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The binomial sequence spaces of nonabsolute type

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Abstract

In this paper, we introduce the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ of nonabsolute type which include the spaces c_0 and c , respectively. Also, we prove that the spaces $b_0^{r,s}$ and $b_c^{r,s}$ are linearly isomorphic to the spaces c_0 and c , in turn, and we investigate some inclusion relations. Moreover, we obtain the Schauder bases of those spaces and determine their α -, β -, and γ -duals. Finally, we characterize some matrix classes related to those spaces.

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Keywords: matrix transformations; matrix domain; Schauder basis; α -, β - and γ -duals; matrix classes

1 The basic definitions and notations

Let w be the set of all real (or complex) valued sequences. Then w becomes a vector space under point-wise addition and scalar multiplication. A sequence space is a vector subspace of w . We use the notations of ℓ_∞ , c_0 , c , and ℓ_p for the spaces of all bounded, null, convergent, and absolutely p -summable sequences, respectively, where $1 \leq p < \infty$.

A Banach sequence space is called a BK -space provided each of the maps $p_n : X \rightarrow \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$ [1]. By taking into account the definition above, one can say that the sequence spaces ℓ_∞ , c_0 , and c are BK -spaces with their usual sup -norm defined by $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$ and ℓ_p is a BK -space with its ℓ_p -norm defined by

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{\frac{1}{p}},$$

where $p \in [0, \infty)$.

Let $A = (a_{nk})$ be an infinite matrix with complex entries and $x \in w$, then the A -transform of x is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \tag{1.1}$$

and is assumed to be convergent for all $n \in \mathbb{N}$ [2]. For brevity in the notation, we henceforth prefer that the summation without limits runs from 0 to ∞ .

Let X and Y be two arbitrary sequence spaces and $A = (a_{nk})$ be an infinite matrix. Then the domain of A is denoted by X_A and defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}, \tag{1.2}$$

which is also a sequence space and the class of all matrices A such that $X \subset Y_A$ is denoted by $(X : Y)$. Moreover, $A = (a_{nk})$ is called conservative if $c \subset c_A$. Furthermore, $A = (a_{nk})$ is called m -multiplicative, if $\lim_{n \rightarrow \infty} (Ax)_n = m \lim_{n \rightarrow \infty} x_n$ for all $x \in c$ and the class of all m -multiplicative matrices is denoted by $(X : Y)_m$. Specially, $A = (a_{nk})$ is called regular, if $A = (a_{nk})$ is 1-multiplicative.

The spaces of all bounded and convergent series are denoted by bs and cs and are defined by aid of the matrix domain of the summation matrix $S = (s_{nk})$ such that $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. A matrix $A = (a_{nk})$ is said to be a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. Furthermore, a triangle matrix uniquely has an inverse, which is also a triangle matrix.

The theory of matrix transformations is of great importance in the summability which was obtained by Cesàro, Borel, Riesz and others. Therefore, many authors have defined new sequence spaces by using this theory. For example, $(\ell_\infty)_{N_q}$ and c_{N_q} in [3], X_p and X_∞ in [4], \tilde{c} and \tilde{c}_0 in [5], a_0^r and a_c^r in [6]. Moreover, many authors have constructed new sequence spaces by using especially the Euler matrix. For instance, e_0^r and e_c^r in [7], e_p^r and e_∞^r in [8] and [9], $e_0^r(\Delta)$, $e_c^r(\Delta)$, and $e_\infty^r(\Delta)$ in [10], $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in [11], $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$, and $e_\infty^r(B^{(m)})$ in [12], $e_0^r(\Delta, p)$, $e_c^r(\Delta, p)$, and $e_\infty^r(\Delta, p)$ in [13], $e_0^r(u, p)$ and $e_c^r(u, p)$ in [14].

In this paper, we introduce the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ of nonabsolute type which include the spaces c_0 and c , respectively. Also, we prove that the spaces $b_0^{r,s}$ and $b_c^{r,s}$ are linearly isomorphic to the spaces c_0 and c , in turn and investigate some inclusion relations. Moreover, we obtain the Schauder basis of those spaces and determine their α -, β -, and γ -duals. Finally, we characterize some matrix classes related to those spaces.

2 The binomial sequence spaces of nonabsolute type

In this chapter, we introduce the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ of nonabsolute type and prove that the spaces $b_0^{r,s}$ and $b_c^{r,s}$ are linearly isomorphic to the spaces c_0 and c , respectively. Moreover, we deal with an inclusion relation related to those spaces.

Let $r, s \in \mathbb{R}$ and $r + s \neq 0$. Then the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined by

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}_0$. For $sr > 0$ we have the following properties of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$:

- (i) $\sup_{n \in \mathbb{N}} \sum_{k=0}^n \left| \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k \right| < \infty,$
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k = 0,$
- (iii) $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k = 1.$

Therefore, the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is regular for $sr > 0$. Unless stated otherwise, we assume that $sr > 0$.

Here, we would like to emphasize that if we take $r + s = 1$, we obtain the Euler matrix $E^r = (e_{nk}^r)$. So the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ generalizes the Euler matrix $E^r = (e_{nk}^r)$.

