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Stability and traveling fronts for a food chain reaction-diffusion systems with nonlocal delays

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Abstract

This paper is purported to investigate a food chain reaction-diffusion predator-prey system with nonlocal delays in a bounded domain with no flux boundary condition. We investigate the global stability and find the sufficient conditions of global stability of the unique positive equilibrium for this system. The derived results show that delays often restrain stability. Using the method of linearizing this system, we see that the zero equilibrium is unstable. Moreover, by constructing upper-lower solutions, we find that there exist traveling wavefronts which connect the zero equilibrium and positive equilibrium when the wave speed is large enough and the prey intrinsic growth rate and the death rate of the predator are relatively big.

Keywords: stability; nonlocal delay; traveling waves

1 Introduction

The work on dynamics of predator-prey systems is one of the dominant topics in mathematical ecology. Among the relationships between the species living in the same environment, the predator-prey theory plays important role. The spatial content of the environment has often been ignored in traditional predator-prey systems. These systems have been formulated and investigated to reveal the time evolution of uniform population distributions in their habitats. However, the spatial distribution of the species is usually inhomogeneous, and ecologists and mathematical ecologists employ the reaction-diffusion predator-prey systems to model the interaction and the tendency of movement between predator and prey which imply that the species diffuse to areas of smaller population concentration during the process of evolution, mainly due to resource limitation. In the past two decades, reaction-diffusion predator-prey systems have been extensively discussed [1–9], but what these models reveal is that the future state of the models is determined only by the present, that is, it is independent of the past. For these systems, the predators must take time to digest their food (preys) before further responses and activities take place. Therefore, the models of species without delays are approximate at best. A realistic model must incorporate the past history of the system, that is, it must include the delay. Recently, some work has studied a delayed diffusive predator-prey system [10–13].

The theory of traveling fronts for the reaction-diffusion equations is one of the fastest developing areas of modern mathematics and has already attracted much attention due to its significance in physics, chemistry, biology and epidemiology. The traveling wave problem for reaction-diffusion systems has been studied by many authors [14–16] in the past ten years. The food chain is a common phenomenon in population ecology. It is also central to understanding of the community structure in ecology. We will explore a basic example of a food chain, namely, a three species food chain in which a resource species is preyed upon by an intermediate predator which in turn is preyed upon by a dominant predator, and this model includes nonlocal delays.

Motivated by the above work, we mainly take into account the following food chain reaction-diffusion predator-prey system with nonlocal delays:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - d_1\Delta u_1 = u_1(r_1 - a_1u_2 - b_1u_1),\\
&\frac{\partial u_2}{\partial t} - d_2\Delta u_2 = u_2\left(r_2 - b_2u_2 - a_2u_3 + a_3\int_\Omega\int_{-\infty}^t K_1(x,y,t-s)u_1(s,y)\,ds\,dy\right),\\
&\frac{\partial u_3}{\partial t} - d_3\Delta u_3 = u_3\left(-\alpha - b_3u_3 + a_4\int_\Omega\int_{-\infty}^t K_2(x,y,t-s)u_2(s,y)\,ds\,dy\right)\\
&\quad \text{in } (0,\infty)\times\Omega,\\
&\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0 \quad \text{on } (0,\infty)\times\partial\Omega,\\
&u_{i0}(\theta,x) = \phi_i(\theta,x) \geq 0 \quad (i=1,2,3) \text{ in } [-\infty,0]\times\Omega,
\end{aligned} \tag{1.1}$$

where Ω is bounded domain in R^N ($N \geq 1$ is an integer) with a smooth boundary $\partial\Omega$; u_1 represents the densities of the prey; u_2 is for the density of the prey, and the same for the predator; u_3 represents the densities of predator; the positive constants d_1, d_2, and d_3 are the diffusion coefficients of the corresponding species; the positive constants r_1, r_2, and α represent the prey intrinsic growth rates and the death rate of predator, respectively. a_1 and a_2 represent interaction rates, respectively; b_1, b_2, and b_3 represent self-limitation rates, respectively; the initial functions $u_{i0}(t,x)$ $(i=1,2,3)$ are Hölder continuous on $[-\infty,0]\times\overline{\Omega}$.

The terms $\int_\Omega\int_{-\infty}^t K_i(x,y,t-s)u_i(s,y)\,ds\,dy$ $(i=1,2)$ represent a time delay because of gestation, that is, predator contributes to the reproduction of predator biomass. In system (1.1), we suppose that the kernels $K_i(x,y,t)$ depend on both the temporal and the spatial variables. The delays in these formulations are nonlocal delays. These formulations reveal that the species drift to their present position (at time t) from all possible positions at all previous times (see [17]). Here, we suppose this drift cannot be viewed as being sufficiently small so as to be purely a local phenomenon.

In this article, we assume that

$$\begin{aligned}
&K_i(x,y,t) = G_i(x,y,t)k_i(t), \quad x,y\in\Omega, k_i(t)\geq 0,\\
&\int_\Omega G_i(x,y,t)\,dx = \int_\Omega G_i(x,y,t)\,dy = 1, \quad t\geq 0,\\
&\int_0^\infty k_i(t)\,dt = 1, \quad tk_i(t)\in L^1\big((0,\infty);R\big), i=1,2,
\end{aligned} \tag{1.2}$$

where $G_i(x,y,t)$ are nonnegative functions which are continuous in $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ for each $t\in[0,\infty)$ and measurable in $t\in[0,\infty)$ for each pair $(x,y)\in\overline{\Omega}\times\overline{\Omega}$.

In this article, by employing the method of eigenvalue and Lyapunov function, we investigate the stability of the unique positive constant solutions and find the sufficient conditions of stability which indicate that nonlocal delays often make impact on the stability of positive constant solution, but they do not impact that of a trivial solution. This result generalizes partially the one proved in [18]. By constructing upper-lower solutions, we establish the existence of the traveling wavefronts when the wave speed is large enough. The novelty of this article is that the system (1.1) incorporates two nonlocal delay terms with which are difficult to deal.

This paper is organized into four sections. In Section 2, the stability of the positive constant solution and the instability of trivial solution of system (1.1) are studied. In Section 3, the existence of traveling waves is established by constructing the upper and lower solution. In the final section, we give a short comment and conclusion.

2 Stability of positive equilibrium

It is easy to check that $(0,0,0)$ and (M_1,M_2,M_3) are a pair of coupled lower-upper solutions of system (1.1), where

$$\begin{aligned} M_1 &= \max\left\{\frac{r_1}{b_1}, \sup_{\theta\le 0}\left\|\phi_1(\theta,\cdot)\right\|_{C(\overline{\Omega},R)}\right\}, \\ M_2 &= \max\left\{\frac{r_1+a_3M_1}{b_2}, \sup_{\theta\le 0}\left\|\phi_2(\theta,\cdot)\right\|_{C(\overline{\Omega},R)}\right\}, \\ M_3 &= \max\left\{\frac{a_4M_2-\alpha}{b_3}, \sup_{\theta\le 0}\left\|\phi_3(\theta,\cdot)\right\|_{C(\overline{\Omega},R)}\right\}. \end{aligned} \tag{2.1}$$

Hence, there exists a unique global solution $(u_1(t,x),u_2(t,x),u_3(t,x))$ satisfying $0\le u_1\le M_1$, $0\le u_2\le M_2$, $0\le u_3\le M_3$ to system (1.1) (see [12]).

