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On approximating the error function

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Abstract

In the article, we present the necessary and sufficient condition for the parameter p on the interval $(7/5,\infty)$ such that the function $x \to \mathrm{erf}(x)/B_p(x)$ is strictly increasing (decreasing) on $(0,\infty)$, and find the best possible parameters p, q on the interval $(7/5,\infty)$ such that the double inequality $B_p(x) < \mathrm{erf}(x) < B_q(x)$ holds for all $x > 0$, where $\mathrm{erf}(x) = 2\int_0^x e^{-t^2}\,dt/\sqrt{\pi}$ is the error function, $B_p(x) = \sqrt{1-\lambda(p)e^{-px^2}-[1-\lambda(p)]e^{-\mu(p)x^2}}$, $\lambda(p) = 16(5p-7)/[(15p^2-40p+28)(45p^2-60p-4)]$ and $\mu(p) = 4(5p-7)/[5(3p-4)]$.

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1 Introduction

It is well known that the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+\frac{1}{2})}x^{2n+1}$$

has numerous applications in probability, statistics, and partial differential equations theory. Recently, the bounds for the error function have attracted the attention of many researchers. In particular, many remarkable inequalities for the error function can be found in the literature [1–13].

Pólya [14] proved that the inequality

$$\mathrm{erf}(x) < \sqrt{1-e^{-4x^2/\pi}}$$

holds for all $x > 0$.

In [15], Chu proved that the double inequality

$$\sqrt{1-e^{-px^2}} < \mathrm{erf}(x) < \sqrt{1-e^{-qx^2}} \tag{1.1}$$

holds for all $x > 0$ if and only if $p \in (0,1]$ and $q \in [4/\pi,\infty)$.

Alzer [16] presented the double inequality

$$\left(1-e^{-\beta(p)x^p}\right)^{1/p} < \frac{1}{\Gamma(1+\frac{1}{p})}\int_0^x e^{-t^p}\,dt < \left(1-e^{-\alpha(p)x^p}\right)^{1/p}$$

for $x > 0$ and $p > 0$ with $p \neq 1$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is the classical gamma function, and $\alpha(p)$ and $\beta(p)$ are, respectively, given by

$$\alpha(p) = \frac{1}{\Gamma^p(1+\frac{1}{p})} \quad (p > 1), \qquad \alpha(p) = 1 \quad (0 < p < 1),$$

and

$$\beta(p) = \frac{1}{\Gamma^p(1+\frac{1}{p})} \quad (0 < p < 1), \qquad \beta(p) = 1 \quad (p > 1).$$

Let $n \geq 2$, and α_n, β_n, α_n^*, β_n^* be, respectively, defined by

$$\alpha_2 = 0.90686\ldots, \qquad \alpha_n = 1 \quad (n \geq 3), \qquad \beta_n = n-1,$$
$$\alpha_n^* = n+1 \quad (n = 2k), \qquad \alpha_n^* = n-1 \quad (n = 2k-1), \qquad \beta_n^* = 1.$$

In [17, 18], Alzer proved that the double inequalities

$$\lambda_n \operatorname{erf}\left(\sum_{i=1}^n x_i\right) \leq \sum_{i=1}^n \operatorname{erf}(x_i) - \prod_{i=1}^n \operatorname{erf}(x_i) \leq \mu_n \operatorname{erf}\left(\sum_{i=1}^n x_i\right), \tag{1.2}$$
$$\lambda \operatorname{erf}\bigl(y + \operatorname{erf}(x)\bigr) < \operatorname{erf}\bigl(x + \operatorname{erf}(y)\bigr) < \mu \operatorname{erf}\bigl(y + \operatorname{erf}(x)\bigr),$$
$$\lambda^* \operatorname{erf}\bigl(y \operatorname{erf}(x)\bigr) < \operatorname{erf}\bigl(x \operatorname{erf}(y)\bigr) \leq \mu^* \operatorname{erf}\bigl(y \operatorname{erf}(x)\bigr),$$

hold for all $x_i \geq 0$ and $y \geq x > 0$ if and only if $\lambda_n \leq \alpha_n$, $\mu_n \geq \beta_n$, $\lambda \leq \operatorname{erf}(1) = 0.8427\ldots$, $\mu \geq 2/\sqrt{\pi} = 1.1283\ldots$, $\lambda^* \leq 0$ and $\mu^* \geq 1$, and inequality (1.2) holds for all $x_i \leq 0$ if and only if $\lambda_n \geq \alpha_n^*$ and $\mu_n \leq \beta_n^*$.

Recently, Neuman [19] proved that the double inequality

$$\frac{2x}{\sqrt{\pi}} e^{-\frac{x^2}{3}} \leq \operatorname{erf}(x) \leq \frac{2x}{\sqrt{\pi}} \frac{e^{-x^2}+2}{3} \tag{1.3}$$

holds for all $x > 0$.

Let $x \in (0,\infty)$, $p \in (7/5,\infty)$, $\lambda(p)$, $\mu(p)$, and $B_p(x)$ be, respectively, defined by

$$\lambda(p) = \frac{16(5p-7)}{(15p^2-40p+28)(45p^2-60p-4)}, \qquad \mu(p) = \frac{4(5p-7)}{5(3p-4)}, \tag{1.4}$$
$$B_p(x) = \sqrt{1-\lambda(p)e^{-px^2} - \bigl[1-\lambda(p)\bigr]e^{-\mu(p)x^2}}. \tag{1.5}$$

The main purpose of this paper is to present the best possible parameters p and q on the interval $(7/5,\infty)$ such that the double inequality

$$B_p(x) < \operatorname{erf}(x) < B_q(x)$$

holds for all $x > 0$.

2 Lemmas

In order to prove our main results, we need to introduce an auxiliary function at first.

Let $-\infty \le a < b \le \infty$, f and g be differentiable on (a,b), and $g' \neq 0$ on (a,b). Then the function $H_{f,g}$ [20, 21] is defined by

$$H_{f,g} \equiv \frac{f'}{g'} g - f. \tag{2.1}$$

It is not difficult to verify that the auxiliary function $H_{f,g}$ has the following properties:

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2} H_{f,g} \tag{2.2}$$

if $g \neq 0$ on (a,b), and

$$H'_{f,g} = \left(\frac{f'}{g'}\right)' g \tag{2.3}$$

if both f and g are twice differentiable on (a,b).

Lemma 2.1 ([20], Theorem 8) *Let $-\infty \le a < b \le \infty$, f and g be differentiable on (a,b) with $f(a^+) = g(a^+) = 0$, $g'(x) \neq 0$ and $g'(x)H_{f,g}(b^-) < (>) \, 0$ for all $x \in (a,b)$. If there exists $\lambda_0 \in (a,b)$ such that f'/g' is strictly increasing (decreasing) on (a,λ_0) and strictly decreasing (increasing) on (λ_0,b), then there exists $\mu_0 \in (a,b)$ such that f/g is strictly increasing (decreasing) on (a,μ_0) and strictly decreasing (increasing) on (μ_0,b).*

Lemma 2.2 *Let $p \in (7/5,\infty)$, $p_0^* = (50 + 2\sqrt{30})/35 = 1.74155\ldots$, $\lambda(p)$ and $\mu(p)$ be defined by* (1.4), *and u_n be defined by*

$$u_n = (5p-6)(5p-8)n - \left(15p^2 - 40p + 28\right). \tag{2.4}$$

Then the following statements are true:

1. $p > \mu(p)$, $0 < \lambda(p) \le (8+\sqrt{14})/16 = 0.73385\ldots$ *and* $0 < \mu(p) < 4/3$ *for* $p \in (7/5,\infty)$, $0 < \mu(p) \le 1$ *for* $p \in (7/5,8/5]$ *and* $1 < \mu(p) < 4/3$ *for* $p \in (8/5,\infty)$;
2. $u_n < 0$ *for all* $n \ge 2$ *if* $p \in (7/5,8/5]$;
3. $u_2 \ge 0$ *and* $u_n > 0$ *for all* $n \ge 3$ *if* $p \in [p^*,\infty)$;
4. *there exists* $n_0 \ge 2$ *such that* $u_{n_0+1} \ge 0$, $u_n < 0$ *for* $2 \le n \le n_0$ *and* $u_n > 0$ *for* $n \ge n_0 + 2$ *and* $u_n > 0$ *for* $n > n_0$ *if* $p \in (8/5,p_0^*)$.

