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An extended Halanay inequality with unbounded coefficient functions on time scales

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Abstract
In this paper, we obtain an extended Halanay inequality with unbounded coefficient functions on time scales, which extends an earlier result in Wen *et al.* (J. Math. Anal. Appl. 347:169-178, 2008). Two illustrative examples are also given.

Keywords: time scales; delay dynamic equation; coefficient functions; inequality

1 Introduction and preliminaries

As is well known, Halanay-type differential inequalities have been very useful in the stability analysis of time-delay systems and these have led to some interesting new stability conditions (see [1–4] and the references therein).

In [3], Halanay proved the following inequality.

Lemma 1.1 (Halanay's inequality) *If*

$$x'(t) \le -\alpha x(t) + \beta \sup_{s\in[t-\tau,t]} x(s), \quad \text{for } t \ge t_0, \tau > 0, \tag{1.1}$$

and $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$x(t) \le Ke^{-\gamma(t-t_0)}, \quad \text{for } t \ge t_0. \tag{1.2}$$

In [5], Baker and Tang obtained the following Halanay-type inequality with unbounded coefficient functions.

Lemma 1.2 (see [5]) *Let* $x(t) > 0, t \in (-\infty, +\infty)$, *and*

$$\frac{dx(t)}{dt} \le -a(t)x(t) + b(t) \sup_{t-\tau(t)\le s\le t} x(s), \quad t > t_0, \tag{1.3}$$

$$x(t) = |\varphi(t)|, \quad t \le t_0, \tag{1.4}$$

where $\varphi(t)$ is bounded and continuous for $t \le t_0$, and $a(t) \ge 0, b(t) \ge 0$ for $t \in [t_0, \infty), \tau(t) \ge 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$. If there exists $\sigma > 0$ such that

$$-a(t) + b(t) \le -\sigma < 0 \quad \text{for } t \ge t_0, \tag{1.5}$$

then

$$\text{(i)} \quad x(t) \le \|\varphi\|^{(-\infty,t_0]}, \quad t \ge t_0 \quad \text{and} \quad \text{(ii)} \quad x(t) \to 0 \quad \text{as } t \to \infty, \tag{1.6}$$

where $\|\varphi\|^{(-\infty,t_0]} = \sup_{t\in(-\infty,t_0]} |\varphi(t)| < \infty$.

In [1], Wen et al. obtained an extension of Lemma 1.2.

In this paper, we extend the main results of [5] to time scale. As an application, we consider the stability of the following delay dynamic equation:

$$\begin{cases} x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)x(t-\tau(t)) + c(t), & t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(s) = |\varphi(s)| & \text{for } s \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \tag{1.7}$$

where $\varphi(s)$ is bounded rd-continuous for $s \in (-\infty, t_0]_{\mathbb{T}}$ and $\tau(t), a(t), b(t), c(t)$ are nonnegative, rd-continuous functions for $t \in [t_0, \infty)_{\mathbb{T}}$ and $c(t)$ is bounded. We prove that the zero solution of the delay difference equation

$$\triangle x(n) = -2(n+1)x(n+1) + \frac{n^2}{2n+1}x(n-2), \quad n \ge 0, \tag{1.8}$$

is stable.

For completeness, we introduce the following concepts related to the notions of time scales. We refer to [6] for additional details concerning the calculus on time scales.

Definition 1.1 (see [6]) A function $h : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)h(t) \ne 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathfrak{R}$ while the set $\mathfrak{R}^+$ is given by $\mathfrak{R}^+ = \{\varphi \in \mathfrak{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$. If $\varphi \in \mathfrak{R}$, the exponential function is defined by

$$e_{\varphi}(t,s) = \exp\biggl(\int_s^t \xi_{\mu(r)}\bigl(\varphi(r)\bigr)\triangle r\biggr), \quad \text{for } t \in \mathbb{T}, s \in \mathbb{T}^{\kappa}, \tag{1.9}$$

where $\xi_{\mu(s)}$ is the cylinder transformation given by

$$\xi_{\mu(r)}\bigl(\varphi(r)\bigr) := \begin{cases} \frac{1}{\mu(r)} \operatorname{Log}(1 + \mu(r)\varphi(r)), & \mu(r) > 0, \\ \varphi(r), & \mu(r) = 0, \end{cases}$$

and some properties of the exponential function are given in the following lemma.