By considering the definition of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$, we define the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ as follows:

$$b_0^{r,s} = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k = 0 \right\}$$

and

$$b_c^{r,s} = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k \text{ exists} \right\}.$$

The sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ can be redefined by using the notion of (1.2) as follows:

$$b_0^{r,s} = (c_0)_{B^{r,s}} \quad \text{and} \quad b_c^{r,s} = c_{B^{r,s}}. \tag{2.1}$$

It is clear that $b_0^{r,s} \subset b_c^{r,s}$. Let $x = (x_k)$ be an arbitrary sequence. Then the $B^{r,s}$ -transform of $x = (x_k)$ is defined by

$$(B^{r,s}x)_k = y_k = \frac{1}{(s+r)^k} \sum_{j=0}^k \binom{k}{j} s^{k-j} r^j x_j \tag{2.2}$$

for all $k \in \mathbb{N}$.

Now, we want to start with the following theorem related to the theory of *BK*-spaces, which is of great importance in the characterization of matrix transformations between sequence spaces.

Theorem 2.1 *The binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ are BK-spaces with their sup-norms defined by*

$$\|x\|_{b_0^{r,s}} = \|x\|_{b_c^{r,s}} = \|B^{r,s}x\|_\infty = \sup_{n \in \mathbb{N}} |(B^{r,s}x)_n|.$$

Proof The sequence spaces c_0 and c are *BK*-spaces according to their *sup-norms*. Moreover, the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is a triangle matrix and (2.1) holds. By combining these three facts and Theorem 4.3.12 of Wilansky [2], we deduce that the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ are *BK*-spaces. This completes the proof. \square

Let $|x| = (|x_k|)$ for all $k \in \mathbb{N}$. Because of $\|x\|_{b_0^{r,s}} \neq \| |x| \|_{b_0^{r,s}}$ and $\|x\|_{b_c^{r,s}} \neq \| |x| \|_{b_c^{r,s}}$ for at least one sequence in the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$, $b_0^{r,s}$ and $b_c^{r,s}$ are sequence spaces of nonabsolute type.

Theorem 2.2 *The binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ are linearly isomorphic to the sequence spaces c_0 and c , respectively.*

Proof Because a repetition of similar statements is redundant, the proof of the theorem is given for only the space $b_c^{r,s}$. For this purpose, we should show the existence of a linear bijection between the spaces $b_c^{r,s}$ and c . Let us consider the transformation L defined by $L : b_c^{r,s} \rightarrow c, L(x) = B^{r,s}x$. Then it is obvious that, for all $x \in b_c^{r,s}, L(x) = B^{r,s}x \in c$. Moreover, it is clear that L is a linear transformation and $x = 0$ whenever $L(x) = 0$. Because of this, L is injective.

Let $y = (y_k) \in c$ be given. We define a sequence $x = (x_k)$ by

$$x_k = \frac{1}{r^k} \sum_{j=0}^k \binom{k}{j} (-s)^{k-j} (s+r)^j y_j$$

for all $k \in \mathbb{N}$. Then we obtain

$$\begin{aligned} (B^{r,s}x)_n &= \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k \\ &= \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} \sum_{j=0}^k \binom{k}{j} (-s)^{k-j} (s+r)^j y_j \\ &= y_n \end{aligned}$$

for all $n \in \mathbb{N}$. So, $B^{r,s}x = y$ and since $y \in c$, we conclude that $B^{r,s}x \in c$. Hence, we deduce that $x \in b_c^{r,s}$ and $L(x) = y$. On account of this L is surjective.

Moreover, we have for all $x \in b_c^{r,s}$

$$\|L(x)\|_\infty = \|B^{r,s}x\|_\infty = \|x\|_{b_c^{r,s}}.$$

So L is norm preserving. Consequently, L is a linear bijection. Then we obtain the fact that the spaces $b_c^{r,s}$ and c are linearly isomorphic, that is, $b_c^{r,s} \cong c$. This completes the proof. \square

Theorem 2.3 *The inclusions $e_0^r \subset b_0^{r,s}$ and $e_c^r \subset b_c^{r,s}$ strictly hold, where e_0^r and e_c^r are Euler sequence spaces of nonabsolute type.*

Proof If $r + s = 1$, we obtain $E^r = B^{r,s}$. So, the inclusion $e_0^r \subset b_0^{r,s}$ holds. Assume that $0 < r < 1$ and $s = 4$. Now, we define a sequence $x = (x_k)$ such that $x_k = (-\frac{3}{r})^k$ for all $k \in \mathbb{N}$. Then it is obvious that $x = ((-\frac{3}{r})^k) \notin c_0$ and $E^r x = ((-2-r)^k) \notin c_0$. On the other hand, $B^{r,s}x = ((\frac{1}{4+r})^k) \in c_0$. As a consequence, $x = (x_k) \in b_0^{r,s} \setminus e_0^r$.