Proof For part (1), from (1.4) we clearly see that

$$p > \mu(p),$$

$$\lambda\left(\frac{7}{5}\right) = \lambda(\infty) = 0, \qquad \lambda\left(\frac{14+2\sqrt{14}}{15}\right) = \frac{8+\sqrt{14}}{16} = 0.73385\ldots, \tag{2.5}$$

$$\mu\left(\frac{7}{5}\right) = 0, \qquad \mu(\infty) = \frac{4}{3}, \qquad \mu\left(\frac{8}{5}\right) = 1, \tag{2.6}$$

$$\lambda'(p) = -\frac{80(3p-4)^2}{9(15p^2-40p+28)^2(45p^2-60p-4)^2}$$

$$\times \left(p - \frac{14 - 2\sqrt{14}}{15}\right)\left(p - \frac{14 + 2\sqrt{14}}{15}\right), \tag{2.7}$$

$$\mu'(p) = \frac{4}{5(3p-4)^2} > 0 \tag{2.8}$$

for $p > 7/5$.

Equation (2.7) implies that $\lambda(p)$ is strictly increasing on $(7/5, (14+2\sqrt{14})/15]$ and strictly decreasing on $[(14+2\sqrt{14})/15, \infty)$. Therefore, $0 < \lambda(p) \le (8+\sqrt{14})/16$ for $p \in (7/5, \infty)$ as follows from (2.5) and the piecewise monotonicity of $\lambda(p)$ on the interval $(7/5, \infty)$, and the remaining desired results for $\mu(p)$ follow easily from (2.6) and (2.8).

For parts (2) and (3), let $x \ge 2$, $p_1(x)$ and $p_2(x)$ be defined by

$$p_1(x) = \frac{35x - 20 - \sqrt{5}\sqrt{5x^2+4x-4}}{5(5x-3)}, \qquad p_2(x) = \frac{35x - 20 + \sqrt{5}\sqrt{5x^2+4x-4}}{5(5x-3)}.$$

Then simple computations lead to

$$p_1(2) = \frac{50 - 2\sqrt{30}}{35} = 1.11558\ldots, \qquad p_1(\infty) = \frac{6}{5}, \tag{2.9}$$

$$p_2(2) = p_0^*, \qquad p_2(3) = \frac{85+\sqrt{265}}{60} = 1.68798\ldots, \qquad p_2(\infty) = \frac{8}{5}, \tag{2.10}$$

$$u_n = 5(5n-3)\big[p - p_1(n)\big]\big[p - p_2(n)\big], \tag{2.11}$$

$$p_1'(x) = \frac{25x - 14 - \sqrt{5}\sqrt{5x^2+4x-4}}{\sqrt{5}(5x-3)^2\sqrt{5x^2+4x-4}} > 0, \tag{2.12}$$

$$p_2'(x) = -\frac{25x - 14 + \sqrt{5}\sqrt{5x^2+4x-4}}{\sqrt{5}(5x-3)^2\sqrt{5x^2+4x-4}} < 0 \tag{2.13}$$

for $x \ge 2$.

It follows from (2.9)-(2.13) that

$$u_2 = 35\big[p - p_0^*\big]\left[p - \frac{50 - 2\sqrt{30}}{35}\right], \tag{2.14}$$

$$\frac{50 - 2\sqrt{30}}{35} \le p_1(n) < \frac{6}{5}, \qquad \frac{8}{5} < p_2(n) \le p_0^* \tag{2.15}$$

for $n \ge 2$ and

$$\frac{8}{5} < p_2(n) \le \frac{85+\sqrt{265}}{60} \tag{2.16}$$

for $n \ge 3$

Therefore, parts (2) and (3) follow easily from (2.11) and (2.14)-(2.16).

For part (4), if $p \in (8/5, p_0^*)$, then from (2.4) and (2.14) we clearly see that the sequence $\{u_n\}_{n=2}^{\infty}$ is strictly increasing and

$$u_2 < 0, \qquad u_\infty = \infty. \tag{2.17}$$

Therefore, part (4) follows from (2.17) and the monotonicity of the sequence $\{u_n\}_{n=2}^{\infty}$. □

Lemma 2.3 *Let $x \in (0,\infty)$, $p \in (7/5,\infty)$, $\lambda(p)$, $\mu(p)$ and $B_p(x)$ be defined by* (1.4) *and* (1.5), *and $C_p(x)$ and $\alpha(p)$ be defined by*

$$C_p(x) = \frac{p\lambda(p)e^{-px^2} + \mu(p)(1-\lambda(p))e^{-\mu(p)x^2}}{p\lambda(p) + \mu(p)(1-\lambda(p))}$$

and

$$\alpha(p) = \sqrt{\frac{4}{\pi[p\lambda(p) + \mu(p)(1-\lambda(p))]}} = \sqrt{\frac{45p^2-60p-4}{3\pi p(5p-7)}}. \tag{2.18}$$

Then the following statements are true:

(1) *the function $p \to B_p(x)$ is strictly increasing on* $(7/5,\infty)$;
(2) *the function $p \to C_p(x)$ is strictly decreasing on* $(7/5,\infty)$;
(3) *the function $p \to \alpha(p)B_p(x)$ is strictly decreasing on* $(7/5,\infty)$.

Proof For part (1), it suffices to show that $\partial B_p^2(x)/\partial p > 0$ for $x \in (0,\infty)$ and $p \in (7/5,\infty)$. Let $t = (p-\mu(p))x^2$ and

$$F_1(t) = -\big(p-\mu(p)\big)\lambda'(p) + \lambda(p)t + \big(p-\mu(p)\big)\lambda'(p)e^t + \mu'(p)\big(1-\lambda(p)\big)te^t. \tag{2.19}$$

Then it follows from (1.4), (1.5), (2.19), and Lemma 2.2(1) that

$$\frac{\partial B_p^2(x)}{\partial p} = \frac{e^{-px^2}}{p-\mu(p)}F_1(t), \tag{2.20}$$

$$p-\mu(p) > 0, \quad t > 0, \tag{2.21}$$

$$F_1(0) = 0, \tag{2.22}$$

$$F_1'(t) = \lambda(p) + \big(p-\mu(p)\big)\lambda'(p)e^t + \mu'(p)\big(1-\lambda(p)\big)e^t + \mu'(p)\big(1-\lambda(p)\big)te^t,$$

$$F_1'(0) = \frac{12(15p^2-40p+28)}{(45p^2-60p-4)^2} > 0, \tag{2.23}$$

$$F_1''(t) = \frac{40(3p-2)(3p-4)}{(45p^2-60p-4)^2}e^t + \frac{20p(3p-4)}{(15p^2-40p+28)(45p^2-60p-4)}te^t > 0 \tag{2.24}$$

for $p \in (7/5,\infty)$ and $t > 0$.