Note that (1.1) admits the following six equilibria: $(0,0,0)$, $(r_1/b_1,0,0)$, $(0,r_2/b_2,0)$, $(\frac{b_2r_1-a_2r_2}{a_1a_3+b_1b_2},\frac{a_3r_1+b_1r_2}{a_1a_3+b_1b_2},0)$, $(0,\frac{r_2b_3+\alpha a_2}{a_2a_4+b_2b_3},\frac{r_2a_4-\alpha b_2}{a_2a_4+b_2b_3})$, and (k_1^*,k_2^*,k_3^*), where

$$k_1^*=\frac{(r_1a_4-\alpha a_1)a_2+(r_1b_3-r_2a_1)b_3}{a_2a_4b_1+(a_1a_3+b_1b_3)b_3},\qquad k_2^*=\frac{(r_2b_3+\alpha a_2)b_1+r_1a_3b_3}{a_2a_4b_1+(a_1a_3+b_1b_3)b_3},$$
$$k_3^*=\frac{(r_1a_4-\alpha a_1)a_3+(r_2a_4-\alpha b_3)b_1}{a_2a_4b_1+(a_1a_3+b_1b_3)b_3}.$$

For the existence of a positive constant solution (k_1^*,k_2^*,k_3^*), it is necessary to assume that

$$r_1a_4>\alpha a_1,\qquad r_1b_3>r_2a_1,\qquad r_2a_4>\alpha b_3.$$

Let $0=\mu_1<\mu_2<\cdots\to+\infty$ denote the eigenvalues of $-\Delta$ in Ω under a homogeneous Neuman boundary condition and φ be the set of eigenfunctions corresponding to μ.

Notation 2.1

(i) $X_{ij}:=\{C\varphi_{ij}:C\in R^3\}$, where φ_{ij} are orthonormal basis of $S(\mu_i)$ for $j=1,\dots,\dim[S(\mu_i)]$.

(ii) $X:=\{(u,v,w)\in C^1(\overline{\Omega})\times C^1(\overline{\Omega}):\frac{\partial u}{\partial\nu}=\frac{\partial u}{\partial\nu}=\frac{\partial w}{\partial\nu}=0 \text{ on } \partial\Omega\}$, so that

$$X=\bigoplus_{i=0}^{\infty}\bigoplus_{j=1}^{\dim[S(\mu_i)]}X_{ij}.$$

Lemma 2.1 ([19, 20]) *Let a and b be positive constants. Suppose that $\xi, \eta \in C^1[a, +\infty)$, $\eta \geq 0$, and ξ is bounded from below. If $\xi'(t) \leq -b\eta(t)$ and $\eta'(t)$ is bounded from above in $[a, +\infty)$, then $\lim t_{t\to\infty}\eta(t) = 0$.*

The following theorem is the global stability result of the positive constant solution (k_1^*, k_2^*, k_3^*) of (1.1). This result extends partially the one proved in [18].

Theorem 2.1 *Suppose that $b_2b_3 > 2a_2a_4$, $b_1b_2 > 2a_1a_3$. Then the positive constant solution (k_1^*, k_2^*, k_3^*) is globally stable for the system* (1.1).

Proof We use the approach developed by [18] to find the proof. Let $(u_1(t,x), u_2(t,x), u_3(t,x))$ be positive solution of (1.1) and define the following Lyapunov function:

$$V_1(t) = \sum_{i=1}^{3} \beta_i \int_\Omega \left(u_i - k_i^* - k_i^* \ln \frac{u_i}{k_i^*} \right) dx,$$

where $\beta_3 = 1$, β_1 and β_2 are positive constant to be determined. By calculating the derivative of $V_1(t)$ along positive solutions of system (1.1), we obtain

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \sum_{i=1}^{2} \beta_i \int_\Omega \frac{\partial u_i}{\partial t}\left(1 - \frac{k_i^*}{u_i}\right) dx + \int_\Omega \frac{\partial u_3}{\partial t}\left(1 - \frac{k_3^*}{u_3}\right) dx \\
&= -\sum_{i=1}^{3} \beta_i d_i k_i^* \int_\Omega \frac{|\nabla u_i|}{u_i^2} dx + \int_\Omega \beta_1 \left(u_1 - k_1^*\right)(r_1 - a_1u_2 - b_1u_1)\, dx \\
&\quad + \int_\Omega \beta_2\left(u_2 - k_2^*\right)\left(r_2 - b_2u_2 - a_2u_3 + a_3 \int_\Omega \int_{-\infty}^{t} K_1(x,y,t-s)u_1(s,y)\, ds\, dy\right) dx \\
&\quad + \int_\Omega \left(u_3 - k_3^*\right)\left(-\alpha - b_3u_3 + a_4 \int_\Omega \int_{-\infty}^{t} K_2(x,y,t-s)u_2(s,y)\, ds\, dy\right) dx.
\end{aligned} \tag{2.2}$$

By using the inequality $ab \leq \frac{1}{2}\lambda a^2 + \frac{1}{2\lambda} b^2$, we obtain

$$\begin{aligned}
\frac{dV_1(t)}{dt} &\leq -\sum_{i=1}^{3} \beta_i d_i k_i^* \int_\Omega \frac{|\nabla u_i|}{u_i^2} dx - \beta_1 b_1 \int_\Omega \left(u_1 - k_1^*\right)^2 dx \\
&\quad + \beta_1 a_1 \int_\Omega \left[\frac{1}{2}\lambda_1\left(u_1 - k_1^*\right)^2 + \frac{1}{2\lambda_1}\left(u_2 - k_2^*\right)^2\right] dx - \beta_2 b_2 \int_\Omega \left(u_2 - k_2^*\right)^2 dx \\
&\quad + \beta_2 a_2 \int_\Omega \left[\frac{1}{2}\lambda_2\left(u_2 - k_2^*\right)^2 + \frac{1}{2\lambda_2}\left(u_3 - k_3^*\right)^2\right] dx - b_3 \int_\Omega \left(u_3 - k_3^*\right)^2 dx \\
&\quad + \frac{1}{2}\lambda_3 a_3 \beta_2 \int_\Omega \int_\Omega \int_{-\infty}^{t} K_1(x,y,t-s)\left(u_1(s,y) - k_1^*\right)^2 ds\, dy\, dx \\
&\quad + \frac{1}{2\lambda_3} a_3 \beta_2 \int_\Omega \int_\Omega \int_{-\infty}^{t} K_1(x,y,t-s)\left(u_2(t,y) - k_2^*\right)^2 ds\, dy\, dx \\
&\quad + \frac{1}{2}\lambda_4 a_4 \int_\Omega \int_\Omega \int_{-\infty}^{t} K_2(x,y,t-s)\left(u_2(s,y) - k_2^*\right)^2 ds\, dy\, dx \\
&\quad + \frac{1}{2\lambda_4} a_4 \int_\Omega \int_\Omega \int_{-\infty}^{t} K_2(x,y,t-s)\left(u_3(t,y) - k_3^*\right)^2 ds\, dy\, dx.
\end{aligned} \tag{2.3}$$

Employing the property of $K_i(x,y,t)$ $(i=1,2)$ as described in (1.2), we obtain

$$\begin{aligned}\frac{dV_1(t)}{dt} \le{}& -\sum_{i=1}^{3}\beta_i d_i k_i^* \int_\Omega \frac{|\nabla u_i|}{u_i^2}\,dx - \beta_1(b_1 - a_1\lambda_1/2)\int_\Omega \big(u_1-k_1^*\big)^2\,dx \\ &- (\beta_2 b_2 - \beta_1 a_1/2\lambda_1 - \lambda_2\beta_2 a_2/2 - a_3\beta_2/2\lambda_3)\int_\Omega \big(u_2-k_2^*\big)^2\,dx \\ &- (b_3 - \beta_2 a_2/2\lambda_2 - a_4/2\lambda_4)\int_\Omega \big(u_3-k_3^*\big)^2\,dx \\ &+ \frac{1}{2}\lambda_3 a_3\beta_2 \int_\Omega\int_\Omega\int_0^\infty K_1(x,y,r)\big(u_1(t-r,y)-k_1^*\big)^2\,dr\,dy\,dx \\ &+ \frac{1}{2}\lambda_4 a_4 \int_\Omega\int_\Omega\int_0^\infty K_2(x,y,r)\big(u_2(t-r,y)-k_2^*\big)^2\,dr\,dy\,dx. \end{aligned} \tag{2.4}$$

