 (1982)

20. Zheng, XY: Proper efficiency in locally convex topological vector spaces. J. Optim. Theory Appl. **94**(2), 469-486 (1997)
21. Borwein, JM, Zhuang, D: Super efficiency in vector optimization. Trans. Am. Math. Soc. **338**(1), 105-122 (1993)
22. Benson, HP: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. **71**, 232-241 (1979)
23. Qiu, QS, Yang, XM: Connectedness of Henig weakly efficient solution set for set-valued optimization problems. J. Optim. Theory Appl. **152**, 439-449 (2012)
24. Aubin, JP, Frankowska, H: Set-Valued Analysis. Birkhäuser, Boston (1990)