Lemma 1.3 (see [7]) *Let $\varphi \in \mathfrak{R}$, Then*

(i) $e_0(s,t) \equiv 1, e_{\varphi}(t,t) \equiv 1$ *and* $e_{\varphi}(\sigma(t),s) = (1 + \mu(t)\varphi(t))e_{\varphi}(t,s)$;

(ii) $\frac{1}{e_{\varphi}(t,s)} = e_{\ominus\varphi}(t,s)$, *where* $\ominus\varphi(t) = -\frac{\varphi(t)}{1+\mu(t)\varphi(t)}$;

(iii) $(\frac{1}{e_{\varphi}(t,s)})^{\triangle} = -\frac{\varphi(t)}{e_{\varphi}(\sigma(t),s)}$;

(iv) $[e_\varphi(c,t)]^\Delta = -\varphi(t)e_\varphi(c,\sigma(t))$, *where* $c \in \mathbb{T}$;
(v) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$.

Lemma 1.4 (see [8]) *For a nonnegative* φ *with* $-\varphi \in \mathfrak{R}^+$, *we have the inequalities*

$$1 - \int_s^t \varphi(u)\Delta u \le e_{-\varphi}(t,s) \le \exp\left\{-\int_s^t \varphi(u)\Delta u\right\} \quad \textit{for all } t \ge s. \tag{1.10}$$

If φ *is rd-continuous and nonnegative, then*

$$1 + \int_s^t \varphi(u)\Delta u \le e_{\varphi}(t,s) \le \exp\left\{\int_s^t \varphi(u)\Delta u\right\} \quad \textit{for all } t \ge s. \tag{1.11}$$

Remark 1.1 If $\varphi \in \mathfrak{R}^+$ and $\varphi(r) > 0$ for all $r \in [s,t]_{\mathbb{T}}$, then

$$e_\varphi(t,r) \le e_\varphi(t,s) \quad \text{and} \quad e_\varphi(a,b) < 1 \quad \text{for } s \le a < b \le t. \tag{1.12}$$

Proof By $\varphi(r) > 0$, $\varphi \in \mathfrak{R}^+$ and Lemma 1.3(iv) we have $[e_\varphi(c,t)]^\Delta = -\varphi(t)e_\varphi(c,\sigma(t)) < 0$, so

$$e_\varphi(t,r) \le e_\varphi(t,s).$$

Since $a < b$, from the above result, we have

$$e_\varphi(a,b) < e_\varphi(a,a) = 1.$$ □

2 Main results

Throughout this paper, we assume that the following conditions hold:

(H_1) Let $x(t)$ be a nonnegative right-dense function satisfying

$$\begin{cases} x^\triangle(t) \le -a(t)x(t) + b(t)\sup_{t-\tau(t)\le s\le t} x(s) + c(t) \\ \qquad\qquad + d(t)\int_0^\infty K(t,s)x(t-s)\Delta s, \quad t \in [t_0,\infty), \\ x(t) = |\varphi(t)|, \quad t \in (-\infty, t_0], \end{cases}$$

where $\varphi(t)$ is bounded rd-continuous for $t \in (-\infty, t_0]_{\mathbb{T}}$ and $\sup_{t\le t_0} |\varphi(t)| = M$.

(H_2) $a(t)$, $b(t)$, $c(t)$, $\tau(t)$ are nonnegative, rd-continuous functions for $t \in [t_0,\infty)_{\mathbb{T}}$ and $c(t)$ is bounded, such that $\sup_{t\ge t_0} c(t) = \overline{c}$, $\lim_{t\to\infty}(t-\tau(t)) = +\infty$.

(H_3) There exists $\delta > 0$ such that $a(t) - b(t) - d(t)\int_0^\infty K(t,s)\Delta s > \delta > 0$, for $t \in [t_0,\infty)_{\mathbb{T}}$, where the delay kernel $K(t,s)$ is a nonnegative, rd-continuous for $(t,s) \in \mathbb{T} \times [0,\infty)$ and satisfies $\forall t \in \mathbb{T}$, $\int_0^\infty K(t,s)\Delta s < \infty$.