This shows that the inclusion $e_0^r \subset b_0^{r,s}$ strictly holds. Another part of the theorem can be proved in a similar way. This completes the proof. \square

Theorem 2.4 *The inclusions $c_0 \subset b_0^{r,s}$ and $c \subset b_c^{r,s}$ strictly hold. But the sequence spaces $b_0^{r,s}$ and ℓ_∞ do not include each other.*

Proof If we consider regularity of the binomial matrix $B^{r,s}$, we can easily conclude that $B^{r,s}x \in c_0$ whenever $x \in c_0$. This means that $x \in b_0^{r,s}$ for all $x \in c_0$, namely $c_0 \subset b_0^{r,s}$. Now

we define a sequence $u = (u_k)$ such that $u_k = (-1)^k$ for all $k \in \mathbb{N}$. Then we obtain $B^{r,s}x = ((\frac{s-r}{s+r})^k) \in c_0$. As a consequence, u is in $b_0^{r,s}$ but not in c_0 . So, the inclusion $c_0 \subset b_0^{r,s}$ is strict. By using a similar way, one can show that the inclusion $c \subset b_c^{r,s}$ is strict.

To prove the second part of the theorem, we consider the sequences $e = (1, 1, 1, \dots)$ and $v = (v_k)$ defined by $v_k = (-\frac{s}{r})^k$ for all $k \in \mathbb{N}$, where $|\frac{s}{r}| > 1$. Then we obtain $B^{r,s}e = e$ and $B^{r,s}v = (1, 0, 0, \dots)$. Hence, e is in ℓ_∞ but not in $b_0^{r,s}$ and v is in $b_0^{r,s}$ but not in ℓ_∞ . This shows that the sequence spaces $b_0^{r,s}$ and ℓ_∞ overlap but these spaces do not include each other. This completes the proof. □

Definition 2.1 (see [2]) An infinite matrix $A = (a_{nk})$ is called coregular, if $A = (a_{nk})$ is conservative and $\chi(A) = \lim_{n \rightarrow \infty} \sum_k a_{nk} - \sum_k \lim_{n \rightarrow \infty} a_{nk} \neq 0$.

By taking into account the regularity of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$, we obtain $\chi(B^{r,s}) = 1 \neq 0$. So, the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is coregular.

Definition 2.2 (see [15]) Let $A = (a_{nk})$ be an infinite matrix with bounded columns. Then A is defined to be of type M if $tA = 0$ implies $t = 0$ for every $t \in \ell$.

Definition 2.3 (see [2]) For a conservative triangle $A = (a_{nk})$, $c \subset c_A$. Its closure \bar{c} in c_A is called the perfect part of c_A . If c is dense, $A = (a_{nk})$ is called perfect.

Now we give the following two theorems, which are needed.

Theorem 2.5 (see [15]) *A regular triangle $A = (a_{nk})$ is of type M if there exists a $(z_i) \in \ell_\infty$ with $z_i \neq z_j$ ($i \neq j$) and $\|(z_i)\|_\infty < 1$ such that*

$$\forall i \in \mathbb{N}_0, \exists (x_k)_k \in \ell_\infty, \quad \forall n \in \mathbb{N} : z_i^n = \sum_{k=0}^n a_{nk} x_{k_i}.$$

Theorem 2.6 (see [2]) *A coregular triangle is perfect if and only if it is of type M .*

Theorem 2.7 *Each regular binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is perfect.*

Proof We know that the regular binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is coregular. So, for the proof, we should show that $B^{r,s} = (b_{nk}^{r,s})$ is of type M .

Let $D^{r,s} = (d_{nk}^{r,s})$ be the inverse of $B^{r,s} = (b_{nk}^{r,s})$. Then we have for every $z \in \mathbb{C}$

$$\begin{aligned} u_k(z) &= \sum_{v=0}^k d_{kv}^{r,s} z^v = \frac{1}{r^k} \sum_{v=0}^k \binom{k}{v} (-s)^{k-v} (s+r)^v z^v \\ &= \frac{1}{r^k} ((s+r)z - s)^k. \end{aligned}$$

One can easily verify that $\sup_{k \in \mathbb{N}} \sup\{|u_k(z)| : |(s+r)z - s| < |r|\} \leq 1$ and that

$$\sum_{k=0}^n b_{nk}^{r,s} u_k(z) = \sum_{k=0}^n b_{nk}^{r,s} \sum_{v=0}^k d_{kv}^{r,s} z^v = z^n$$

for all $z \in \mathbb{C}$. Therefore, if we choose a sequence (z_i) with $z_i \neq z_j$ ($i \neq j$), $\|(z_i)\|_\infty < 1$ and $|(s+r)z_i - s| < |r|$ ($i \in \mathbb{N}_0$), then $x_{ki} = u_k(z_i)$ ($i, k \in \mathbb{N}_0$) fit Theorem 2.7. Thus, the regular binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is perfect. This completes the proof. \square

3 The Schauder basis and α -, β -, γ - and continuous duals

In this chapter, we construct the Schauder basis and designate the α -, β -, γ -, and continuous duals of the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$.

Given a normed space $(X, \|\cdot\|_X)$, a set $\{x_k : x_k \in X, k \in \mathbb{N}\}$ is called a Schauder basis for X if for all $x \in X$ there exist unique scalars $\lambda_k, k \in \mathbb{N}$, such that $x = \sum_k \lambda_k x_k$; i.e.,

$$\left\| x - \sum_{k=0}^n \lambda_k x_k \right\|_X \rightarrow 0$$

as $n \rightarrow \infty$ [1].