From (2.22)-(2.24) we clearly see that

$$F_1(t) > 0 \tag{2.25}$$

for $p \in (7/5,\infty)$ and $t > 0$.

Therefore, part (1) follows from (2.20), (2.21), and (2.25).

For part (2), it is enough to prove that $\partial C_p(x)/\partial p < 0$ for $x \in (0,\infty)$ and $p \in (7/5,\infty)$. Let $t = (p-\mu(p))x^2$ and

$$F_2(t) = -\frac{15p^2-40p+28}{p-\mu(p)}t - 10(3p-4) + \left[-\frac{15p^2-40p+28}{p-\mu(p)}t + 10(3p-4)\right]e^t.$$

Then elaborated computations lead to

$$\begin{aligned}
C_p(x) &= \frac{1}{3}\left[\frac{4e^{-px^2}}{15p^2-40p+28} + \frac{5(3p-4)^2e^{-\mu(p)x^2}}{15p^2-40p+28}\right], \\
\frac{\partial C_p(x)}{\partial p} &= -\frac{1}{3}\left[\frac{4x^2}{15p^2-40p+28} + \frac{40(3p-4)}{(15p^2-40p+28)^2}\right]e^{-px^2} \\
&\quad + \frac{1}{3}\left[-\frac{4x^2}{15p^2-40p+28} + \frac{40(3p-4)}{(15p^2-40p+28)^2}\right]e^{-\mu(p)x^2} \\
&= \frac{4e^{-px^2}}{3(15p^2-40p+28)^2}F_2(t), \qquad (2.26)\\
F_2(t) &= -5(3p-4)t - 10(3p-4) + \bigl[-5(3p-4)t + 10(3p-4)\bigr]e^t \\
&= -5(3p-4)\bigl[(t+2)+(t-2)e^t\bigr] \\
&= -5(3p-4)\sum_{n=2}^{\infty}\frac{(n-2)}{n!}t^n < 0 \qquad (2.27)
\end{aligned}$$

for $p \in (7/5, \infty)$ and $t > 0$.

Therefore, part (2) follows from (2.21), (2.26), and (2.27).

For part (3), let $G_p(x)$ be defined by

$$G_p(x) = \frac{\pi}{4}\alpha^2(p)B_p^2(x) = \frac{1-\lambda(p)e^{-px^2}-(1-\lambda(p))e^{-\mu(p)x^2}}{p\lambda(p)+\mu(p)(1-\lambda(p))}. \qquad (2.28)$$

Then elaborated computations lead to

$$\frac{\partial G_p(x)}{\partial x} = \frac{2x[p\lambda(p)e^{-px^2}+\mu(p)(1-\lambda(p))e^{-\mu(p)x^2}]}{p\lambda(p)+\mu(p)(1-\lambda(p))} = 2xC_p(x). \qquad (2.29)$$

It follows from Lemma 2.3(2) and (2.29) that

$$\frac{\partial}{\partial x}\left(\frac{\partial G_p(x)}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{\partial G_p(x)}{\partial x}\right) = 2x\frac{\partial C_p(x)}{\partial p} < 0 \qquad (2.30)$$

for $x > 0$ and $p \in (7/5, \infty)$.

Inequality (2.30) implies that the function $x \to \partial G_p(x)/\partial p$ is strictly decreasing on $(0, \infty)$ and

$$\begin{aligned}
\frac{\partial G_p(x)}{\partial p} &< \left.\frac{\partial G_p(x)}{\partial p}\right|_{x=0} \\
&= \left[-\frac{1-\lambda(p)e^{-px^2}-(1-\lambda(p))e^{-\mu(p)x^2}}{(p\lambda(p)+\mu(p)(1-\lambda(p)))^2}\frac{d(p\lambda(p)+\mu(p)(1-\lambda(p)))}{dp}\right]_{x=0} \\
&\quad + \left[\frac{-\lambda'(p)e^{-px^2}+\lambda(p)x^2e^{-px^2}+\lambda'(p)e^{-\mu(p)x^2}+\mu'(p)x^2(1-\lambda(p))e^{-\mu(p)x^2}}{p\lambda(p)+\mu(p)(1-\lambda(p))}\right]_{x=0} \\
&= 0 \qquad (2.31)
\end{aligned}$$

for $x > 0$ and $p \in (7/5, \infty)$.

Therefore, part (3) follows from (2.28) and (2.31). □

Lemma 2.4 *Let $p \in (7/5, \infty)$, $x \in (0, \infty)$, $\lambda(p)$, $\mu(p)$, $B_p(x)$ and $H_{f,g}(x)$ be, respectively, defined by* (1.4), (1.5) *and* (2.1), *and $f_1(x)$, $g_1(x)$, $f_2(x)$ and $g_2(x)$ be, respectively, defined by*

$$f_1(x) = B_p^2(x) = 1 - \lambda(p)e^{-px^2} - \big(1 - \lambda(p)\big)e^{-\mu(p)x^2}, \qquad g_1(x) = \operatorname{erf}^2(x), \tag{2.32}$$

$$f_2(x) = \big[p\lambda(p)e^{(1-p)x^2} + \mu(p)\big(1 - \lambda(p)\big)e^{(1-\mu(p))x^2}\big]x, \qquad g_2(x) = \frac{2}{\sqrt{\pi}} \operatorname{erf}(x). \tag{2.33}$$

Then

$$H_{f_2,g_2}(\infty) = \lim_{x\to\infty}\left(\frac{f_2'(x)}{g_2'(x)}g_2(x) - f_2(x)\right) = \begin{cases} \infty, & p \in (\frac{7}{5}, \frac{8}{5}], \\ -\infty, & p \in (\frac{8}{5}, \infty), \end{cases} \tag{2.34}$$

$$H_{f_1,g_1}(\infty) = \lim_{x\to\infty}\left(\frac{f_1'(x)}{g_1'(x)}g_1(x) - f_1(x)\right) = \begin{cases} \infty, & p \in (\frac{7}{5}, \frac{8}{5}], \\ -1, & p \in (\frac{8}{5}, \infty). \end{cases} \tag{2.35}$$

Proof Let $t = (p - \mu(p))x^2$ and

$$\begin{aligned} h_1(t) &= p\lambda(p)\big(p - \mu(p)\big) - 2p\lambda(p)(p-1)t \\ &\quad + \mu(p)\big(p - \mu(p)\big)\big(1 - \lambda(p)\big)e^t - 2\mu(p)\big(\mu(p) - 1\big)\big(1 - \lambda(p)\big)te^t. \end{aligned} \tag{2.36}$$