Define a new Lyapunov function

$$\begin{aligned}V(t) ={}& V_1(t) + \frac{1}{2}\lambda_3 a_3\beta_2 \int_\Omega\int_\Omega\int_0^\infty\int_{t-r}^{t} K_1(x,y,r)\big(u_1(l,y)-k_1^*\big)^2\,dl\,dr\,dy\,dx \\ &+ \frac{1}{2}\lambda_4 a_4 \int_\Omega\int_\Omega\int_0^\infty\int_{t-r}^{t} K_2(x,y,r)\big(u_2(l,y)-k_2^*\big)^2\,dl\,dr\,dy\,dx. \end{aligned} \tag{2.5}$$

Then, combining (2.4) and (2.5), we get

$$\begin{aligned}\frac{dV_1(t)}{dt} \le{}& -\sum_{i=1}^{3}\beta_i d_i k_i^* \int_\Omega \frac{|\nabla u_i|}{u_i^2}\,dx - \beta_1(b_1 - a_1\lambda_1/2)\int_\Omega \big(u_1-k_1^*\big)^2\,dx \\ &- (\beta_2 b_2 - \beta_1 a_1/2\lambda_1 - \lambda_2\beta_2 a_2/2 - a_3\beta_2/2\lambda_3)\int_\Omega \big(u_2-k_2^*\big)^2\,dx \\ &- (b_3 - \beta_2 a_2/2\lambda_2 - a_4/2\lambda_4)\int_\Omega \big(u_3-k_3^*\big)^2\,dx \\ &+ \frac{1}{2}\lambda_3 a_3\beta_2 \int_\Omega\int_\Omega\int_0^\infty K_1(x,y,r)\big(u_1(t,y)-k_1^*\big)^2\,dr\,dy\,dx \\ &+ \frac{1}{2}\lambda_4 a_4 \int_\Omega\int_\Omega\int_0^\infty K_2(x,y,r)\big(u_2(t,y)-k_2^*\big)^2\,dr\,dy\,dx. \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}&\int_\Omega\int_\Omega\int_0^\infty K_i(x,y,r)\big(u_i(t,y)-k_i^*\big)^2\,dr\,dy\,dx \\ &\quad= \int_\Omega \big(u_i(t,y)-k_i^*\big)^2\,dy \quad (i=1,2),\end{aligned}$$

it follows from (2.6) that

$$\begin{aligned}\frac{dV_1(t)}{dt} \le{}& -\sum_{i=1}^{3}\beta_i d_i k_i^* \int_\Omega \frac{|\nabla u_i|}{u_i^2}\,dx \\ &- \big(\beta_1(b_1 - a_1\lambda_1/2) - \lambda_3 a_3\beta_2/2\big)\int_\Omega \big(u_1-k_1^*\big)^2\,dx\end{aligned}$$

$$
\begin{aligned}
&-(\beta_2 b_2 - \beta_1 a_1/2\lambda_1 - \lambda_2\beta_2 a_2/2 - a_3\beta_2/2\lambda_3 - \lambda_4 a_4/2)\int_\Omega (u_2 - k_2^*)^2\,dx \\
&-(b_3 - \beta_2 a_2/2\lambda_2 - a_4/2\lambda_4)\int_\Omega (u_3 - k_3^*)^2\,dx. \qquad (2.7)
\end{aligned}
$$

Integrating (2.7) over $[0,T]$ ($T > 0$), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{3}\beta_i d_i k_i^*\left\|\frac{|\nabla u_i|}{u_i}\right\|^2_{L^2_{(\Omega_T)}} + \beta_1 b_1\|u_1 - k_1^*\|^2_{L^2_{(\Omega_T)}} + \beta_2 b_2\|u_2 - k_2^*\|^2_{L^2_{(\Omega_T)}} + b_3\|u_3 - k_3^*\|^2_{L^2_{(\Omega_T)}} \\
&\quad\le V(0) + \frac{1}{2}(\beta_1 a_1\lambda_1 + \lambda_3\beta_2 a_3)\|u_1 - k_1^*\|^2_{L^2_{(\Omega_T)}} + \frac{1}{2}(\beta_1 a_1/\lambda_1 + \lambda_2\beta_2 a_2 + a_3\beta_2/\lambda_3 \\
&\qquad + \lambda_4 a_4)\|u_2 - k_2^*\|^2_{L^2_{(\Omega_T)}} + \frac{1}{2}(\beta_2 a_2/\lambda_2 + a_4/\lambda_4)\|u_3 - k_3^*\|^2_{L^2_{(\Omega_T)}}. \qquad (2.8)
\end{aligned}
$$

Taking

$$
\lambda_1 = \lambda_3 = \frac{2\beta_1 b_1}{a_1\beta_1 + a_3} \quad\text{and}\quad \lambda_2 = \lambda_4 = \frac{\beta_2 a_2 + a_4}{2b_3},
$$

it is derived from (2.8) that

$$
\begin{aligned}
&\sum_{i=1}^{3}\frac{\beta_i d_i k_i^*}{M_i}\||\nabla u_i|\|^2_{L^2_{(\Omega_T)}} + \beta_2 b_2\|u_2 - k_2^*\|^2_{L^2_{(\Omega_T)}} \\
&\quad\le V(0) + \bigl((a_1\beta_1 + a_3\beta_2)^2/4\beta_1 b_1 + (a_2\beta_2 + a_4)^2/4b_3\bigr)\|u_2 - k_2^*\|^2_{L^2_{(\Omega_T)}}. \qquad (2.9)
\end{aligned}
$$

Using the conditions $b_2 b_3 > 2a_2 a_4$, $b_1 b_2 > 2a_1 a_3$, one can choose $\beta_1, \beta_2 > 0$ such that

$$
\beta_2 b_2 > \frac{1}{4}\bigl[(a_1\beta_1 + a_3\beta_2)^2/\beta_1 b_1 + (a_2\beta_2 + a_4)^2/b_3\bigr],
$$

because the inequalities $2b_1 b_2\beta_1\beta_2 > a_1^2\beta_1^2 + 2a_1 a_3\beta_1\beta_2 + a_2^2\beta_2^2$ and $2b_2 b_3\beta_2 > a_2^2\beta_2^2 + 2a_2 a_4\beta_2 + a_4^2$ hold for certain β_1 and β_2.

Therefore, we obtain

$$
\||\nabla u_i|\|_{L^2_{(\Omega_T)}} \le C_1, \qquad \|u_2 - k_2^*\|_{L^2_{(\Omega_T)}} \le C_1, \qquad (2.10)
$$

where C_1 is constant independent of T. In a similar way, by taking

$$
\lambda_1 = \lambda_3 = \frac{\beta_1 a_1 + a_3\beta_2}{\beta_2 b_2} \quad\text{and}\quad \lambda_2 = \lambda_4 = \frac{\beta_2 b_2}{a_2\beta_2 + a_4},
$$

it is derived from (2.8) that

$$
\begin{aligned}
&\sum_{i=1}^{3}\frac{\beta_i d_i k_i^*}{M_i}\||\nabla u_i|\|^2_{L^2_{(\Omega_T)}} + \beta_1 b_1\|u_1 - k_1^*\|^2_{L^2_{(\Omega_T)}} + b_3\|u_3 - k_3^*\|^2_{L^2_{(\Omega_T)}} \\
&\quad\le V(0) + \bigl((\beta_1 a_1 + a_3\beta_2)^2/2\beta_2 b_2\bigr)\|u_1 - k_1^*\|^2_{L^2_{(\Omega_T)}} \\
&\qquad + \bigl((\beta_2 a_2 + a_4)^2/2\beta_2 b_2\bigr)\|u_3 - k_3^*\|^2_{L^2_{(\Omega_T)}}. \qquad (2.11)
\end{aligned}
$$

Using the conditions $b_2b_3 > 2a_2a_4$, $b_1b_2 > 2a_1a_2$, one can choose $\beta_1, \beta_2 > 0$ again such that

$$\beta_1 b_1 > \frac{(a_1\beta_1 + a_3\beta_2)^2}{2\beta_2 b_2} \quad \text{and} \quad b_3 > \frac{(a_2\beta_2 + a_4)^2}{2\beta_2 b_2}.$$

Therefore, we see that

$$\left\|u_1 - k_1^*\right\|^2_{L^2_{(\Omega_T)}} \le C_2 \quad \text{and} \quad \left\|u_3 - k_3^*\right\|^2_{L^2_{(\Omega_T)}} \le C_2, \tag{2.12}$$

where C_2 is constant independent of T.