Theorem 3.1 *Let $\mu_k = \{B^{r,s}x\}_k$ for all $k \in \mathbb{N}$ and $l = \lim_{k \rightarrow \infty} \mu_k$. We define a sequence $g^{(k)}(r, s) = \{g_n^{(k)}(r, s)\}_{n \in \mathbb{N}}$ as follows:*

$$g_n^{(k)}(r, s) = \begin{cases} 0, & 0 \leq n < k, \\ \frac{1}{r^n} \binom{n}{k} (-s)^{n-k} (s+r)^k, & n \geq k, \end{cases}$$

for all fixed $k \in \mathbb{N}$. Then the following statements hold.

- (a) *The sequence $\{g^{(k)}(r, s)\}_{k \in \mathbb{N}}$ is a Schauder basis for the binomial sequence space $b_0^{r,s}$, and every $x \in b_0^{r,s}$ has a unique representation of the form*

$$x = \sum_k \mu_k g^{(k)}(r, s).$$

- (b) *The set $\{e, g^{(0)}(r, s), g^{(1)}(r, s), \dots\}$ is a Schauder basis for the binomial sequence space $b_c^{r,s}$, and any $x \in b_c^{r,s}$ has a unique representation of the form*

$$x = le + \sum_k [\mu_k - l] g^{(k)}(r, s).$$

Proof (a) Obviously we have

$$B^{r,s} g^{(k)}(r, s) = e^{(k)} \in c_0, \quad k \in \mathbb{N},$$

where $e^{(k)}$ is a sequence with 1 in the k th place and zeros elsewhere. So, the inclusion $\{g^{(k)}(r, s)\} \subset b_0^{r,s}$ holds.

Given a sequence $x = (x_k) \in b_0^{r,s}$ and $m \in \mathbb{N}$, we define

$$x^{[m]} = \sum_{k=0}^m \mu_k g^{(k)}(r, s).$$

Then, if we apply the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ to $x^{[m]}$, we have

$$B^{r,s} x^{[m]} = \sum_{k=0}^m \mu_k B^{r,s} g^{(k)}(r, s) = \sum_{k=0}^m (B^{r,s} x)_k e^{(k)}$$

and

$$\{B^{r,s}(x - x^{[m]})\}_n = \begin{cases} 0, & 0 \leq n \leq m, \\ (B^{r,s}x)_n, & n > m, \end{cases}$$

for all $m, n \in \mathbb{N}$.

Now, let $\epsilon > 0$ be arbitrarily given. Then we may choose a $m_0 \in \mathbb{N}$ such that

$$|(B^{r,s}x)_m| < \frac{\epsilon}{2}$$

for all $m \geq m_0$. Therefore,

$$\|x - x^{[m]}\|_{b_0^{r,s}} = \sup_{n \geq m} |(B^{r,s}x)_n| < \sup_{n \geq m_0} |(B^{r,s}x)_n| \leq \frac{\epsilon}{2} < \epsilon$$

for all $m \geq m_0$. This shows that

$$x = \sum_k \mu_k g^{(k)}(r, s).$$

To complete the proof of part (a), we should show the uniqueness of this representation.

We assume that there exists a representation

$$x = \sum_k \lambda_k g^{(k)}(r, s).$$

Due to the transformation, L defined in the proof of Theorem 2.2 is continuous; we can write

$$(B^{r,s}x)_n = \sum_k \lambda_k \{B^{r,s}g^{(k)}(r, s)\}_n = \sum_k \lambda_k e_n^{(k)} = \lambda_n$$

for all $n \in \mathbb{N}$, which contradicts the fact that $(B^{r,s}x)_n = \mu_n$ for all $n \in \mathbb{N}$. Hence, every $x \in b_0^{r,s}$ has a unique representation, as desired.

(b) We know that $\{g^{(k)}(r, s)\} \subset b_0^{r,s}$ and $B^{r,s}e = e \in c$. So, the inclusion $\{e, g^{(k)}(r, s)\} \subset b_c^{r,s}$ trivially holds.

For a given arbitrary sequence $x = (x_k) \in b_c^{r,s}$, we define a sequence $y = (y_k)$ such that $y = x - le$, where $l = \lim_{k \rightarrow \infty} \mu_k$. Then it is obvious that $y = (y_k) \in b_0^{r,s}$. By considering the part (a), one can say that $y = (y_k)$ has a unique representation. This implies that $x = (x_k)$ has a unique representation, as desired in part (b). This completes the proof. \square

By taking into account the results of Theorems 2.1 and 3.1, we give the following result.

Corollary 3.2 *The binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ are separable.*

Let X and Y be two arbitrary sequence spaces. The multiplier space of X and Y is symbolized with $M(X, Y)$ and defined by

$$M(X, Y) = \{y = (y_k) \in w : xy = (x_k y_k) \in Y \text{ for all } x = (x_k) \in X\}.$$

By considering the definition of $M(X, Y)$, the α -, β -, and γ -duals of a sequence space X are defined by

$$X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, cs) \quad \text{and} \quad X^\gamma = M(X, bs),$$

respectively.

By X^* , we denote the space of all bounded linear functionals on X . X^* is called the continuous dual of a normed space X .