Then (2.1) and (2.33) lead to

$$f_2(x) = \frac{\sqrt{t}}{\sqrt{p - \mu(p)}}\big(p\lambda(p)e^{\frac{1-p}{p-\mu(p)}t} + \mu(p)\big(1 - \lambda(p)\big)e^{\frac{1-\mu(p)}{p-\mu(p)}t}\big), \tag{2.37}$$

$$\frac{f_2'(x)}{g_2'(x)} = \frac{\pi}{4(p - \mu(p))}e^{\frac{2-p}{p-\mu(p)}t}h_1(t),$$

$$\begin{aligned} H_{f_2,g_2}(x) &= \frac{f_2'(x)}{g_2'(x)}g_2(x) - f_2(x) \\ &= \frac{\sqrt{\pi}}{2(p - \mu(p))}\operatorname{erf}(x)e^{\frac{2-p}{p-\mu(p)}t}h_1(t) \\ &\quad - \frac{\sqrt{t}}{\sqrt{p - \mu(p)}}\big(p\lambda(p)e^{\frac{1-p}{p-\mu(p)}t} + \mu(p)\big(1 - \lambda(p)\big)e^{\frac{1-\mu(p)}{p-\mu(p)}t}\big). \end{aligned} \tag{2.38}$$

If $p \in (8/5, \infty)$, then Lemma 2.2(1), (2.36), and (2.37) lead to

$$p > \mu(p), \qquad 0 < \lambda(p) < 1, \qquad 1 < \mu(p) < \frac{4}{3}, \tag{2.39}$$

$$f_2(\infty) = 0, \tag{2.40}$$

$$\begin{aligned} &\lim_{t\to\infty}\frac{h_1(t)}{te^t} = -2\mu(p)\big(\mu(p) - 1\big)\big(1 - \lambda(p)\big) < 0, \\ &\lim_{t\to\infty} e^{\frac{2-p}{p-\mu(p)}t}h_1(t) = \lim_{t\to\infty} te^{\frac{2-\mu(p)}{p-\mu(p)}t}\lim_{t\to\infty}\frac{h_1(t)}{te^t} = -\infty. \end{aligned} \tag{2.41}$$

Therefore, $H_{f_2,g_2}(\infty) = -\infty$ for $p \in (8/5, \infty)$ follows from (2.38), (2.39), and (2.41).

If $p \in (7/5, 8/5]$, then it follows from Lemma 2.2(1) and (2.36) together with (2.37) that

$$p > \mu(p), \qquad 0 < \lambda(p) < 1, \qquad 0 < \mu(p) \le 1, \tag{2.42}$$

$$\lim_{t\to\infty}\frac{f_2(x)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}}=\lim_{t\to\infty}\frac{p\lambda(p)e^{-t}+\mu(p)(1-\lambda(p))}{\sqrt{(p-\mu(p))t}}=0, \tag{2.43}$$

$$\begin{aligned}
&\lim_{t\to\infty}\frac{h_1(t)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}}\\
&=\lim_{t\to\infty}\left[\frac{p\lambda(p)(p-\mu(p))}{t}e^{\frac{\mu(p)-1}{p-\mu(p)}t}-2p\lambda(p)(p-1)e^{\frac{\mu(p)-1}{p-\mu(p)}t}\right]\\
&\quad+\lim_{t\to\infty}\left[\frac{\mu(p)(1-\lambda(p))(p-\mu(p))}{t}e^{\frac{p-1}{p-\mu(p)}t}+2\mu(p)\big(1-\lambda(p)\big)\big(1-\mu(p)\big)e^{\frac{p-1}{p-\mu(p)}t}\right]\\
&=\infty.
\end{aligned} \tag{2.44}$$

Therefore,

$$H_{f_2,g_2}(\infty)=\lim_{t\to\infty}te^{\frac{1-\mu(p)}{p-\mu(p)}t}\left[\frac{\sqrt{\pi}e^{\frac{2-p}{p-\mu(p)}t}}{2(p-\mu(p))}\operatorname{erf}(x)\frac{h_1(t)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}}-\frac{f_2(x)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}}\right]=\infty$$

for $p\in(7/5,8/5]$ as follows from (2.38) and (2.42)-(2.44).

Similarly, from (2.1) and (2.32) we have

$$\begin{aligned}
H_{f_1,g_1}(x)={}&\frac{\sqrt{\pi}}{2}x\operatorname{erf}(x)\big[p\lambda(p)e^{(1-p)x^2}+\mu(p)\big(1-\lambda(p)\big)e^{(1-\mu(p))x^2}\big]\\
&-\big[1-\lambda(p)e^{-px^2}-\big(1-\lambda(p)\big)e^{-\mu(p)x^2}\big].
\end{aligned} \tag{2.45}$$

If $p\in(7/5,8/5]$, then Lemma 2.2(1) gives

$$0<\lambda(p)<1,\qquad 0<\mu(p)\le 1. \tag{2.46}$$

Therefore, $H_{f_1,g_1}(\infty)=\infty$ for $p\in(7/5,8/5]$ as follows from (2.45) and (2.46).

If $p\in(8/5,\infty)$, then Lemma 2.2(1) leads to

$$0<\lambda(p)<1,\qquad 1<\mu(p)\le\frac{4}{3}. \tag{2.47}$$

Therefore, $H_{f_1,g_1}(\infty)=-1$ for $p\in(8/5,\infty)$ as follows from (2.45) and (2.47). □

Lemma 2.5 *Let* $p\in(7/5,\infty)$, $p_0^*=(50+2\sqrt{30})/35=1.74155\ldots$, $x\in(0,\infty)$, $\lambda(p)$, $\mu(p)$, $B_p(x)$, $H_{f,g}(x)$, $f_1(x)$, $g_1(x)$, $f_2(x)$ *and* $g_2(x)$ *be, respectively, defined by* (1.4), (1.5), (2.1), (2.32) *and* (2.33). *Then the following statements are true*:

(1) *if* $f_1(x)/g_1(x)$ *is strictly increasing on* $(0,\infty)$, *then* $p\in(7/5,8/5]$;

(2) *if* $f_1(x)/g_1(x)$ *is strictly decreasing on* $(0,\infty)$, *then* $p\in[p_0^*,\infty)$.

Proof (1) It follows from (2.1) and (2.32) that

$$\lim_{x\to\infty}e^{x^2}\left(\frac{f_1(x)}{g_1(x)}\right)'=\lim_{x\to\infty}e^{x^2}\frac{g_1'(x)}{g_1^2(x)}H_{f_1,g_1}(x)=\frac{4}{\sqrt{\pi}}\lim_{x\to\infty}H_{f_1,g_1}(x). \tag{2.48}$$

If $f_1(x)/g_1(x)$ is strictly increasing on $(0,\infty)$, then (2.48) leads to

$$\lim_{x\to\infty}H_{f_1,g_1}(x)\ge 0 \tag{2.49}$$

and we assert that $p \in (7/5, 8/5]$. Otherwise, $p > 8/5$ and (2.35) lead to the conclusion $H_{f_1,g_1}(\infty) = -1$, which contradicts with (2.49).