Choosing

$$\lambda_1 = \lambda_3 = \frac{\beta_1 a_1 + a_3\beta_2}{\beta_2 b_2} + \varepsilon, \qquad \lambda_2 = \lambda_4 = \frac{\beta_2 b_2}{a_2\beta_2 + a_4},$$

where ε is sufficiently small positive constant. Using the conditions of Theorem 2.1 and (2.7), one can easily verify that there exists a positive constant δ ($\delta > 0$) such that

$$\begin{aligned} &\frac{dV}{dt} \le -\delta \int_\Omega \left[\left(u_1 - k_1^*\right)^2 + \left(u_2 - k_2^*\right)^2 + \left(u_3 - k_3^*\right)^2\right] dx, \\ &\frac{dV}{dt} \le 0, \qquad (u_1, u_2, u_3) \ne \left(k_1^*, k_2^*, k_3^*\right). \end{aligned} \tag{2.13}$$

Using integration by parts, the Hölder inequality, (2.1), and (2.10), one can easily check that $\frac{d}{dt}\int_\Omega [(u_1 - k_1^*)^2 + (u_2 - k_2^*)^2 + (u_3 - k_3^*)^2]\, dx$ is bounded from above. Then, using Lemma 2.1, (2.10), and (2.13), we see that

$$\begin{aligned} &\left\|u_1(t,\cdot) - k_1^*\right\|_{L^2_{(\Omega)}} \to 0, \\ &\left\|u_2(t,\cdot) - k_2^*\right\|_{L^2_{(\Omega)}} \to 0, \\ &\left\|u_3(t,\cdot) - k_3^*\right\|_{L^2_{(\Omega)}} \to 0. \end{aligned} \tag{2.14}$$

Obviously,

$$\left\|u(t,x)\right\|_{L^\infty_{(\Omega)}} \le C_3 \|u\|^{\frac{1}{2}}_{W_2^1} \|u\|_{L^2(\Omega)}. \tag{2.15}$$

It follows from (2.10), (2.12), (2.14), (2.15), and (2.1) that

$$\begin{aligned} &\left\|u_1(t,\cdot) - k_1^*\right\|_{L^\infty_{(\Omega)}} \to 0, \\ &\left\|u_2(t,\cdot) - k_2^*\right\|_{L^\infty_{(\Omega)}} \to 0, \\ &\left\|u_3(t,\cdot) - k_3^*\right\|_{L^\infty_{(\Omega)}} \to 0. \end{aligned}$$

Namely, (u_1, u_2, u_3) converges uniformly to (k_1^*, k_2^*, k_3^*). Using the fact that $V(u_1, u_2, u_3)$ is decreasing for t, one can derive that (k_1^*, k_2^*, k_3^*) is globally stable. This completes the proof. □

Theorem 2.2 *The trivial equilibrium* $(0,0,0)$ *is unstable for the system* (1.1).

Proof The linearized problem of (1.1) at $(0,0,0)$ can be expressed by

$$\mathbf{w}_t = \big(D\Delta + F_{\mathbf{w}}(0,0,0)\big)\mathbf{w},$$

where $\mathbf{w} = (u_1(t,x), u_2(t,x), u_3(t,x))^T$, and $F = (u_1(r_1 - a_1u_2 - b_1u_1), u_2(r_2 - b_2u_2 - a_2u_3 + a_3\int_\Omega\int_{-\infty}^t K_1(x,y,t-s)u_1(s,y)\,ds\,dy), u_3(-\alpha - b_3u_3 + a_4\int_\Omega\int_{-\infty}^t K_2(x,y,t-s)u_2(s,y)\,ds\,dy))$. By direct calculations, we obtain

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \qquad F_{\mathbf{w}}(0,0,0) = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Consider the following eigenvalue problem:

$$\big(D\Delta + F_{\mathbf{w}}(\mathbf{w}_*)\big)\begin{pmatrix} \phi \\ \varphi \\ \psi \end{pmatrix} = \tilde{\lambda}\begin{pmatrix} \phi \\ \varphi \\ \psi \end{pmatrix},$$

where $\mathbf{w}_*$ is constant solution of (1.1). Using the eigenfunction expansions (2.6) in [3] for ϕ, φ and ψ, $\tilde{\lambda}$ is an eigenvalue of $D\Delta + F_{\mathbf{w}}(\mathbf{w}_*)$ if and only if $\tilde{\lambda}$ is an eigenvalue of the matrix $-\mu_k D + F_{\mathbf{w}}(0,0,0)$ for each $k \geq 1$. Therefore, to study the local stability at $(0,0,0)$, it is necessary to investigate the characteristic equation

$$\det\big(\tilde{\lambda}I + \mu_k D - F_{\mathbf{w}}(0,0,0)\big) = (\tilde{\lambda} + \mu_k d_1 - r_1)(\tilde{\lambda} + \mu_k d_2 - r_2)(\tilde{\lambda} + \mu_k d_2 + \alpha) = 0.$$

If $i = 1$, then $\mu_k = 0$. Therefore, there exist two positive characteristic roots, which, in view of Theorem 5.13 in [21], yields the desired result. □

3 The existence of traveling waves

In this section, we assume that $\Omega \subset R^1$. Denote $f_1(u_1,u_2,u_3) = u_1(r_1 - a_1u_2 - b_1u_1)$, $f_2(u_1,u_2,u_3) = u_2(r_2 - b_2u_2 - a_2u_3 + a_3\int_\Omega\int_{-\infty}^t K_1(x,y,t-s)u_1(s,y)\,ds\,dy)$, $f_3(u_1,u_2,u_3) = u_3(-\alpha - b_3u_3 + a_4\int_\Omega\int_{-\infty}^t K_2(x,y,t-s)u_2(s,y)\,ds\,dy)$. Let $(u_1(t,x), u_2(t,x), u_3(t,x)) = (\phi(x+ct), \varphi(x+ct), \psi(x+ct))$ be a traveling wave solution of (1.1), where $\phi, \varphi, \psi \in C^2(R, R^2)$ and $c > 0$ is a constant accounting for the wave speed, and denote the traveling wave coordinate $x + ct$ still by t. Then the system (1.1) can be rewritten in the form

$$\begin{aligned} &d_1\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t, \psi_t) = 0, \\ &d_2\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t, \psi_t) = 0, \\ &d_3\psi''(t) - c\psi'(t) + f_{c3}(\phi_t, \varphi_t, \psi_t) = 0, \end{aligned} \tag{3.1}$$

where f_{ci} $(i = 1,2,3)$ are defined by

$$\begin{aligned} &f_{ci}(\phi,\varphi,\psi) = f_i\big(\phi^c, \varphi^c, \psi^c\big), \qquad \phi^c(s) = \phi(cs), \qquad \varphi^c(s) = \varphi(cs), \\ &\psi^c(s) = \psi(cs), \quad s \in (-\infty, 0], i = 1,2,3. \end{aligned}$$

If (3.1) has a solution satisfying the following asymptotic boundary conditions:

$$\lim_{t\to-\infty}\phi(t)=\phi_-,\qquad \lim_{t\to-\infty}\varphi(t)=\varphi_-,\qquad \lim_{t\to-\infty}\psi(t)=\psi_-,$$
$$\lim_{t\to+\infty}\phi(t)=\phi_+,\qquad \lim_{t\to+\infty}\varphi(t)=\varphi_+,\qquad \lim_{t\to+\infty}\psi(t)=\psi_+,$$

then system (1.1) has a traveling wave solution (see [15, 16]). Without loss of generality, we assume that $(\phi_-,\varphi_-,\psi_-)=(0,0,0)$ and $(\phi_+,\varphi_+,\psi_+)=(k_1^*,k_2^*,k_3^*)$.