Theorem 2.1 *Assume that* (H_1)-(H_3) *and* $-a(t) \in \mathfrak{R}^+$ *hold, then we have*

(i)
$$x(t) \le \frac{\overline{c}}{\delta} + M, \quad t \in [t_0, +\infty). \tag{2.1}$$

If we assume further that $d(t) = 0$ *in* (H_1), (H_3) *and there exists* $0 < \kappa < 1$ *such that*

$$\kappa a(t) - b(t) > 0 \quad \textit{for } t \in [t_0, +\infty)_{\mathbb{T}}, \tag{2.2}$$

then we have

(ii) *for any given* $\epsilon > 0$, *there exists* $\widetilde{t} = \widetilde{t}(M,\epsilon) > t_0$, *such that*

$$x(t) \le \frac{\overline{c}}{\delta} + \epsilon, \quad t \in [\widetilde{t}, \infty). \tag{2.3}$$

Proof We now consider the following two cases successively.

Case 1. $\overline{c} > 0$.

Proof of Theorem 2.1(i).

For any $\varepsilon > 1$, we have from (H_1)

$$\forall t \le t_0, \quad x(t) = \left|\varphi(t)\right| \le \sup_{t \le t_0}\left|\varphi(t)\right| = M < \frac{\overline{c}}{\delta} + \varepsilon M, \tag{2.4}$$

from this we shall deduce that

$$\forall t \ge t_0, \quad x(t) < \frac{\overline{c}}{\delta} + \varepsilon M. \tag{2.5}$$

To prove (2.5), let $t_1 = \sup\{t | x(s) \le \frac{\overline{c}}{\delta} + \varepsilon M, s \in [t_0, t]_{\mathbb{T}}\} > t_0$, we will show $t_1 = \infty$.

Suppose $t_1 < \infty$. Clearly we have $x(t_1) \le \frac{\overline{c}}{\delta} + \varepsilon M$.

In fact, suppose that $x(t_1) \le \frac{\overline{c}}{\delta} + \varepsilon M$ fails, then we have $x(t_1) > \frac{\overline{c}}{\delta} + \varepsilon M$.

If t_1 is left-dense, there is $\{t_n\}$ satisfying: $t_n < t_1, t_n \to t_1$ $(n \to \infty)$, and $x(t_n) \le \frac{\overline{c}}{\delta} + \varepsilon M$, we have $x(t_1) = \lim_{n\to\infty} x(t_n) \le \frac{\overline{c}}{\delta} + \varepsilon M$, which contradicts $x(t_1) > \frac{\overline{c}}{\delta} + \varepsilon M$.

If t_1 is left-scattered, $\rho(t_1) < t_1$ and $x(\rho(t_1)) \le \frac{\overline{c}}{\delta} + \varepsilon M$; $x(t_1) > \frac{\overline{c}}{\delta} + \varepsilon M$, then we have sup $\{t | x(s) < \frac{\overline{c}}{\delta} + \varepsilon M, s \in [t_0, t]\} = \rho(t_1) < t_1$, which contradicts the definition of t_1.

Therefore we can suppose $t_1 < \infty, x(t_1) \le \frac{\overline{c}}{\delta} + \varepsilon M$. We will discuss two cases:

Case 1.1. Suppose $x(t_1) = \frac{\overline{c}}{\delta} + \varepsilon M, t_1 > t_0$,

$$\forall t \in [t_0, t_1)_{\mathbb{T}}, \quad x(t) \le \frac{\overline{c}}{\delta} + \varepsilon M, \qquad x(t_1) = \frac{\overline{c}}{\delta} + \varepsilon M. \tag{2.6}$$

Clearly we have $x^{\triangle}(t_1) \ge 0$. In fact, suppose that $x^{\triangle}(t_1) \ge 0$ fails, then we have $x^{\triangle}(t_1) < 0$.

If t_1 is right-dense, $\forall s > t_1$, from $x^{\triangle}(t_1) = \lim_{s \to t_1^+} \frac{x(t_1) - x(s)}{t_1 - s} < 0$, we get $x(s) < x(t_1) = \frac{\overline{c}}{\delta} + \varepsilon M$, which contradicts the definition of t_1.

If t_1 is right-scattered, from $x^{\triangle}(t_1) = \frac{x(\sigma(t_1)) - x(t_1)}{\mu(t_1)} < 0$, we get $x(\sigma(t_1)) < x(t_1) = \frac{\overline{c}}{\delta} + \varepsilon M$, which contradicts the definition of t_1.