(2) Let $t = (p - \mu(p))x^2$, u_n and $h_1(t)$ be, respectively, defined by (2.4) and (2.36), and $h_2(t)$ and v_n be, respectively, defined by

$$\begin{aligned} h_2(t) &= 2\mu(p)\big(1-\lambda(p)\big)\big(\mu(p)-1\big)\big(\mu(p)-2\big)te^t - \mu(p)\big(1-\lambda(p)\big)\big(3\mu(p)-4\big) \\ &\quad \times \big(p-\mu(p)\big)e^t + 2p\lambda(p)(p-1)(p-2)t \\ &\quad - p\lambda(p)(3p-4)\big(p-\mu(p)\big), \end{aligned} \tag{2.50}$$

$$v_n = -\frac{4\mu(p)(1-\lambda(p))}{25(3p-4)^2}u_n. \tag{2.51}$$

Then from (2.1)-(2.3), (2.32), (2.33), (2.36), and (2.50) we have

$$\begin{aligned} \left(\frac{f_2'(x)}{g_2'(x)}\right)' &= \frac{\pi}{4(p-\mu(p))}\frac{d}{dt}\big[e^{\frac{2-p}{p-\mu(p)}t}h_1(t)\big]\frac{dt}{dx} \\ &= \frac{\pi x}{2(p-\mu(p))}e^{\frac{2-p}{p-\mu(p)}t}h_2(t), \end{aligned} \tag{2.52}$$

$$\begin{aligned} h_2(t) &= 2\mu(p)\big(1-\lambda(p)\big)\big(\mu(p)-1\big)\big(\mu(p)-2\big)\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!} \\ &\quad - \mu(p)\big(1-\lambda(p)\big)\big(3\mu(p)-4\big)\big(p-\mu(p)\big)\sum_{n=0}^{\infty}\frac{t^n}{n!} \\ &\quad + 2p\lambda(p)(p-1)(p-2)t - p\lambda(p)(3p-4)\big(p-\mu(p)\big) \\ &= -\big(p-\mu(p)\big)\big[\big(p-\mu(p)\big)\big(3p+3\mu(p)-4\big)\lambda(p) + \mu(p)\big(3\mu(p)-4\big)\big] \\ &\quad + \big(p-\mu(p)\big)\big(2p^2+5\mu^2(p)+2p\mu(p)-6p-10\mu(p)+4\big)\lambda(p) \\ &\quad + \mu(p)\big(4p-3p\mu(p)+5\mu^2(p)-10\mu(p)+4\big) + \sum_{n=2}^{\infty}\frac{v_nt^n}{n!} \\ &= \sum_{n=2}^{\infty}\frac{v_nt^n}{n!}, \end{aligned} \tag{2.53}$$

$$\left(\frac{f_1'(x)}{g_1'(x)}\right)' = \frac{g_1'(x)}{g_1^2(x)}\left[\frac{f_1'(x)}{g_1'(x)}g_1(x) - f_1(x)\right] = \frac{g_1'(x)}{g_1^2(x)}H_{f_1,g_1}(x),$$

$$H_{f_1,g_1}'(x) = \left(\frac{f_1'(x)}{g_1'(x)}\right)'g_1(x) = \left(\frac{f_2(x)}{g_2(x)}\right)'g_1(x) = \frac{g_2'(x)}{g_2^2(x)}H_{f_2,g_2}(x)g_1(x),$$

$$H_{f_2,g_2}'(x) = \left(\frac{f_2'(x)}{g_2'(x)}\right)'g_2(x) = \frac{\pi x}{2(p-\mu(p))}e^{\frac{2-p}{p-\mu(p)}t}h_2(t)g_2(x),$$

$$\frac{g_1'(x)}{g_1^2(x)} = \frac{4}{\sqrt{\pi}}\frac{e^{-x^2}}{\operatorname{erf}^3(x)} \sim \frac{\pi}{2x^3}, \qquad \frac{g_2'(x)}{g_2^2(x)}g_1(x) = e^{-x^2} \sim 1 \quad \big(x \to 0^+\big),$$

$$g_2(x) = \frac{2}{\sqrt{\pi}}\operatorname{erf}(x) \sim \frac{4x}{\pi}, \qquad h_2(t) \sim \frac{v_2}{2}t^2 = \frac{(p-\mu(p))^2}{2}v_2x^4 \quad \big(x \to 0^+\big),$$

$$\left(\frac{f_1(x)}{g_1(x)}\right)' \sim \frac{\pi}{2x^3}H_{f_1,g_1}(x) \quad \big(x \to 0^+\big),$$

$$
\begin{aligned}
&H'_{f_1,g_1}(x) \sim H_{f_2,g_2}(x), \\
&H'_{f_2,g_2}(x) \sim \bigl(p-\mu(p)\bigr)\nu_2 x^6 \quad \bigl(x \to 0^+\bigr).
\end{aligned}
$$

Note that $H_{f_1,g_1}(0^+) = H_{f_2,g_2}(0^+) = 0$. Making use of the L'Hôspital rule we get

$$
\begin{aligned}
&\lim_{x\to 0^+} x^{-5}\left(\frac{f_1(x)}{g_1(x)}\right)' \\
&\quad = \frac{\pi}{2}\lim_{x\to 0^+}\frac{H_{f_1,g_1(x)}(x)}{x^8} = \frac{\pi}{16}\lim_{x\to 0^+}\frac{H_{f_2,g_2(x)}(x)}{x^7} \\
&\quad = \frac{\pi}{112}\lim_{x\to 0^+}\frac{H'_{f_2,g_2(x)}(x)}{x^6} = \frac{\pi(p-\mu(p))}{112}\nu_2 \\
&\quad = -\frac{\pi\mu(p)(p-\mu(p))(1-\lambda(p))}{700}\bigl(35p^2-100p+68\bigr). \qquad (2.54)
\end{aligned}
$$

If $p \in (7/5,\infty)$ and $f_1(x)/g_1(x)$ is strictly decreasing on $(0,\infty)$, then it follows from Lemma 2.2(1) and (2.54) that

$$35p^2 - 100p + 68 \geq 0,$$

which leads to $p \geq (50+2\sqrt{30})/35 = p_0^*$. □

3 Main results

Theorem 3.1 *Let* $p \in (7/5,\infty)$, $x > 0$, $p_0^* = (50+2\sqrt{30})/35$, $\lambda(p)$, $\mu(p)$, $B_p(x)$ *and* $\alpha(p)$ *be, respectively, defined by* (1.4), (1.5), *and* (2.18), x_0 *be the unique solution of the equation*

$$\frac{d}{dx}\left(\frac{B_p^2(x)}{\operatorname{erf}^2(x)}\right) = 0$$

on the interval $(0,\infty)$ *and* $\beta(p) = \operatorname{erf}(x_0)/B_p(x_0)$. *Then the following statements are true*:

(1) *the function* $x \to Q_p(x) = \operatorname{erf}(x)/B_p(x)$ *is strictly decreasing on* $(0,\infty)$ *if and only if* $p \in (7/5, 8/5]$, *and the double inequality*

$$1 < \frac{\operatorname{erf}(x)}{B_p(x)} < \alpha(p) \qquad (3.1)$$

holds for all $x > 0$ *with the best possible parameters* 1 *and* $\alpha(p)$ *if* $p \in (7/5, 8/5]$;

(2) *the function* $x \to Q_p(x) = \operatorname{erf}(x)/B_p(x)$ *is strictly increasing on* $(0,\infty)$ *if and only if* $p \in [p_0^*,\infty)$, *and the double inequality*

$$\alpha(p) < \frac{\operatorname{erf}(x)}{B_p(x)} < 1 \qquad (3.2)$$

holds for all $x > 0$ *with the best possible parameters* 1 *and* $\alpha(p)$ *if* $p \in [p_0^*,\infty)$;

(3) *if* $p \in (8/5, p_0^*)$, *then* $Q_p(x)$ *is strictly decreasing on* $(0, x_0]$ *and strictly increasing on* $[x_0,\infty)$, *and the double inequality*

$$\beta(p) \leq \frac{\operatorname{erf}(x)}{B_p(x)} < \max\bigl\{1, \alpha(p)\bigr\} \qquad (3.3)$$

for all $x > 0$.