According to basic theory of the existence of traveling wave solutions (see [15, 16]), we mainly need to check that the system (1.1) satisfies partial quasi-monotonicity conditions, that is, there exist three positive constants $\rho_1,\rho_2,\rho_3>0$ such that

$$\begin{aligned}
&f_1(\phi_1,\varphi_1,\psi_1)-f_1(\phi_2,\varphi_1,\psi_2)+\rho_1\bigl[\phi_1(0)-\phi_2(0)\bigr]\geq 0,\\
&f_1(\phi_1,\varphi_1,\psi_1)-f_1(\phi_1,\varphi_2,\psi_1)\leq 0,\\
&f_2(\phi_1,\varphi_1,\psi_1)-f_2(\phi_2,\varphi_2,\psi_1)+\rho_2\bigl[\varphi_1(0)-\varphi_2(0)\bigr]\geq 0,\\
&f_2(\phi_1,\varphi_1,\psi_1)-f_2(\phi_1,\varphi_1,\psi_2)\leq 0,\\
&f_3(\phi_1,\varphi_1,\psi_1)-f_3(\phi_2,\varphi_2,\psi_2)+\rho_3\bigl[\psi_1(0)-\psi_2(0)\bigr]\geq 0,
\end{aligned}\tag{3.2}$$

with $0\leq\phi_2(s)\leq\phi_1(s)\leq M_1$, $0\leq\varphi_2(s)\leq\varphi_1(s)\leq M_2$, $0\leq\psi_2(s)\leq\psi_1(s)\leq M_3$, and we also need to check that a pair of continuous functions $(\overline{\phi},\overline{\varphi},\overline{\psi})$ and $(\underline{\phi},\underline{\varphi},\underline{\psi})$ is a pair of upper-lower solution of system (3.1), that is,

$$\begin{aligned}
&d_1\overline{\phi}''(t)-c\overline{\phi}'(t)+f_{c1}(\overline{\phi}_t,\underline{\varphi}_t,\overline{\psi}_t)\leq 0,\\
&d_2\overline{\varphi}''(t)-c\overline{\varphi}'(t)+f_{c2}(\overline{\varphi}_t,\overline{\varphi}_t,\underline{\psi}_t)\leq 0,\\
&d_3\overline{\psi}''(t)-c\overline{\psi}'(t)+f_{c3}(\overline{\psi}_t,\overline{\varphi}_t,\overline{\psi}_t)\leq 0
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
&d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+f_{c1}(\underline{\phi}_t,\overline{\varphi}_t,\underline{\psi}_t)\geq 0,\\
&d_2\underline{\varphi}''(t)-c\underline{\varphi}'(t)+f_{c2}(\underline{\varphi}_t,\underline{\varphi}_t,\overline{\psi}_t)\geq 0,\\
&d_3\underline{\psi}''(t)-c\underline{\psi}'(t)+f_{c3}(\underline{\psi}_t,\underline{\varphi}_t,\underline{\psi}_t)\geq 0,
\end{aligned}\tag{3.4}$$

where $(0,0,0)\leq(\underline{\phi},\underline{\varphi},\underline{\psi})\leq(\overline{\phi},\overline{\varphi},\overline{\psi})\leq(M_1,M_2,M_3)$, $t\in R$.

Lemma 3.1 *$f_{c1}(\phi_t,\varphi_t,\psi_t)$, $f_{c2}(\phi_t,\varphi_t,\psi_t)$, and $f_{c3}(\phi_t,\varphi_t,\psi_t)$ of system* (1.1) *satisfy* (3.2).

Proof Let $\phi_1(s)$, $\phi_2(s)$, $\varphi_1(s)$, $\varphi_2(s)$, $\psi_1(s)$, $\psi_2(s)$ satisfy $0\leq\phi_2(s)\leq\phi_1(s)\leq M_1$, $0\leq\varphi_2(s)\leq\varphi_1(s)\leq M_2$, $0\leq\psi_2(s)\leq\psi_1(s)\leq M_3$, $s\in(-\infty,0]$.

For any $\phi_i,\varphi_i,\psi_i\in((-\infty,0],R)$, $i=1,2$, we have

$$\begin{aligned}
&f_{c1}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c1}(\phi_{2t},\varphi_{1t},\psi_{2t})\\
&\quad=\phi_1(0)\bigl(r_1-a_1\varphi_1(0)-b_1\phi_1(0)\bigr)-\phi_2(0)\bigl(r_1-a_1\varphi_1(0)-b_1\phi_2(0)\bigr)
\end{aligned}$$

$$\begin{aligned}
&\geq r_1\big(\phi_1(0)-\phi_2(0)\big)-a_1\varphi_1(0)\big(\phi_1(0)-\phi_2(0)\big)-2b_1M_1\big(\phi_1(0)-\phi_2(0)\big)\\
&\geq (-a_1M_2-2b_1M_1)\big(\phi_1(0)-\phi_2(0)\big).
\end{aligned} \tag{3.5}$$

Let $\rho_1 = a_1M_2 + 2b_1M_1 > 0$, then it is easy to see that

$$\begin{aligned}
&f_{c1}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c1}(\phi_{2t},\varphi_{1t},\psi_{2t})+\rho_1\big(\phi_1(0)-\phi_2(0)\big)\geq 0,\\
&f_{c1}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c1}(\phi_{1t},\varphi_{2t},\psi_{1t})\\
&\quad=\phi_1(0)\big(r_1-a_1\varphi_1(0)-b_1\phi_1(0)\big)-\phi_1(0)\big(r_1-a_1\varphi_2(0)-b_1\phi_1(0)\big)\\
&\quad=-a_1\phi_1(0)\big(\varphi_1(0)-\varphi_2(0)\big)\leq 0.
\end{aligned} \tag{3.6}$$

For $f_{c2}(\phi_t,\varphi_t,\psi_t)$, we have

$$\begin{aligned}
&f_{c2}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c2}(\phi_{2t},\varphi_{2t},\psi_{1t})\\
&\quad=\varphi_1(0)\big(r_2-b_2\varphi_1(0)-a_2\psi_1(0)+a_3\phi_1(0)\big)-\varphi_2(0)\big(r_2-b_2\varphi_2(0)\\
&\qquad-a_2\psi_1(0)+a_3\phi_2(0)\big)\\
&\quad\geq r_2\big(\varphi_1(0)-\varphi_2(0)\big)-2b_2M_2\big(\varphi_1(0)-\varphi_2(0)\big)-a_2M_3\big(\varphi_1(0)-\varphi_2(0)\big)\\
&\qquad+a_3\phi_1(0)\big(\varphi_1(0)-\varphi_2(0)\big)\\
&\quad\geq(-a_2M_3-2b_2M_2)\big(\varphi_1(0)-\varphi_2(0)\big).
\end{aligned} \tag{3.7}$$

Let $\rho_2 = a_2M_3 + 2b_2M_2 > 0$, then it is easy to see that

$$\begin{aligned}
&f_{c2}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c2}(\phi_{2t},\varphi_{2t},\psi_{1t})+\rho_2\big(\varphi_1(0)-\varphi_2(0)\big)\geq 0,\\
&f_{c2}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c2}(\phi_{1t},\varphi_{1t},\psi_{2t})\\
&\quad=\varphi_1(0)\big(r_2-b_1\varphi_1(0)-a_2\psi_1(0)+a_3\phi_1(0)\big)-\varphi_1(0)\big(r_2-b_1\varphi_1(0)\\
&\qquad-a_2\psi_2(0)+a_3\phi_1(0)\big)\\
&\quad=-a_2\varphi_1(0)\big(\psi_1(0)-\psi_2(0)\big).
\end{aligned} \tag{3.8}$$