We have from (2.6), (H_1), and (H_3)

$$\begin{aligned}
x^{\triangle}(t_1) &\overset{(\mathrm{H}_1)}{\le} -a(t_1)x(t_1) + b(t_1) \sup_{t_1 - \tau(t_1) \le s \le t_1} x(s) + c(t_1) + d(t_1)\int_0^\infty K(t_1, s)x(t_1 - s)\triangle s \\
&\overset{(2.6)}{=} -a(t_1)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) + b(t_1) \sup_{t_1 - \tau(t_1) \le s \le t_1} x(s) + c(t_1) + d(t_1)\int_0^\infty K(t_1, s)x(t_1 - s)\triangle s \\
&\overset{(2.6)}{\le} -\left(a(t_1) - b(t_1) + d(t_1)\int_0^\infty K(t_1, s)\triangle s\right)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) + \overline{c} \\
&\overset{(\mathrm{H}_3)}{<} -\delta\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) + \overline{c} = -\delta\varepsilon M < 0,
\end{aligned} \tag{2.7}$$

which contradicts $x^{\triangle}(t_1) \ge 0$.

Case 1.2. Suppose $x(t_1) < \frac{\overline{c}}{\delta} + \varepsilon M$. In this case, t_1 must be right-scattered, for otherwise if t_1 is right-dense, there exists ϵ_1 sufficiently small so that $x(t) < \frac{\overline{c}}{\delta} + \varepsilon M$, for $t \in [t_1, t_1 + \epsilon_1]_{\mathbb{T}}$. Therefore, $x(t) \leq \frac{\overline{c}}{\delta} + \varepsilon M$, for $t \in [t_0, t_1 + \epsilon_1]_{\mathbb{T}}$. This contradicts the definition of t_1. Hence, since t_1 is right-scattered, we have

$$x\bigl(\sigma(t_1)\bigr) > \frac{\overline{c}}{\delta} + \varepsilon M \quad \text{and} \quad x(t) \leq \frac{\overline{c}}{\delta} + \varepsilon M \quad \text{for all } t \leq t_1 < \sigma(t_1). \tag{2.8}$$

We have from (2.8) and (H_1)

$$\begin{aligned} \frac{x(\sigma(t_1)) - x(t_1)}{\mu(t_1)} &= x^{\triangle}(t_1) \\ &\overset{(H_1)}{\leq} -a(t_1)x(t_1) + b(t_1) \sup_{t_1 - \tau(t_1) \leq s \leq t_1} x(s) + c(t_1) \\ &\quad + d(t_1) \int_0^{\infty} K(t_1, s)x(t_1 - s)\triangle s \\ &\overset{(2.8)}{<} -a(t_1)x(t_1) + \left(b(t_1) + d(t_1) \int_0^{\infty} K(t_1, s)\triangle s\right)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) + \overline{c}. \end{aligned} \tag{2.9}$$

By (2.8), (2.9), (H_3), and $1 - \mu(t)a(t) > 0, t \in \mathbb{T}$, we get

$$\begin{aligned} &\frac{\overline{c}}{\delta} + \varepsilon M < x\bigl(\sigma(t_1)\bigr) \\ &\quad \overset{(2.9)}{<} \bigl(1 - \mu(t_1)a(t_1)\bigr)x(t_1) + \mu(t_1)\left(b(t_1) + d(t_1) \int_0^{\infty} K(t_1, s)\triangle s\right)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) \\ &\quad\quad + \mu(t_1)\overline{c} \\ &\quad \overset{(2.8)}{<} \left(1 - \mu(t_1)a(t_1) + \mu(t_1)b(t_1) + \mu(t_1)d(t_1) \int_0^{\infty} K(t_1, s)\triangle s\right)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) \\ &\quad\quad + \mu(t_1)\overline{c} \\ &\quad \overset{(H_3)}{\leq} \bigl(1 - \delta\mu(t_1)\bigr)\left(\frac{\overline{c}}{\delta} + \varepsilon M\right) + \mu(t_1)\overline{c} = \frac{\overline{c}}{\delta} + \varepsilon M - \delta\varepsilon M\mu(t_1), \end{aligned} \tag{2.10}$$

which leads to a contradiction.

Hence the inequality (2.5) must hold.

Since $\varepsilon > 1$ is arbitrary, we let $\varepsilon \to 1^+$ and obtain

$$\forall t \geq t_0, \quad x(t) \leq \frac{\overline{c}}{\delta} + M. \tag{2.11}$$

Proof of Theorem 2.1(ii).

If $M = 0$, it is evident from (2.1) that (2.3) holds. Now we assume $M > 0$. Let $\limsup_{t\to\infty} x(t) = \alpha$, then $0 \leq \alpha \leq \frac{\overline{c}}{\delta} + M$. Now we prove that $\alpha \leq \frac{\overline{c}}{\delta}$.