Proof Let $t = (p - \mu(p))x^2$, $f_1(x)$, $g_1(x)$, $f_2(x)$, $g_2(x)$, u_n, v_n, and $h_2(t)$ be defined by (2.32) and (2.33), (2.4), (2.51), and (2.53). Then

$$Q_p^{-2}(x) = \frac{f_1(x)}{g_1(x)},$$

$$f_1(0^+) = g_1(0^+) = 0, \qquad g_1'(x) > 0, \tag{3.4}$$

$$f_2(0^+) = g_2(0^+) = 0, \qquad g_2'(x) > 0, \tag{3.5}$$

$$\frac{f_1'(x)}{g_1'(x)} = \frac{f_2(x)}{g_2(x)}. \tag{3.6}$$

(1) If $Q_p(x) = \mathrm{erf}(x)/B_p(x)$ is strictly decreasing on $(0,\infty)$, then $f_1(x)/g_1(x)$ is strictly increasing on $(0,\infty)$ and $p \in (7/5, 8/5]$ by Lemma 2.5(1).

If $p \in (7/5, 8/5]$, then it follows from Lemma 2.2(1) and (2) together with (2.51)-(2.53) that the function $f_2'(x)/g_2'(x)$ is strictly increasing on $(0,\infty)$. Then from the monotone form of L'Hôpital's rule [22], Theorem 1.25, and (3.5) together with (3.6) we know that the function $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\infty)$. Therefore, $Q_p(x)$ is strictly decreasing or $f_1(x)/g_1(x)$ is strictly increasing on $(0,\infty)$ as follows from the monotone form of L'Hôpital's rule [22], Theorem 1.25, and (3.4) together with the monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$.

Note that

$$\lim_{x\to 0^+} \frac{f_1(x)}{g_1(x)} = \frac{\pi}{4}\bigl[p\lambda(p) + \mu(p)\bigl(1-\lambda(p)\bigr)\bigr] = \frac{1}{\alpha^2(p)}, \qquad \lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = 1. \tag{3.7}$$

Therefore, the double inequality (3.1) holds for all $x > 0$ and $p \in (7/5, 8/5]$ with the best possible parameters 1 and $\alpha(p)$ as follows from (3.7) and the monotonicity of $f_1(x)/g_1(x)$ on the interval $(0,\infty)$.

(2) If $Q_p(x) = \mathrm{erf}(x)/B_p(x)$ is strictly increasing on $(0,\infty)$, then $f_1(x)/g_1(x)$ is strictly decreasing on $(0,\infty)$ and $p \in [p_0^*, \infty)$ by Lemma 2.5(2).

If $p \in [p_0^*, \infty)$, then it follows from Lemma 2.2(1) and (3) together with (2.51)-(2.53) that the function $f_2'(x)/g_2'(x)$ is strictly decreasing on $(0,\infty)$. Therefore, $Q_p(x)$ is strictly increasing or $f_1(x)/g_1(x)$ is strictly decreasing on $(0,\infty)$ as follows from the monotone form of L'Hôpital's rule and (3.4)-(3.6) together with the monotonicity of the function $f_2'(x)/g_2'(x)$ on the interval $(0,\infty)$, and the double inequality (3.2) holds for all $x > 0$ and $p \in [p_0^*, \infty)$ with the best possible parameters 1 and $\alpha(p)$ as follows from (3.7) and the monotonicity of $f_1(x)/g_1(x)$ on the interval $(0,\infty)$.

(3) If $p \in (8/5, p_0^*)$, then it follows from [23], Lemma 6.4, or [24], Lemma 7, Lemma 2.2(1) and (4), (2.34), (2.35), and (2.51)-(2.53) that there exists $x_1 \in (0,\infty)$ such that $f_2'(x)/g_2'(x)$ is strictly increasing on $(0, x_1)$ and strictly decreasing on (x_1, ∞), and

$$H_{f_1,g_1}(\infty) = -1, \tag{3.8}$$

$$H_{f_2,g_2}(\infty) = -\infty. \tag{3.9}$$

From Lemma 2.1, (3.5), (3.6), (3.9) and the piecewise monotonicity of $f_2'(x)/g_2'(x)$ on the interval $(0,\infty)$ we known that there exists $x_2 \in (0,\infty)$ such that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0, x_2)$ and strictly decreasing on (x_2, ∞). Then (3.4), (3.8) and Lemma 2.1

lead to the conclusion that there exists $x_0 \in (0,\infty)$ such that the function $f_1(x)/g_1(x) = B_p^2(x)/\operatorname{erf}^2(x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on (x_0,∞). We clearly see that x_0 is the unique solution of the equation

$$\frac{d}{dx}\left(\frac{B_p^2(x)}{\operatorname{erf}^2(x)}\right) = 0$$

on the interval $(0,\infty)$. Therefore, $Q_p(x) = \operatorname{erf}(x)/B_p(x) = (f_1(x)/g_1(x))^{-1/2}$ is strictly decreasing on $(0,x_0]$ and strictly increasing on $[x_0,\infty)$, and inequality (3.3) holds for all $x > 0$ as follows easily from (3.7). □

Let $p \in (7/5,\infty)$ and $\alpha(p) = \sqrt{(45p^2 - 60p - 4)/[3p\pi(5p - 7)]} = 1$, then $p = p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)] = 1.71318\ldots \in (7/5, p_0^*)$. Numerical computations show that $x_0 = 1.68913\ldots$ is the unique solution of the equation

$$\frac{d}{dx}\left(\frac{B_{p_0}^2(x)}{\operatorname{erf}^2(x)}\right) = 0 \tag{3.10}$$

on the interval $(0,\infty)$, $\beta(p_0) = \operatorname{erf}(x_0)/B_{p_0}(x_0) = 0.9998\ldots$. Therefore, Theorem 3.1(3) leads to Corollary 3.1 immediately.

Corollary 3.1 *Let* $p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)]$, $B_p(x)$ *be defined by* (1.5) *and* $x_0 = 1.68913\ldots$ *be the unique solution of equation* (3.10) *on the interval* $(0,\infty)$. *Then the double inequality*

$$0.9998 < \frac{\operatorname{erf}(x_0)}{B_{p_0}(x_0)} \le \frac{\operatorname{erf}(x)}{B_{p_0}(x)} < 1 \tag{3.11}$$

holds for all $x > 0$.

Theorem 3.2 *Let* $p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)]$, $p \in (7/5,\infty)$, $x > 0$, $\lambda(p)$, $\mu(p)$ *and* $B_p(x)$ *be, respectively, defined by* (1.4) *and* (1.5). *Then the inequality*

$$\operatorname{erf}(x) > B_p(x) \tag{3.12}$$

holds for all $x > 0$ *if and only if* $p \in (7/5, 8/5]$, *and inequality* (3.12) *is reversed if and only if* $p \in [p_0,\infty)$.