Let us give some properties to use in the next lemma:

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right|^p < \infty, \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{3.2}$$

$$\lim_{n \rightarrow \infty} a_{nk} = \alpha_k \quad \text{for each } k \in \mathbb{N}, \tag{3.3}$$

$$\lim_{n \rightarrow \infty} \sum_k a_{nk} = \alpha, \tag{3.4}$$

where \mathcal{F} is the collection of all finite subsets of \mathbb{N} and $1 \leq p < \infty$.

Lemma 3.3 (see [16]) *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*

- (i) $A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1) \Leftrightarrow$ (3.1) holds with $p = 1$;
- (ii) $A = (a_{nk}) \in (c_0 : c) \Leftrightarrow$ (3.2) and (3.3) hold;
- (iii) $A = (a_{nk}) \in (c : c) \Leftrightarrow$ (3.2), (3.3) and (3.4) hold;
- (iv) $A = (a_{nk}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) \Leftrightarrow$ (3.2) holds;
- (v) $A = (a_{nk}) \in (c : \ell_p) \Leftrightarrow$ (3.1) holds with $1 \leq p < \infty$.

Theorem 3.4 *The α -dual of the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$ is*

$$v_1^{r,s} = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right| < \infty \right\}.$$

Proof Let us consider the sequence $x = (x_n)$ that is defined in the proof of Theorem 2.2. Then, for given $a = (a_n) \in w$, we obtain

$$a_n x_n = \sum_{k=0}^n \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n y_k = (U^{r,s} y)_n$$

for all $n \in \mathbb{N}$. By considering the equality above, we deduce that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in b_0^{r,s}$ or $b_c^{r,s}$ if and only if $U^{r,s} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c . This shows that $a = (a_n) \in \{b_0^{r,s}\}^\alpha = \{b_c^{r,s}\}^\alpha$ if and only if $U^{r,s} \in (c_0 : \ell_1) = (c : \ell_1)$. If we combine this and Lemma 3.3(i), we obtain

$$a = (a_n) \in \{b_0^{r,s}\}^\alpha = \{b_c^{r,s}\}^\alpha \Leftrightarrow \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right| < \infty.$$

Therefore, $\{b_0^{r,s}\}^\alpha = \{b_c^{r,s}\}^\alpha = v_1^{r,s}$. This completes the proof. □

Theorem 3.5 Define the sets $v_2^{r,s}$, $v_3^{r,s}$, and $v_4^{r,s}$ by

$$v_2^{r,s} = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^n \left| \sum_{j=k}^n \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\},$$

$$v_3^{r,s} = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$v_4^{r,s} = \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=k}^n \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \text{ exists} \right\}.$$

Then the following statements hold.

- (I) $\{b_0^{r,s}\}^\beta = v_2^{r,s} \cap v_3^{r,s}$,
- (II) $\{b_c^{r,s}\}^\beta = v_2^{r,s} \cap v_3^{r,s} \cap v_4^{r,s}$,
- (III) $\{b_0^{r,s}\}^\gamma = \{b_c^{r,s}\}^\gamma = v_2^{r,s}$.

Proof Let $a = (a_n) \in w$. If we bear in mind the sequence $x = (x_k)$ that is defined in proof of Theorem 2.2, we have

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \left[\frac{1}{r^k} \sum_{j=0}^k \binom{k}{j} (-s)^{k-j} (r+s)^j y_j \right] a_k \\ &= \sum_{k=0}^n \left[\sum_{j=k}^n \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right] y_k \\ &= (H^{r,s} y)_n \end{aligned}$$

for all $n \in \mathbb{N}$, where the matrix $H^{r,s} = (h_{nk}^{r,s})$ is defined by

$$h_{nk}^{r,s} = \begin{cases} \sum_{j=k}^n \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Then:

(I) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in b_0^{r,s}$ if and only if $H^{r,s}y \in c$ whenever $y = (y_k) \in c_0$. This shows that $a = (a_k) \in \{b_0^{r,s}\}^\beta$ if and only if $H^{r,s} \in (c_0 : c)$. If we combine this and Lemma 3.3(ii), we conclude that

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^n |h_{nk}^{r,s}| < \infty, \tag{3.5}$$

$$\lim_{n \rightarrow \infty} h_{nk}^{r,s} \text{ exists for each } k \in \mathbb{N}. \tag{3.6}$$

These results show that $\{b_0^{r,s}\}^\beta = v_2^{r,s} \cap v_3^{r,s}$.

(II) In a similar way, we obtain $a = (a_k) \in \{b_c^{r,s}\}^\beta$ if and only if $H^{r,s} \in (c : c)$. If we combine this and Lemma 3.3(iii), we deduce that (3.5), (3.6) hold and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n h_{nk}^{r,s} \text{ exists,} \tag{3.7}$$

which shows that $\{b_c^{r,s}\}^\beta = v_2^{r,s} \cap v_3^{r,s} \cap v_4^{r,s}$.