Suppose this is not true, *i.e.* $\alpha > \frac{\overline{c}}{\delta}$, then we can choose $\varepsilon_2 > 0$ such that $\alpha = \frac{\overline{c}}{\delta} + \varepsilon_2$.

Since $\tau(t) \geq 0$, and $\lim_{t\to\infty}(t - \tau(t)) = +\infty$, we have $\limsup_{t\to\infty} \sup_{t-\tau(t)\leq s\leq t} x(s) = \alpha$.

Clearly, there exists a sufficiently large $T > 0$ and T is fixed, such that

$$\lambda := \kappa + (1 - \kappa)\exp(-\delta T) < 1. \tag{2.12}$$

Taking $\theta : 0 < \theta < \frac{1-\lambda}{1+\lambda}\varepsilon_2$, using the properties of the superior limits we see that there exists a sufficiently large $t^* > t_0$, such that

$$\begin{cases} x(t^*) > \alpha - \theta, \\ x(t) < \alpha + \theta, \quad t \in [t^* - T, t^*] \\ \sup_{t-\tau(t)\leq s\leq t} x(s) \leq \alpha + \theta, \quad t \in [t^* - T, t^*]. \end{cases} \tag{2.13}$$

On the other hand, it follows from (H_1) and (H_3) that

$$\begin{aligned} x^{\triangle}(t) &\overset{(\mathrm{H}_1)}{\leq} -a(t)x(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} x(s) + c(t) \\ &\overset{(\mathrm{H}_3)}{\leq} -a(t)x(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} x(s) + \frac{a(t)-b(t)}{\delta}\bar{c} \\ &= -a(t)\left(x(t) - \frac{\bar{c}}{\delta}\right) + b(t) \sup_{t-\tau(t)\leq s\leq t} \left(x(s) - \frac{\bar{c}}{\delta}\right). \end{aligned} \tag{2.14}$$

Denote $y(t) = x(t) - \frac{\bar{c}}{\delta}$, and (2.13) implies that

$$\begin{cases} y(t^*) = x(t^*) - \frac{\bar{c}}{\delta} > \alpha - \theta - \frac{\bar{c}}{\delta} = \varepsilon_2 - \theta, \\ y(t) = x(t) - \frac{\bar{c}}{\delta} \leq \alpha + \theta - \frac{\bar{c}}{\delta} = \varepsilon_2 + \theta, \quad t \in [t^* - T, t^*] \\ \sup_{t-\tau(t)\leq s\leq t} y(s) = \sup_{t-\tau(t)\leq s\leq t} (x(s) - \frac{\bar{c}}{\delta}) \leq \alpha + \theta - \frac{\bar{c}}{\delta} = \varepsilon_2 + \theta, \\ \quad t \in [t^* - T, t^*]. \end{cases} \tag{2.15}$$

By (2.2), (2.14), (2.15), and $y(t) = x(t) - \frac{\bar{c}}{\delta}$, we have

$$\begin{aligned} y^{\triangle}(t) &\overset{(2.14)}{\leq} -a(t)y(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} y(s) \\ &\overset{(2.15)}{\leq} -a(t)y(t) + (\varepsilon_2 + \theta)b(t) \\ &\overset{(2.2)}{<} -a(t)y(t) + \kappa(\varepsilon_2 + \theta)a(t), \end{aligned} \tag{2.16}$$

which implies

$$\bigl(y(t) - \kappa(\varepsilon_2 + \theta)\bigr)^{\triangle} \leq -a(t)\bigl(y(t) - \kappa(\varepsilon_2 + \theta)\bigr); \tag{2.17}$$

then we have

$$\left(\frac{y(t) - \kappa(\varepsilon_2 + \theta)}{e_{-a}(t, t_0)}\right)^{\triangle} = \frac{(y(t) - \kappa(\varepsilon_2 + \theta))^{\triangle} + a(t)(y(t) - \kappa(\varepsilon_2 + \theta))}{e_{-a}(\sigma(t), t_0)} \overset{(2.17)}{\leq} 0, \tag{2.18}$$

where we used the property of the exponential function: if $p \in \mathfrak{R}^+$ and $t_0 \in \mathbb{T}$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