Proof Making use of the L'Hôspital rule and Lemma 2.2(1) we have

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\operatorname{erf}^2(x) - B_p^2(x)}{e^{-\mu(p)x^2}} \\
&\quad = \lim_{x\to\infty}\left[-\frac{2\operatorname{erf}(x)}{\sqrt{\pi}\,\mu(p)}\frac{e^{(\mu(p)-1)x^2}}{x} + \frac{p\lambda(p)}{\mu(p)}e^{-(p-\mu(p))x^2} + 1 - \lambda(p)\right],
\end{aligned} \tag{3.13}$$

$$p > \mu(p), \qquad 0 < \lambda(p) \le \frac{8 + \sqrt{14}}{16} = 0.73385\ldots, \tag{3.14}$$

$$0 < \mu(p) \le 1 \quad (7/5 < p \le 8/5), \tag{3.15}$$

$$1 < \mu(p) < \frac{4}{3} \quad (8/5 < p < \infty). \tag{3.16}$$

It follows from (3.13)-(3.16) that

$$\lim_{x\to\infty} \frac{\operatorname{erf}^2(x) - B_p^2(x)}{e^{-\mu(p)x^2}} = \begin{cases} 1 - \lambda(p) > 0, & p \in (\frac{7}{5}, \frac{8}{5}], \\ -\infty, & p \in (\frac{8}{5}, \infty). \end{cases} \tag{3.17}$$

If inequality (3.12) holds for all $x > 0$, then

$$\lim_{x\to\infty} \frac{\operatorname{erf}^2(x) - B_p^2(x)}{e^{-\mu(p)x^2}} \geq 0, \tag{3.18}$$

and $p \in (7/5, 8/5]$ as follows easily from (3.17) and (3.18).

If $p \in (7/5, 8/5]$, then inequality (3.12) holds for all $x > 0$ as follows directly from Theorem 3.1(1).

If $\operatorname{erf}(x) < B_p(x)$ for all $x > 0$, then $p \geq p_0$ as follows easily from

$$\lim_{x\to 0^+} \frac{\operatorname{erf}(x)}{B_p(x)} = \alpha(p) = \sqrt{\frac{45p^2 - 60p - 4}{3\pi p(5p - 7)}} \leq 1.$$

If $p \in [p_0, \infty)$, then we divide the proof into two cases.

Case 1. $p \in [p_0^*, \infty)$. Then $\operatorname{erf}(x) < B_p(x)$ for all $x > 0$ as follows from Theorem 3.1(2).

Case 2. $p \in [p_0, p_0^*)$. Then

$$\alpha(p) = \sqrt{\frac{45p^2 - 60p - 4}{3\pi p(5p - 7)}} \leq \alpha(p_0) = 1, \tag{3.19}$$

and $\operatorname{erf}(x) < B_p(x)$ for all $x > 0$ as follows from Theorem 3.1(3) and (3.19). □

Remark 3.1 Let $p_0^* = (50 + 2\sqrt{30})/35$, and $f_2(x)$ and $g_2(x)$ be defined by (2.33). Then from (3.6) and the proof of Theorem 3.1 we know that the function $f_2(x)/g_2(x)$ is strictly increasing on $(0, \infty)$ if $p \in (7/5, 8/5]$ and strictly decreasing on $(0, \infty)$ if $p \in [p_0^*, \infty)$. Therefore, we have

$$\frac{\pi}{4}\left[p\lambda(p) + \mu(p)\bigl(1 - \lambda(p)\bigr)\right] = \lim_{x\to 0^+} \frac{f_2(x)}{g_2(x)} < \frac{f_2(x)}{g_2(x)} < \lim_{x\to\infty} \frac{f_2(x)}{g_2(x)} = \infty$$

for all $x \in (0, \infty)$ and $p \in (7/5, 8/5]$, and

$$0 = \lim_{x\to\infty} \frac{f_2(x)}{g_2(x)} < \frac{f_2(x)}{g_2(x)} < \lim_{x\to 0^+} \frac{f_2(x)}{g_2(x)} = \frac{\pi}{4}\left[p\lambda(p) + \mu(p)\bigl(1 - \lambda(p)\bigr)\right]$$

for all $x \in (0, \infty)$ and $p \in [p_0^*, \infty)$.

Remark 3.1 can be restated as Theorem 3.3.

Theorem 3.3 *Let* $p_0^* = (50 + 2\sqrt{30})/35$ *and* $C_p(x)$ *be defined by Lemma* 2.3. *Then the inequality*

$$\operatorname{erf}(x) < \frac{2xe^{x^2}}{\sqrt{\pi}} C_p(x) \tag{3.20}$$

holds for all $x > 0$ *if* $p \in (7/5, 8/5]$, *and inequality* (3.20) *is reversed for all* $x > 0$ *if* $p \in [p_0^*, \infty)$.

Remark 3.2 Let $p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)]$, $p, q \in (7/5, \infty)$, $x > 0$, $\lambda(p)$, $\mu(p)$ and $B_p(x)$ be, respectively, defined by (1.4) and (1.5). Then it follows from Lemma 2.3(1) and Theorem 3.2 that the double inequality

$$\begin{aligned}&\sqrt{1 - \lambda(p)e^{-px^2} - \left[1 - \lambda(p)\right]e^{-\mu(p)x^2}}\\&\quad = B_p(x) < \operatorname{erf}(x) < B_q(x) = \sqrt{1 - \lambda(q)e^{-qx^2} - \left[1 - \lambda(q)\right]e^{-\mu(q)x^2}}\end{aligned}$$

holds for all $x > 0$ with the best possible parameters $p = 8/5$ and $q = p_0$.

Corollary 3.2 *Let* $p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)]$, *and* $\lambda(p)$ *and* $\mu(p)$ *be defined by* (1.4). *Then the inequalities*

$$\begin{aligned}&\sqrt{1 - e^{-x^2}} < \sqrt{1 - \frac{25}{57}e^{-8x^2/5} - \frac{32}{57}e^{-x^2}} < \operatorname{erf}(x)\\&\quad < \sqrt{1 - \lambda(p_0)e^{-p_0x^2} - \left[1 - \lambda(p_0)\right]e^{-\mu(p_0)x^2}} < \sqrt{1 - e^{4x^2/\pi}}\end{aligned}$$

hold for all $x > 0$.

Proof From (1.4) one has

$$\lambda\left(\frac{8}{5}\right) = \frac{25}{57}, \qquad \mu\left(\frac{8}{5}\right) = 1. \tag{3.21}$$

Note that p_0 satisfies the identity

$$\frac{45p_0^2 - 60p_0 - 4}{p_0(5p_0 - 7)} = 3\pi. \tag{3.22}$$

It follows from Remark 3.2 and (3.21) that

$$\sqrt{1 - \frac{25}{57}e^{-8x^2/5} - \frac{32}{57}e^{-x^2}} < \operatorname{erf}(x) < \sqrt{1 - \lambda(p_0)e^{-p_0x^2} - \left[1 - \lambda(p_0)\right]e^{-\mu(p_0)x^2}}$$

for all $x > 0$. Therefore, it suffices to prove that

$$\frac{25}{57}e^{-8x^2/5} + \frac{32}{57}e^{-x^2} < e^{-x^2}, \tag{3.23}$$

$$\lambda(p_0)e^{-p_0x^2} + \left[1 - \lambda(p_0)\right]e^{-\mu(p_0)x^2} > e^{-4x^2/\pi} \tag{3.24}$$

for all $x > 0$.