For $f_{c3}(\phi_t,\varphi_t,\psi_t)$, we have

$$\begin{aligned}
&f_{c3}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c3}(\phi_{2t},\varphi_{2t},\psi_{2t})\\
&\quad=\psi_1(0)\big(-\alpha-b_3\psi_1(0)+a_4\varphi_1(0)\big)\\
&\qquad-\psi_2(0)\big(-\alpha-b_3\psi_2(0)+a_4\varphi_2(0)\big)\\
&\quad\geq-\alpha\big(\psi_1(0)-\psi_2(0)\big)-2b_3M_3\big(\psi_1(0)-\psi_2(0)\big)\\
&\quad=(-\alpha-2b_3M_3)\big(\psi_1(0)-\psi_2(0)\big).
\end{aligned} \tag{3.9}$$

Let $\rho_3 = 2b_3M_3 + \alpha > 0$, then it is easy to see that

$$f_{c3}(\phi_{1t},\varphi_{1t},\psi_{1t})-f_{c3}(\phi_{2t},\varphi_{2t},\psi_{2t})+\rho_3\big(\psi_1(0)-\psi_2(0)\big)\geq 0.$$

This completes the proof of Lemma 3.1. □

We assume that $c^2 > c_*^2 \triangleq \max\{4d_1r_1, 4d_2(r_2+a_3M_1), 4d_3(\alpha+a_4M_2)\}$, $c_* > 0$. Using this assumption, one can see that there exist $\eta_i > 0$ $(i=1,2,3)$ such that

$$\begin{aligned} &d_1\eta_1^2 - c\eta_1 + r_1 = 0, \\ &d_2\eta_2^2 - c\eta_2 + r_2 + a_3M_1 = 0, \\ &d_3\eta_3^2 - c\eta_3 + \alpha + a_4M_2 = 0. \end{aligned} \tag{3.10}$$

Assume that $r_1 > \max\{a_1k_2^*, a_1k_1^*\}$ and $\alpha > \frac{b_2b_3k_2^*}{a_2}$, then one can choose positive constants ε_5 and ε_6 such that

$$\begin{aligned} &a_1(\varepsilon_5 - k_2^*) > b_1k_1^*, \qquad a_2(\varepsilon_6 - k_3^*) > b_2k_2^*, \qquad \alpha > b_3(\varepsilon_6 - k_3^*), \\ &r_1b_2 > a_1a_2(\varepsilon_6 - k_3^*), \qquad r_2 + b_2(\varepsilon_5 - k_2^*) > a_2k_3^*, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &(k_1^*+\varepsilon_1)\bigl(r_1 - a_1(k_2^*-\varepsilon_5) - b_1(k_1^*+\varepsilon_1)\bigr) < 0, \\ &(k_2^*+\varepsilon_2)\bigl(r_2 - b_2(k_2^*+\varepsilon_2) - a_2(k_3^*-\varepsilon_6) + a_3M_1\bigr) < 0, \\ &(k_3^*+\varepsilon_3)\bigl(-\alpha - b_3(k_3^*+\varepsilon_3) + a_4M_2\bigr) < 0, \\ &(k_1^*-\varepsilon_4)\bigl[r_1 - a_1(k_2^*+\varepsilon_2) - b_1(k_1^*-\varepsilon_4)\bigr] > 0, \\ &(k_2^*-\varepsilon_5)\bigl[r_2 - b_2(k_2^*-\varepsilon_5) - a_2(k_3^*+\varepsilon_3)\bigr] > 0, \\ &(k_3^*-\varepsilon_6)\bigl(-\alpha - b_3(k_3^*-\varepsilon_6)\bigr) > 0, \end{aligned} \tag{3.12}$$

for ε_i $(i=1,2,3)$ being relatively big.

For the above constants and suitable constants $\tilde{t}_i > 0$ $(i=1,2,3,4,5,6)$ satisfying $\tilde{t}_5 < \min\{\tilde{t}_1, \tilde{t}_3\}$, $\tilde{t}_2 > \max\{\tilde{t}_4, \tilde{t}_6\}$, we define the continuous functions $\overline{\Phi}(t) = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ and $\underline{\Phi}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ as follows:

$$\overline{\phi}(t) = \begin{cases} k_1^*e^{\eta_1 t}, & t \le \tilde{t}_1, \\ k_1^* + \varepsilon_1 e^{-\eta t}, & t > \tilde{t}_1, \end{cases} \qquad \overline{\varphi}(t) = \begin{cases} k_2^*e^{\eta_2 t}, & t \le \tilde{t}_2, \\ k_2^* + \varepsilon_2 e^{-\eta t}, & t > \tilde{t}_2, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} k_3^*e^{\eta_3 t}, & t \le \tilde{t}_3, \\ k_3^* + \varepsilon_3 e^{-\eta t}, & t > \tilde{t}_3, \end{cases} \qquad \underline{\phi}(t) = \begin{cases} 0, & t \le \tilde{t}_4, \\ k_1^* - \varepsilon_4 e^{-\eta t}, & t > \tilde{t}_4, \end{cases}$$

$$\underline{\varphi}(t) = \begin{cases} 0, & t \le \tilde{t}_5, \\ k_2^* - \varepsilon_5 e^{-\eta t}, & t > \tilde{t}_5, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} 0, & t \le \tilde{t}_6, \\ k_3^* - \varepsilon_6 e^{-\eta t}, & t > \tilde{t}_6, \end{cases}$$

where

$$\begin{aligned} &\tilde{t}_2 = \frac{1}{\eta_2}\ln\frac{r_1}{a_1k_2^*}, \qquad \tilde{t}_6 = \frac{1}{\eta_2}\ln\frac{a_2(\varepsilon_6 - k_3^*)}{b_2k_2^*}, \\ &\tilde{t}_5 = \max\left\{\frac{1}{\eta_1}\ln\frac{a_1(\varepsilon_5 - k_2^*)}{b_1k_1^*}, \frac{1}{\eta_3}\ln\frac{r_2 + b_2(\varepsilon_5 - k_2^*)}{a_2k_3^*}\right\}. \end{aligned} \tag{3.13}$$

Lemma 3.2 *Assume that $r_1 > \max\{a_1k_2^*, a_1k_1^*\}$ and $\alpha > \frac{b_2b_3k_2^*}{a_2}$, then $(\overline{\phi}, \overline{\varphi}, \overline{\psi})$ is an upper solution of system* (3.1).

Proof If $t \le \tilde{t}_5 < \tilde{t}_1$, then $\overline{\phi}(t) = k_1^* e^{\eta_1 t}$, and $\underline{\varphi}(t) = 0$. Therefore, we have

$$\begin{aligned} &d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + \overline{\phi}\bigl(r_1 - a_1\underline{\varphi}(t) - b_1\overline{\phi}(t)\bigr) \\ &\quad\le \bigl(d_1\eta_1^2 - c\eta_1 + r_1\bigr)k_1^* e^{\eta_1 t} = 0. \end{aligned} \tag{3.14}$$

If $t > \tilde{t}_5$ and $t \le \tilde{t}_1$, then $\overline{\phi}(t) = k_1^* e^{\eta_1 t}$ and $\underline{\varphi}(t) = k_2^* - \varepsilon_5 e^{-\eta t}$. Therefore, we obtain

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + \overline{\phi}(t)\bigl(r_1 - a_1\underline{\varphi}(t) - b_1\overline{\phi}(t)\bigr) = I_1(\eta), \tag{3.15}$$

where $I_1(\eta) = k_1^* e^{\eta_1 t}[a_1(\varepsilon_5 e^{-\eta t} - k_2^*) - b_1 k_1^* e^{\eta_1 t}]$, $I_1(0) = k_1^* e^{\eta_1 t}[a_1(\varepsilon_5 - k_2^*) - b_1 k_1^* e^{\eta_1 t}] < 0$ when $t > \tilde{t}_5$. Therefore, there exists $\eta_1^* > 0$ such that $d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + \overline{\phi}(t)(r_1 - a_1\underline{\varphi}(t) - b_1\overline{\phi}(t)) \le 0$ for all $\eta \in (0, \eta_1^*)$.