Integrating both sides of (2.18) from $t^* - T$ to t^* and by (1.11) we obtain

$$\begin{aligned} (1-\kappa)(\varepsilon_2 + \theta) - 2\theta &= \varepsilon_2 - \theta - \kappa(\varepsilon_2 + \theta) \\ &< y\bigl(t^*\bigr) - \kappa(\varepsilon_2 + \theta) \end{aligned}$$

$$\begin{aligned}
&\overset{\text{(Lemma 1.3(v))}}{\leq} e_{-a}\big(t^*, t^*-T\big)\big[y\big(t^*-T\big) - \kappa(\varepsilon_2+\theta)\big] \\
&\overset{\text{(1.11), (2.15)}}{\leq} \big(\varepsilon_2+\theta-\kappa(\varepsilon_2+\theta)\big)\exp\biggl(-\int_{t^*-T}^{t^*} a(u)\Delta u\biggr) \\
&< (1-\kappa)(\varepsilon_2+\theta)\exp(-\delta T),
\end{aligned} \tag{2.19}$$

where we used $a(t) \geq a(t) - b(t) \geq \delta > 0$ in the last step.

By (2.19), we have

$$\theta \geq \frac{1-[\kappa+(1-\kappa)\exp(-\delta T)]}{1+[\kappa+(1-\kappa)\exp(-\delta T)]}\varepsilon_2 = \frac{1-\lambda}{1+\lambda}\varepsilon_2.$$

This contradicts the choice of θ, so we get $\alpha \leq \frac{\bar{c}}{\delta}$. From the definition of the superior limits we obtain (2.3).

Case 2. $\bar{c} = 0$.

If only we replace $\bar{c}$ in the proof of Case 1 by $\bar{c} + \epsilon_3$ for any given $\epsilon_3 > 0$, then let $\epsilon_3 \to 0^+$, we find that (2.1) and (2.3) hold.

A combination of Cases 1 and 2 completes the proof of Theorem 2.1. □

Remark 2.1 When $M = 0$, from (2.7), (H_3) must have the form that there exists $\delta > 0$ such that

$$a(t) - b(t) - d(t)\int_0^\infty K(t,s)\Delta s > \delta > 0, \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

When $M > 0$, (H_3) may have the form that there exists $\delta > 0$ such that

$$a(t) - b(t) - d(t)\int_0^\infty K(t,s)\Delta s \geq \delta > 0, \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

Similarly, in [1] when $G = 0$, (2.10) must have the form that

$$\alpha(t) + \beta(t) < -\sigma < 0 \quad \text{for } t \geq t_0;$$

when $G > 0$, (2.10) may have the form that

$$\alpha(t) + \beta(t) \leq -\sigma < 0 \quad \text{for } t \geq t_0.$$

Theorem 2.1 can be regarded as the extension of the main theorem of [5], Theorem 2.3 of [1].

3 Applications and examples

Consider the delay dynamic equation

$$\begin{cases} x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)x(t-\tau(t)) + c(t) + d(t)\int_0^\infty K(t,s)x(t-s)\triangle s, \\ \quad t \in [t_0,+\infty)_{\mathbb{T}}, \\ x(t) = |\varphi(t)| \quad \text{for } t \in (-\infty,t_0]_{\mathbb{T}}, \end{cases} \tag{3.1}$$

where $\varphi(t)$ is bounded rd-continuous for $s \in (-\infty, t_0]_{\mathbb{T}}$ and $\tau(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$ are nonnegative, rd-continuous functions for $t \in [t_0, \infty)_{\mathbb{T}}$ and $c(t)$ is bounded,

$$\sup_{t\leq t_0}\left|\varphi(t)\right| = M, \qquad \sup_{t\geq t_0} c(t) = \overline{c}, \qquad \lim_{t\to\infty}\bigl(t-\tau(t)\bigr) = +\infty.$$

Assume there exists $\delta > 0$ such that

$$a(t) - b(t) - d(t)\int_0^{\infty} K(t,s)\triangle s > \delta > 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.2}$$

where the delay kernel $K(t,s)$ is a nonnegative, rd-continuous for $(t,s) \in [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}}$.