(III) $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in b_0^{r,s}$ or $b_c^{r,s}$ if and only if $H^{r,s}y \in \ell_\infty$ whenever $y = (y_k) \in c_0$ or c . This means that $a = (a_k) \in \{b_0^{r,s}\}^\gamma = \{b_c^{r,s}\}^\gamma$ if and only if $H^{r,s} \in (c_0 : \ell_\infty) = (c : \ell_\infty)$. By combining this and Lemma 3.3(iv), we deduce that (3.5) holds. Hence, $\{b_0^{r,s}\}^\gamma = \{b_c^{r,s}\}^\gamma = v_2^{r,s}$. This completes the proof. \square

Theorem 3.6 $\{b_0^{r,s}\}^*$ and $\{b_c^{r,s}\}^*$ are equivalent to ℓ_1 .

Proof To avoid a repetition of similar statements, the proof of the theorem is given for only the binomial sequence space $b_c^{r,s}$. For the proof, the existence of a linear surjective norm preserving mapping $L : \{b_c^{r,s}\}^* \rightarrow \ell_1$ should be shown.

Suppose that $f \in \{b_c^{r,s}\}^*$. Now from Theorem 3.1(b) we know that $\{e, g^{(0)}(r, s), g^{(1)}(r, s), \dots\}$ is a basis for $b_c^{r,s}$, and each $x \in b_c^{r,s}$ has a unique representation of the form

$$x = le + \sum_k [\mu_k - l]g^{(k)}(r, s).$$

By the linearity and continuity of f , we get

$$f(x) = lf(e) + \sum_k [\mu_k - l]f(g^{(k)}(r, s)) \tag{3.8}$$

for all $x \in b_c^{r,s}$. Now define a sequence $x = (x_k) \in b_c^{r,s}$ such that $\|x\|_{b_c^{r,s}} = 1$ as follows:

$$x_k = \begin{cases} \frac{1}{r^k} \sum_{j=0}^k \binom{k}{j} (-s)^{k-j} (r+s)^j \operatorname{sgn} f(g^{(j)}(r, s)), & 0 \leq k \leq n, \\ \frac{1}{r^k} \sum_{j=0}^n \binom{k}{j} (-s)^{k-j} (r+s)^j \operatorname{sgn} f(g^{(j)}(r, s)), & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Hence,

$$|f(x)| = \sum_{k=0}^n |f(g^{(k)}(r, s))| \leq \|f\| \tag{3.9}$$

since $|f(x)| \leq \|f\| \cdot \|x\|$ on $b_c^{r,s}$. It follows from (3.9) that

$$\sum_k |f(g^{(k)}(r, s))| = \sup_{n \in \mathbb{N}} \sum_{k=0}^n |f(g^{(k)}(r, s))| \leq \|f\|.$$

Now we write (3.8) as

$$f(x) = al + \sum_k a_k \mu_k,$$

where

$$a = f(e) - \sum_k f(g^{(k)}(r, s)), \quad a_k = f(g^{(k)}(r, s))$$

and the series $\sum_k f(g^{(k)}(r, s))$ is absolutely convergent.

By taking into account $|\lim_{k \rightarrow \infty} (B^{r,s}x)_k| \leq \|x\|_{b_c^{r,s}} = 1$, we get

$$|f(x)| \leq \|x\|_{b_c^{r,s}} \left(|a| + \sum_k |a_k| \right)$$

whence

$$\|f(x)\| \leq |a| + \sum_k |a_k| \tag{3.10}$$

and $|f(x)| \leq \|f\|$, so we define for all $n \geq 0$,

$$x_k = \begin{cases} \frac{1}{r^k} \sum_{j=0}^k \binom{k}{j} (-s)^{k-j} (r+s)^j \operatorname{sgn} a_j, & 0 \leq k \leq n, \\ \frac{1}{r^k} \sum_{j=0}^n \binom{k}{j} (-s)^{k-j} (r+s)^j \operatorname{sgn} a_j + \frac{1}{r^k} \sum_{j=n+1}^k \binom{k}{j} (-s)^{k-j} (r+s)^j \operatorname{sgn} a, & k > n. \end{cases}$$

Then $x \in b_c^{r,s}$, $\|x\|_{b_c^{r,s}} = 1$, $\lim_{k \rightarrow \infty} (B^{r,s}x)_k = \operatorname{sgn} a$ and so

$$|f(x)| = \left| |a| + \sum_{k=0}^n |a_k| + \operatorname{sgn} a \sum_{k=n+1}^{\infty} a_k \right| \leq \|f\|.$$

We know that $\lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} a_k = 0$ whenever $a = (a_k) \in \ell_1$. Then, if we pass to the limit as $n \rightarrow \infty$ in the last inequality, we have

$$|a| + \sum_k |a_k| \leq \|f\|. \tag{3.11}$$

By combining (3.10) and (3.11), we conclude that

$$\|f\| = |a| + \sum_k |a_k|,$$

which is the norm on ℓ_1 .

Let us define a transformation L such that $L : \{b_c^{r,s}\}^* \rightarrow \ell_1, L(f) = (a, a_0, a_1, \dots)$. Then we have

$$\|L(f)\| = |a| + |a_0| + |a_1| + \dots = \|f\|$$

$\|L(f)\|$ being the ℓ_1 norm. Therefore, L is norm preserving. It is obvious that L is surjective and linear. This completes the proof. \square

4 Some matrix classes related to the binomial sequence spaces

In this chapter, we characterize some matrix classes related to the binomial sequence spaces $b_0^{r,s}$ and $b_c^{r,s}$. Now, we start with two lemmas which are required in the proof of the *theorems.