Inequality (3.23) follows easily from

$$\frac{25}{57}e^{-8x^2/5} + \frac{32}{57}e^{-x^2} - e^{-x^2} = \frac{25}{57}e^{-8x^2/5}\left(1 - e^{3x^2/5}\right).$$

Making use of (1.4) and (3.22) together with the arithmetic-geometric mean inequality one has

$$\begin{aligned}\lambda(p_0)e^{-p_0x^2}+\bigl[1-\lambda(p_0)\bigr]e^{-\mu(p_0)x^2} &> e^{-[p_0\lambda(p_0)+\mu(p_0)(1-\lambda(p_0))]x^2}\\ &= e^{-\frac{12p_0(5p_0-7)}{45p_0^2-60p_0-4}x^2}=e^{-4x^2/\pi}\end{aligned}$$

for all $x > 0$. □

Remark 3.3 We clearly see that the results given in Theorem 3.2, Remark 3.2, and Corollary 3.2 are improvements and refinements of inequality (1.1).

Let $p_0^* = (50 + 2\sqrt{30})/35$ and $\alpha(p)$ be defined by (2.18). Then

$$\alpha\left(\frac{8}{5}\right)=\sqrt{\frac{19}{6\pi}}=1.00398\ldots,\qquad \alpha\left(\frac{3}{2}\right)=\sqrt{\frac{29}{9\pi}}=1.01275\ldots,\tag{3.25}$$

$$\begin{aligned}&\alpha(\infty)=\sqrt{\frac{3}{\pi}}=0.97720\ldots,\qquad \alpha(2)=\sqrt{\frac{28}{9\pi}}=0.99513\ldots,\\ &\alpha\left(p_0^*\right)=\sqrt{\frac{160}{51\pi}}=0.99930\ldots.\end{aligned}\tag{3.26}$$

Corollary 3.3 *Let* $p = 3/2, 8/5$ *in Theorem* 3.1(1) *and* $p = p_0^*, 2, \infty$ *in Theorem* 3.1(2). *Then Lemma* 2.3(1) *and* (3) *together with* (3.25) *and* (3.26) *leads to*

$$\begin{aligned}&\sqrt{1-\frac{128}{203}e^{-3x^2/2}-\frac{75}{203}e^{-4x^2/5}}\\ &\quad<\sqrt{1-\frac{25}{57}e^{-8x^2/5}-\frac{32}{57}e^{-x^2}}<\operatorname{erf}(x)\\ &\quad<\sqrt{\frac{19}{6\pi}}\sqrt{1-\frac{25}{57}e^{-8x^2/5}-\frac{32}{57}e^{-x^2}}\\ &\quad<\sqrt{\frac{29}{9\pi}}\sqrt{1-\frac{128}{203}e^{-3x^2/2}-\frac{75}{203}e^{-4x^2/5}},\end{aligned}\tag{3.27}$$

$$\begin{aligned}&\sqrt{\frac{3}{\pi}}\sqrt{1-e^{-4x^2/3}}\\ &\quad<\sqrt{\frac{28}{9\pi}}\sqrt{1-\frac{3}{28}e^{-2x^2}-\frac{25}{28}e^{-6x^2/5}}\\ &\quad<\sqrt{\frac{160}{51\pi}}\sqrt{1-\frac{480-43\sqrt{30}}{960}e^{-(50+2\sqrt{30})x^2/35}-\frac{480+43\sqrt{30}}{960}e^{-(50-2\sqrt{30})x^2/35}}\\ &\quad<\operatorname{erf}(x)<\sqrt{1-\frac{480-43\sqrt{30}}{960}e^{-(50+2\sqrt{30})x^2/35}-\frac{480+43\sqrt{30}}{960}e^{-(50-2\sqrt{30})x^2/35}}\\ &\quad<\sqrt{1-\frac{3}{28}e^{-2x^2}-\frac{25}{28}e^{-6x^2/5}}\\ &\quad<\sqrt{1-e^{-4x^2/3}}.\end{aligned}\tag{3.28}$$

Remark 3.4 From inequality (3.27) we clearly see that the double inequalities

$$0 < \frac{\operatorname{erf}(x) - B_{3/2}(x)}{\operatorname{erf}(x)} < \sqrt{\frac{29}{9\pi}} - 1 = 0.01275\ldots,$$
$$0 < \frac{\operatorname{erf}(x) - B_{3/2}(x)}{\operatorname{erf}(x)} < \sqrt{\frac{19}{6\pi}} - 1 = 0.00398\ldots,$$

hold for all $x > 0$.

Remark 3.5 Let $p_0 = (21\pi - 60 + \sqrt{3}\sqrt{147\pi^2 - 920\pi + 1440})/[30(\pi - 3)]$, $p_0^* = (50 + 2\sqrt{30})/35$, $B_p(x)$ be defined by (1.5) and $x_0 = 1.68913\ldots$ be the unique solution of equation (3.10) on the interval $(0, \infty)$, $\beta(p_0) = \operatorname{erf}(x_0)/B_{p_0}(x_0) = 0.9998\ldots$. Then Corollary 3.1 and (3.28) lead to

$$-0.00013\ldots = 1 - \frac{1}{\beta(p_0)} < \frac{\operatorname{erf}(x) - B_{p_0}(x)}{\operatorname{erf}(x)} \leq 0,$$
$$-0.00069\ldots = 1 - \sqrt{\frac{51\pi}{160}} < \frac{\operatorname{erf}(x) - B_{p_0^*}(x)}{\operatorname{erf}(x)} < 0,$$
$$-0.00488\ldots = 1 - \sqrt{\frac{9\pi}{28}} < \frac{\operatorname{erf}(x) - B_2(x)}{\operatorname{erf}(x)} < 0,$$
$$-0.02332\ldots = 1 - \sqrt{\frac{\pi}{3}} < \frac{\operatorname{erf}(x) - \sqrt{1 - e^{-4x^2/3}}}{\operatorname{erf}(x)} < 0$$

for all $x > 0$.

Corollary 3.4 *Let $p = (7/5)^+, 3/2, 8/5$ and $p = 2, \infty$ in Theorem 3.3. Then it follows from Lemma 2.3(2) that the inequalities*

$$\frac{2x}{\sqrt{\pi}} e^{-x^2/3} < \frac{2x}{\sqrt{\pi}} \frac{e^{-x^2} + 5e^{-x^2/5}}{6} < \operatorname{erf}(x) < \frac{2x}{\sqrt{\pi}} \frac{5e^{-3x^2/5} + 4}{9}$$
$$< \frac{2x}{\sqrt{\pi}} \frac{16e^{-x^2/2} + 5e^{x^2/5}}{21} < \frac{2x}{\sqrt{\pi}} \frac{20e^{-2x^2/5} + e^{x^2}}{21}$$

hold for all $x > 0$.

Remark 3.6 From the identities

$$\frac{e^{-x^2} + 5e^{-x^2/5}}{6} - e^{-x^2/3}$$
$$= \frac{1}{6} e^{-x^2} \bigl(e^{2x^2/15} - 1\bigr)^2 \bigl(5e^{8x^2/15} + 4e^{2x^2/5} + 3e^{4x^2/15} + 2e^{2x^2/15} + 1\bigr)$$

and

$$\frac{5e^{-3x^2/5} + 4}{9} - \frac{e^{-x^2} + 2}{3}$$
$$= -\frac{e^{-x^2}}{9} \bigl(e^{x^2/5} - 1\bigr)^2 \bigl(2e^{3x^2/5} + 4e^{2x^2/5} + 6e^{x^2/5} + 3\bigr)$$

we know that the results given in Theorem 3.3 or Corollary 3.4 are better than that in (1.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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