From (3.1), we have

$$\begin{aligned} x(t) &= x(t_0)e_{\ominus a}(t,t_0) \\ &\quad + \int_{t_0}^{t} e_{\ominus a}(t,s)\left[b(s)x\bigl(s-\tau(s)\bigr) + c(s) + d(s)\int_0^{\infty} K(s,v)x(s-v)\triangle v\right]\triangle s. \end{aligned} \tag{3.3}$$

Let the functions $y(t)$ be defined as follows: $y(t) = |x(t)|$ for $t \in (-\infty, t_0]_{\mathbb{T}}$, and

$$\begin{aligned} y(t) &= \left|x(t_0)\right|e_{\ominus a}(t,t_0) \\ &\quad + \int_{t_0}^{t} e_{\ominus a}(t,s)\left[b(s)\left|x\bigl(s-\tau(s)\bigr)\right| + c(s) + d(s)\int_0^{\infty} K(s,v)\left|x(s-v)\right|\triangle v\right]\triangle s, \end{aligned}$$

for $t > t_0$. Then we have $|x(t)| \leq y(t)$, for all $t \in (-\infty, +\infty)_{\mathbb{T}}$.

By [9], Theorem 5.37, we get

$$\begin{aligned} y^{\triangle}(t) &= \ominus a(t)\biggl\{\left|x(t_0)\right|e_{\ominus a}(t,t_0) + \int_{t_0}^{t} e_{\ominus a}(t,s)\biggl[b(s)\left|x\bigl(s-\tau(s)\bigr)\right| + c(s) \\ &\quad + d(s)\int_0^{\infty} K(s,v)\left|x(s-v)\right|\triangle v\biggr]\triangle s\biggr\} \\ &\quad + e_{\ominus a}\bigl(\sigma(t),t\bigr)\biggl\{b(t)\left|x\bigl(t-\tau(t)\bigr)\right| + c(t) + d(t)\int_0^{\infty} K(t,v)\left|x(t-v)\right|\triangle v\biggr\} \\ &\leq \frac{1}{1+\mu(t)a(t)}\biggl\{-a(t)y(t) + b(t)\sup_{t-\tau(t)\leq\theta\leq t} y(\theta) + c(t) \\ &\quad + d(t)\int_0^{\infty} K(t,v)y(t-v)\triangle v\biggr\}, \quad t \in [t_0,\infty)_{\mathbb{T}}. \end{aligned} \tag{3.4}$$

Example 1 Let $\mathbb{T} = \mathbb{R}^+$, then system (H_1) is expressed as

$$\begin{cases} x'(t) \leq -a(t)x(t) + b(t)\sup_{t-\tau(t)\leq s\leq t} x(s) + c(t) \\ \qquad\quad + d(t)\int_0^{\infty} K(t,s)x(t-s)\,ds, \quad t \geq 0, \\ x(t) = |\varphi(t)|, \quad t \leq 0, \end{cases} \tag{3.5}$$

where $\varphi(t)$ is bounded continuous for $t \in (-\infty, 0]$ and $\sup_{t\leq 0}|\varphi(t)| = M$.

We choose some explicit nonnegative, continuous functions for $a(t), b(t), c(t), d(t), \tau(t)$, $K(t,s)$. Let

$$a(t) = \frac{(t+1)^2}{t+2}, \qquad b(t) = \frac{t^2+t}{2e(t+2)}, \qquad c(t) = \left(\frac{t+2}{t+1}\right)^t, \qquad d(t) = \frac{t}{e(2-e^{-t^2})\sqrt{\pi}},$$
$$K(t,s) = (2-\cos 2ts)e^{-s^2}, \quad (t,s) \in [0,\infty) \times [0,\infty), \tau(t) < t \text{ and } \lim_{t\to\infty}\bigl(t-\tau(t)\bigr) = +\infty.$$

Obviously, $a(t), b(t), d(t)$ are unbounded for $t \geq 0$ and $\sup_{t\geq 0} c(t) = \overline{c} = e$.

(1) $\forall t \in [0,\infty), g(t) := \int_0^\infty K(t,s)\,ds = \int_0^\infty (2-\cos 2ts)e^{-s^2}\,ds$, then since $\forall (t,s) \in [0,\infty) \times [0,\infty)$,

$$\bigl|K(t,s)\bigr| \leq 3e^{-s^2} \quad \text{and} \quad \left|\frac{\partial K(t,s)}{\partial t}\right| = \bigl|2se^{-s^2}\sin 2ts\bigr| \leq 2se^{-s^2},$$
$$\int_0^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2} = g(0) \quad \text{and} \quad \int_0^\infty 2se^{-s^2}\,ds = 1,$$

we have $g(t) = \int_0^\infty K(t,s)\,ds$ is convergent for $t \in [0,\infty)$ and $\int_0^\infty K_t(t,s)\,ds$ is uniformly convergent for $t \in [0,\infty)$.