Lemma 4.1 (see [2]) *Each matrix map between BK-spaces is continuous.*

Lemma 4.2 (see [17]) *Let X, Y be any two sequence spaces, A be an infinite matrix and B be a triangle matrix. Then $A \in (X : Y_B)$ if and only if $BA \in (X : Y)$.*

For brevity of notations, here and in the following, we prefer to use

$$t_{nk}^{r,s} = \sum_{j=k}^n \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj}$$

for all $n, k \in \mathbb{N}$.

Theorem 4.3 *Let $A = (a_{nk})$ be an infinite matrix with complex entries. Then the following statements hold.*

(I) $A \in (b_c^{r,s} : \ell_p)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} t_{nk}^{r,s} \right|^p < \infty, \tag{4.1}$$

$$t_{nk}^{r,s} \text{ exists for all } k, n \in \mathbb{N}, \tag{4.2}$$

$$\sum_k t_{nk}^{r,s} \text{ converges for all } n \in \mathbb{N}, \tag{4.3}$$

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^m \left| \sum_{j=k}^m \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj} \right| < \infty, \quad n \in \mathbb{N}, \tag{4.4}$$

where $1 \leq p < \infty$.

(II) $A \in (b_c^{r,s} : \ell_\infty)$ if and only if (4.2) and (4.4) hold, and

$$\sup_{n \in \mathbb{N}} \sum_k |t_{nk}^{r,s}| < \infty. \tag{4.5}$$

Proof Given a sequence $x = (x_k) \in b_c^{r,s}$, we suppose that the conditions (4.1)-(4.4) hold. Then, by taking into account Theorem 3.5(II), we conclude that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{b_c^{r,s}\}^\beta$ for all $n \in \mathbb{N}$. Thus, the A -transform of x exists. Let us consider a matrix $U^{r,s} = (u_{nk}^{r,s})$ defined by $u_{nk}^{r,s} = t_{nk}^{r,s}$ for all $n, k \in \mathbb{N}$. Since $U^{r,s} = (u_{nk}^{r,s})$ satisfies Lemma 3.3(v), we deduce that $U^{r,s} = (u_{nk}^{r,s}) \in (c : \ell_p)$.

Now, we consider the following equality:

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^m \sum_{j=k}^m \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj} y_k \tag{4.6}$$

for all $n, m \in \mathbb{N}$. If we take the limit (4.6) side by side as $m \rightarrow \infty$, we obtain

$$\sum_k a_{nk} x_k = \sum_k t_{nk}^{r,s} y_k. \tag{4.7}$$

By taking the ℓ_p -norm (4.7) side by side, we have

$$\|Ax\|_{\ell_p} = \|U^{r,s}y\|_{\ell_p} < \infty.$$

Therefore $Ax \in \ell_p$ and so $A \in (b_c^{r,s} : \ell_p)$.

Conversely, suppose that $A \in (b_c^{r,s} : \ell_p)$. It is known that $b_c^{r,s}$ and ℓ_p are BK-spaces. If we combine this fact and Lemma 4.1, we deduce that there is a constant $M > 0$ such that

$$\|Ax\|_{\ell_p} \leq M\|x\|_{b_c^{r,s}} \tag{4.8}$$

for all $x \in b_c^{r,s}$.

Now, we define a sequence $x = (x_k)$ such that $x = \sum_{k \in K} g^{(k)}(r, s)$ for all fixed $k \in \mathbb{N}$, where $g^{(k)}(r, s) = \{g_n^{(k)}(r, s)\}_{n \in \mathbb{N}}$ and $K \in \mathcal{F}$. Since inequality (4.8) holds for all $x \in b_c^{r,s}$, we have

$$\|Ax\|_{\ell_p} = \left(\sum_n \left| \sum_{k \in K} t_{nk}^{r,s} \right|^p \right)^{\frac{1}{p}} \leq M\|x\|_{b_c^{r,s}} = M.$$

Therefore (4.1) holds.

According to the assumption, A can be applied to the binomial sequence space $b_c^{r,s}$. So, it is trivial that the conditions (4.2)-(4.4) hold. This completes the proof of part (I).

If we take Lemma 3.3(iv) instead of Lemma 3.3(v), then part (II) can be proved in a similar way. □

Theorem 4.4 *Let $A = (a_{nk})$ be an infinite matrix with complex entries. Then, $A \in (b_c^{r,s} : c)$ if and only if (4.2), (4.4), and (4.5) hold, and*

$$\lim_{n \rightarrow \infty} t_{nk}^{r,s} = \alpha_k \quad \text{for all } k \in \mathbb{N}, \tag{4.9}$$

$$\lim_{n \rightarrow \infty} \sum_k t_{nk}^{r,s} = \alpha. \tag{4.10}$$

Proof Suppose that A satisfies the conditions (4.2), (4.4), (4.5), (4.9), and (4.10). Given an arbitrary sequence $x = (x_k) \in b_c^{r,s}$ with $\lim_{k \rightarrow \infty} x_k = l$, then Ax exists. Since $B^{r,s} = (b_{nk}^{r,s})$ is regular and $y = (y_k)$ is connected with the sequence $x = (x_k)$ by equation (2.2), we obtain $y = (y_k) \in c$ such that $\lim_{k \rightarrow \infty} y_k = l$.