If $t > \tilde{t}_1 > \tilde{t}_5$, then $\overline{\phi}(t) = k_1^* + \varepsilon_1 e^{-\eta t}$ and $\underline{\varphi}(t) = k_2^* - \varepsilon_5 e^{-\eta t}$. Therefore, we have

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + \overline{\phi}(t)\bigl(r_1 - a_1\underline{\varphi}(t) - b_1\overline{\phi}(t)\bigr) = I_2(\eta), \tag{3.16}$$

where $I_2(\eta) = (d_1\varepsilon_1\eta^2 + c\varepsilon_1\eta)e^{-\eta t} + (k_1^* + \varepsilon_1 e^{-\eta t})(r_1 - a_1(k_2^* - \varepsilon_5 e^{-\eta t}) - b_1(k_1^* + \varepsilon_1 e^{-\eta t}))$. It follows from (3.12) that $I_2(0) = (k_1^* + \varepsilon_1)(r_1 - a_1(k_2^* - \varepsilon_5) - b_1(k_1^* + \varepsilon_1)) < 0$. Therefore, there exists $\eta_2^* > 0$ such that $d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + \overline{\phi}(t)(r_1 - a_1\underline{\varphi}(t) - b_1\overline{\phi}(t)) \le 0$ for all $\eta \in (0, \eta_2^*)$.

If $t \le \tilde{t}_6 < \tilde{t}_2$, then $\overline{\varphi}(t) = k_2^* e^{\eta_2 t}$, and $\underline{\psi}(t) = 0$. It follows that

$$\begin{aligned} &d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}\bigl(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)\bigr) \\ &\quad\le \bigl(d_2\eta_2^2 - c\eta_2 + r_2\bigr)k_2^* e^{\eta_2 t} + k_2^* e^{\eta_2 t}\bigl(-b_2 k_2^* e^{\eta_2 t} + a_3 M_1\bigr) \\ &\quad\le \bigl(d_2\eta_2^2 - c\eta_2 + r_2 + a_3 M_1\bigr)k_2^* e^{\eta_2 t} = 0. \end{aligned} \tag{3.17}$$

If $t > \tilde{t}_6$ and $t \le \tilde{t}_2$, then $\overline{\varphi}(t) = k_2^* e^{\eta_2 t}$ and $\underline{\psi} = k_3^* - \varepsilon_6 e^{-\eta t}$. By calculating, we have

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}\bigl(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)\bigr) \le I_3(\eta), \tag{3.18}$$

where $I_3(\eta) = k_2^* e^{\eta_2 t}(a_2(\varepsilon_6 e^{-\eta t} - k_3^*) - b_2 k_2^* e^{\eta_2 t})$. Thus, $I_3(0) = k_2^* e^{\eta_2 t}(a_2(\varepsilon_6 - k_3^*) - b_2 k_2^* e^{\eta_2 t}) < 0$ when $t > \tilde{t}_6$. Hence, there exists $\eta_3^* > 0$ such that $d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)) \le 0$ for all $\eta \in (0, \eta_3^*)$.

If $t > \tilde{t}_2$, then $\overline{\varphi}(t) = k_2^* + \varepsilon_2 e^{-\eta t}$ and $\underline{\psi}(t) = k_3^* - \varepsilon_6 e^{-\eta t}$. By calculating, we have

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}\bigl(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)\bigr) \le I_4(\eta), \tag{3.19}$$

where $I_4(\eta) = (d_2\varepsilon_2\eta^2 + c\varepsilon_2\eta)e^{-\eta t} + (k_2^* + \varepsilon_2 e^{-\eta t})(r_2 + a_3 M_1 - b_2(k_2^* + \varepsilon_2 e^{-\eta t}) - a_2(k_3^* - \varepsilon_6 e^{-\eta t}))$. It follows from (3.12) that $I_4(0) = (k_2^* + \varepsilon_2)(r_2 + a_3 M_1 - b_2(k_2^* + \varepsilon_2) - a_2(k_3^* - \varepsilon_6)) < 0$. Hence, there exists $\eta_4^* > 0$ such that $d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)) \le 0$ for all $\eta \in (0, \eta_4^*)$.

If $t \le \tilde{t}_3$, then $\overline{\psi}(t) = k_3^* e^{\eta_3 t}$. By calculating, we have

$$\begin{aligned} &d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c3}\bigl(\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t)\bigr) \\ &\quad\le \bigl(d_3\eta_3^2 - c\eta_3\bigr)k_3^* e^{\eta_3 t} + k_3^* e^{\eta_3 t}\bigl(-\alpha - b_3 k_3^* e^{\eta_3 t} + a_4 M_2\bigr) \\ &\quad\le \bigl(d_3\eta_3^2 - c\eta_3 - \alpha + a_4 M_2\bigr)k_3^* e^{\eta_3 t} = 0. \end{aligned} \tag{3.20}$$

If $t > \tilde{t}_3$, then $\overline{\psi}(t) = k_3^* + \varepsilon_3 e^{-\eta_3 t}$. By calculating, we get

$$d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c3}\bigl(\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t)\bigr) \le I_5(\eta), \tag{3.21}$$

where $I_5(\eta) = (d_3\varepsilon_3\eta^2 + c\varepsilon_3\eta)e^{-\eta t} + (k_3^* + \varepsilon_3 e^{-\eta t})(-\alpha - b_3(k_3^* + \varepsilon_2 e^{-\eta t}) + a_4 M_2)$. It follows from (3.12) that $I_5(0) = (k_3^* + \varepsilon_3)(-\alpha - b_3(k_3^* + \varepsilon_3) + a_4 M_2) < 0$. Hence, there exists $\eta_5^* > 0$ such that $d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c3}(\overline{\phi}(t), \overline{\varphi}(t), \underline{\psi}(t)) \le 0$ for all $\eta \in (0, \eta_5^*)$.

Finally, for any $\eta \in (0, \min\{\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*, \eta_5^*\})$, we see that (3.3) holds. This completes the proof. □

Lemma 3.3 *Assume that* $r_1 > \max\{a_1 k_2^*, a_1 k_1^*\}$ *and* $\alpha > \frac{b_2 b_3 k_2^*}{a_2}$, *then* $(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ *is a pair of lower solution of system* (3.1).

Proof If $t \le \tilde{t}_4$, then $\underline{\phi}(t) = 0$. We have $d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)(r_1 - a_1\overline{\varphi}(t) - b_1\underline{\phi}(t)) = 0$.

If $\tilde{t}_4 < t \le \tilde{t}_2$, then $\underline{\phi}(t) = k_1^* - \varepsilon_4 e^{-\eta t}$, $\overline{\varphi} = k_2^* e^{\eta_2 t}$. We have

$$\begin{aligned} &d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)\bigl(r_1 - a_1\overline{\varphi}(t) - b_1\underline{\phi}(t)\bigr) \\ &\quad = (-d_1\eta - c)\varepsilon_4\eta e^{-\eta t} + \bigl(k_1^* - \varepsilon_4 e^{-\eta t}\bigr)\bigl(r_1 - a_1 k_2^* e^{\eta_2 t} - b_1\bigl(k_1^* - \varepsilon_4 e^{-\eta t}\bigr)\bigr) \\ &\quad \triangleq I_6(\eta). \end{aligned} \tag{3.22}$$

Using (3.13), we see that $I_6(0) = (k_1^* - \varepsilon_4)(r_1 - a_1 k_2^* e^{\eta_2 t} - b_1(k_1^* - \varepsilon_4)) > 0$ when $\tilde{t}_4 < t \le \tilde{t}_2$. Therefore, there exists η_6^* such that $d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)(r_1 - a_1\underline{\varphi}(t) - b_1\underline{\phi}(t)) \ge 0$ for $\eta \in (0, \eta_6^*)$.