So

$$g'(t) = \int_0^\infty K_t(t,s)\,ds = \int_0^\infty 2se^{-s^2}\sin 2ts\,ds = -2tg(t) + 2\sqrt{\pi}t.$$

Rearrange terms and obtain

$$\frac{(g(t)-\sqrt{\pi})'}{g(t)-\sqrt{\pi}} = -2t. \tag{3.6}$$

Solving (3.6) for $g(t)$, we have

$$g(t) = \int_0^\infty K(t,s)\,ds = \frac{\sqrt{\pi}}{2}\bigl(2-e^{-t^2}\bigr) < \sqrt{\pi}.$$

(2) There exists $\delta = \frac{1}{2} > 0$, such that

$$a(t) - b(t) = \frac{(t+1)^2}{t+2} - \frac{t^2+t}{2e(t+2)} \geq \frac{1}{2} = \delta > 0, \quad t \in [0,\infty).$$

By (i) of Theorem 2.1, we have $|x(t)| \leq \frac{\overline{c}}{\delta} + M = 2e + M, t \geq 0$.

Take $\kappa = \frac{1}{2} \in (0,1)$, it is easy to see that

$$\kappa a(t) - b(t) = \frac{1}{e} \cdot \frac{(t+1)^2}{t+2} \geq \frac{1}{4} > 0, \quad t \in [0,\infty).$$

By (ii) of Theorem 2.1, for any given $\epsilon > 0$, there exists $\widetilde{t} = \widetilde{t}(M,\epsilon) > 0$, such that $|x(t)| \leq \frac{\overline{c}}{\delta} + \epsilon = 2e + \epsilon, t \geq \widetilde{t} > 0$.

Taking $c(t) \equiv 0$, we have $|x(t)| \leq \epsilon$, $t \geq \widetilde{t} > 0$. So the zero solution of the system (3.1) is stable.

Example 2 Consider the delay dynamic equation

$$\begin{cases} x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)x(t-\tau(t)) + c(t), & t \in [t_0, +\infty)_{\mathbb{T}}, \\ x(t) = |\varphi(t)| \quad \text{for } t \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \tag{3.7}$$

where $\varphi(t)$ is bounded, rd-continuous for $t \leq t_0$ and $\sup_{t \leq t_0} |\varphi(t)| = M$, $a(t), b(t), c(t), \tau(t)$ are nonnegative, rd-continuous functions for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\sup_{t \geq t_0} c(t) = \overline{c}$.

If there exists $\delta > 0$ such that

$$a(t) - b(t) \geq \delta > 0 \quad \text{for } t \geq t_0. \tag{3.8}$$

Similar to Example 1, we get

$$y^{\triangle}(t) \leq \frac{1}{1 + \mu(t)a(t)} \Big\{ -a(t)y(t) + b(t) \sup_{t-\tau(t) \leq s \leq t} y(s) + c(t) \Big\}, \quad t \in [t_0, +\infty)_{\mathbb{T}}.$$

In particular, we take $\mathbb{T} = \mathbb{N}$, (3.7) reduces to

$$\triangle x(n) = -a(n)x(n+1) + b(n)x(n-2) + c(n), \quad n \geq 2.$$

Let $a(n) = 2(n+1), b(n) = \frac{n^2}{2n+1}, c(n) = \frac{5n}{\sqrt[n]{n!}}, \tau(n) = 2$.

Obviously, $a(n), b(n)$ are unbounded for $n \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} c(n) = \lim_{n \to \infty} \frac{5n}{\sqrt[n]{n!}} = \overline{c} = 5e$.

(1) $\forall n \geq 2$, $\frac{a(n)-b(n)}{1+a(t)} = \frac{2(n+1) - \frac{n^2}{2n+1}}{2n+3} = \frac{3n^2+6n+2}{(2n+1)(2n+3)} \geq \frac{2}{3} = \delta$.

(2) Take $\kappa = 0.9 \in (0,1)$, it is easy to see that

$$\frac{\kappa a(n) - b(n)}{1 + a(n)} > 0.$$

By (ii) of Theorem 2.1, for any given $\epsilon > 0$, there exists $\widetilde{t} = \widetilde{t}(M, \epsilon) > 0$, such that $|x(t)| \leq \frac{\overline{c}}{\delta} + \epsilon = \frac{15e}{2} + \epsilon, t \geq \widetilde{t} > 0$.

Taking $c(n) \equiv 0$, we have $|x(t)| \leq \epsilon$, $t \geq \widetilde{t} > 0$. So the zero solution of the system (3.7) is stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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