If $t > \tilde{t}_2$, then $\underline{\phi}(t) = k_1^* - \varepsilon_4 e^{-\eta t}$, $\overline{\varphi} = k_2^* + \varepsilon_2 e^{-\eta t}$. We have

$$\begin{aligned} &d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)\bigl(r_1 - a_1\overline{\varphi}(t) - b_1\underline{\phi}(t)\bigr) \\ &\quad \ge (-d_1\eta - c)\varepsilon_4\eta e^{-\eta t} + \bigl(k_1^* - \varepsilon_4 e^{-\eta t}\bigr)\bigl(r_1 - a_1\bigl(k_2^* + \varepsilon_2 e^{-\eta t}\bigr) - b_1\bigl(k_1^* - \varepsilon_4 e^{-\eta t}\bigr)\bigr) \\ &\quad \triangleq I_7(\eta). \end{aligned} \tag{3.23}$$

Using (3.12), we see that $I_7(0) = (k_1^* - \varepsilon_4)(r_1 - a_1(k_2^* + \varepsilon_2) - b_1(k_1^* - \varepsilon_4)) > 0$. Therefore, there exists η_7^* such that $d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)(r_1 - a_1\underline{\varphi}(t) - b_1\underline{\phi}(t)) \ge 0$ for $\eta \in (0, \eta_7^*)$.

If $t \le \tilde{t}_5$, then $\underline{\varphi} = 0$. Therefore we have $d_2\underline{\varphi}''(t) - c\underline{\varphi}' + f_{c2}(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)) = 0$.

If $\tilde{t}_5 < t \le \tilde{t}_3$, then $\underline{\varphi}(t) = k_2^* - \varepsilon_5 e^{-\eta t}$, $\overline{\psi} = k_3^* e^{\eta_3 t}$. We get

$$\begin{aligned} &d_2\underline{\varphi}''(t) - c\underline{\varphi}' + f_{c2}\bigl(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)\bigr) \\ &\quad = (-d_2\eta - c)\varepsilon_5\eta e^{-\eta t} + \bigl(k_2^* - \varepsilon_5 e^{-\eta t}\bigr)\bigl(r_2 - b_2\bigl(k_2^* - \varepsilon_5 e^{-\eta t}\bigr) - a_2 k_3^* e^{\eta_3 t}\bigr) \\ &\quad \triangleq I_8(\eta). \end{aligned} \tag{3.24}$$

Using (3.13), we see that $I_8(0) = (k_2^* - \varepsilon_5)(r_2 - b_2(k_2^* - \varepsilon_5) - a_2 k_3^* e^{\eta_3 t}) > 0$ when $t > \tilde{t}_5$. Hence, there exists η_8^* such that $d_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_{c2}(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)) \ge 0$ for all $\eta \in (0, \eta_8^*)$.

If $t > \tilde{t}_3$, then $\underline{\varphi}(t) = k_2^* - \varepsilon_5 e^{-\eta t}$. We get

$$\begin{aligned} &d_2\underline{\varphi}''(t) - c\underline{\varphi}' + f_{c2}\bigl(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)\bigr) \\ &\quad \ge (-d_2\eta - c)\varepsilon_5\eta e^{-\eta t} + \bigl(k_2^* - \varepsilon_5 e^{-\eta t}\bigr)\bigl(r_2 - b_2\bigl(k_2^* - \varepsilon_5 e^{-\eta t}\bigr) - a_2\bigl(k_3^* + \varepsilon_3 e^{-\eta t}\bigr)\bigr) \\ &\quad \triangleq I_9(\eta). \end{aligned} \tag{3.25}$$

Using (3.12), we see that $I_9(0) = (k_2^* - \varepsilon_5)(r_2 - b_2(k_2^* - \varepsilon_5) - a_2(k_3^* + \varepsilon_3)) > 0$. Hence, there exists η_9^* such that $d_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_{c2}(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)) \geq 0$ for all $\eta \in (0, \eta_9^*)$.

If $t \leq \tilde{t}_6$, then $\underline{\psi} = 0$. Therefore, we have $d_3\underline{\psi}''(t) - c\underline{\psi}' + f_{c3}(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) = 0$.

If $t > \tilde{t}_6$, then $\underline{\psi}(t) = k_3^* - \varepsilon_6 e^{-\eta t}$. Hence, we get

$$\begin{aligned} & d_3\underline{\psi}''(t) - c\underline{\psi}' + f_{c3}\big(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)\big) \\ & \quad \geq (-d_3\eta - c)\varepsilon_6\eta e^{-\eta t} + \big(k_3^* - \varepsilon_6 e^{-\eta t}\big)\big(-\alpha - b_3\big(k_3^* - \varepsilon_6 e^{-\eta t}\big)\big) \\ & \quad \triangleq I_{10}(\eta). \end{aligned} \tag{3.26}$$

Using (3.12), we see that $I_{10}(0) = (k_3^* - \varepsilon_6)(-\alpha - b_3(k_3^* - \varepsilon_6)) > 0$. Hence, there exists η_{10}^* such that $d_3\underline{\psi}''(t) - c\underline{\psi}'(t) + f_{c3}(\underline{\phi}(t), \underline{\varphi}(t), \overline{\psi}(t)) \geq 0$ for all $\eta \in (0, \eta_{10}^*)$.

Finally, for any $\eta \in (0, \min\{\eta_6^*, \eta_7^*, \eta_8^*, \eta_9^*, \eta_{10}^*\})$, we see that (3.4) holds. This completes the proof. □

By using Lemmas 3.1-3.3, we have the following conclusion.

Theorem 3.1 *Assume that* $r_1 > \max\{a_1k_2^*, a_1k_1^*\}$ *and* $\alpha > \frac{b_2b_3k_2^*}{a_2}$, *then, for any* $c > c^* > 0$, *system* (1.1) *always has a traveling wave solution with speed c connecting the trivial steady state* $(0, 0)$ *and the positive steady state* (k_1^*, k_2^*, k_3^*).

4 Results and discussion

In this article, we investigate a food chain reaction-diffusion predator-prey systems with nonlocal delay in a bounded domain with no flux boundary condition incorporating delay respecting gestation of the predators. By using the methods of the Lyapunov function, we prove global stability of positive equilibrium of the system (1.1). If the nonlocal delay terms are replaced by the general terms without delay, one can easily verify that the positive constant solution of (1.1) is globally stable without any condition when a positive constant equilibrium exists.

The above result shows that if the intra-specific competitions of the predators and preys dominate their inter-specific interaction, then the unique positive equilibrium is globally stable if it exists, which implies that predators and preys are permanent from biologic view. If the nonlocal delay terms are replaced by a general term without delay, without any condition the positive equilibrium is globally stable when it exists. This implies that nonlocal delays often impact the global stability of a positive constant solution. By using the method of the upper-lower solutions, we also see that there exists a traveling wavefront connecting the zero solution to the positive equilibrium of the system when the wave speed is large enough and the prey intrinsic growth rate and the death rate of predator are relatively big.

In modern mathematics, the theory and methods of traveling waves solutions develop quickly and they have attracted much attention due to their significance in the real word. We would like to extend this theory and these methods to the integral equation, and we refer to [22–26], but this question is still an open problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CL carried out collecting data, participated in computation and drafted the manuscript. GH helped to draft the manuscript. All authors read and approved the final manuscript.

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