By considering the conditions (4.5) and (4.9), we have

$$\sum_{j=0}^k |\alpha_j| \leq \sup_{n \in \mathbb{N}} \sum_j |t_{nj}^{r,s}| < \infty$$

for all $k \in \mathbb{N}$. This shows us that $(\alpha_k) \in \ell_1$. Bearing in mind the condition (4.7), we have

$$\sum_k a_{nk} x_k = \sum_k t_{nk}^{r,s} (y_k - l) + l \sum_k t_{nk}^{r,s}, \quad n \in \mathbb{N}. \tag{4.11}$$

By taking into account the conditions (4.5), (4.9), and (4.10), if we take the limit (4.11) side by side as $n \rightarrow \infty$, we write

$$\lim_{n \rightarrow \infty} (Ax)_n = \sum_k \alpha_k (y_k - l) + l\alpha, \tag{4.12}$$

which means that $A \in (b_c^{r,s} : c)$.

On the contrary, suppose that $A \in (b_c^{r,s} : c)$. It is well known that every convergent sequence is also bounded, namely $c \subset \ell_\infty$. By combining this fact and Theorem 4.3(II), we deduce that the necessity of the conditions (4.2), (4.4), and (4.5) holds. Since Ax exists and belongs to c for all $x \in b_c^{r,s}$, if we take $g^{(k)}(r, s) = \{g_n^{(k)}(r, s)\}_{n \in \mathbb{N}}$ instead of an arbitrary sequence $x = (x_k)$, we deduce that $Ag^{(k)}(r, s) = \{t_{nk}^{r,s}\}_{n \in \mathbb{N}} \in c$ for all $k \in \mathbb{N}$. This shows us that the necessity of (4.9) holds.

Moreover, if we take $x = e$ in (4.7), we obtain $Ax = \{\sum_k t_{nk}^{r,s}\}_{n \in \mathbb{N}} \in c$. The last result is the necessity of (4.10). This completes the proof. □

Corollary 4.5 *Let $A = (a_{nk})$ be an infinite matrix with complex entries. Then $A \in (b_c^{r,s} : c)_m$ if and only if the conditions (4.2), (4.4), and (4.5) hold, and the conditions (4.9) and (4.10) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$ and $\alpha = m$, in turn.*

Lemma 4.6 (see [18]) *Let $A = (a_{nk})$ be an infinite matrix with complex entries. Then $A \in (b_\infty^{r,s} : c)$ if and only if (4.5) and (4.9) hold, and*

$$\lim_{n \rightarrow \infty} \sum_k |t_{nk}^{r,s}| = \sum_k \left| \lim_{n \rightarrow \infty} t_{nk}^{r,s} \right|, \tag{4.13}$$

$$\lim_{m \rightarrow \infty} \sum_k \left| \sum_{j=k}^m \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_{nj} \right| = \sum_k |t_{nk}^{r,s}|, \quad n \in \mathbb{N}. \tag{4.14}$$

Theorem 4.7 $(b_c^{r,s} : c)_m \cap (b_\infty^{r,s} : c) = \emptyset$.

Proof We suppose that $(b_c^{r,s} : c)_m \cap (b_\infty^{r,s} : c) \neq \emptyset$. Then there is at least a matrix $A = (a_{nk})$ such that the conditions of Corollary 4.5 and Lemma 4.6 hold for $A = (a_{nk})$. If we consider the conditions (4.9) and (4.13), we conclude that

$$\lim_{n \rightarrow \infty} \sum_k |t_{nk}^{r,s}| = 0.$$

This result contradicts the condition (4.10). So, the classes $(b_c^{r,s} : c)_m$ and $(b_\infty^{r,s} : c)$ are disjoint. This last step completes the proof of the theorem. □

Now, by using Lemma 4.2, we can give some more results.

Corollary 4.8 *Given an infinite matrix $A = (a_{nk})$ with complex entries, we define a matrix $E^{u,v} = (e_{nk}^{u,v})$ as follows:*

$$e_{nk}^{u,v} = \frac{1}{(u+v)^n} \sum_{j=0}^n \binom{n}{j} v^{n-j} u^j a_{jk}$$

for all $n, k \in \mathbb{N}$, where $u, v \in \mathbb{R}$ and $uv > 0$. Then the necessary and sufficient conditions in order that A belongs to any of the classes $(b_c^{r,s} : b_\infty^{u,v})$, $(b_c^{r,s} : b_p^{u,v})$, $(b_c^{r,s} : b_c^{u,v})$ and $(b_c^{r,s} : b_c^{u,v})_m$ are obtained by taking $E^{u,v} = (e_{nk}^{u,v})$ instead of $A = (a_{nk})$ in the required ones in Theorems 4.3, 4.4 and Corollary 4.5, where $b_p^{u,v}$ and $b_\infty^{u,v}$ are defined in [18].

Corollary 4.9 *Given an infinite matrix $A = (a_{nk})$ with complex entries, we define a matrix $C = (c_{nk})$ as follows:*

$$c_{nk} = \frac{1}{n+1} \sum_{j=0}^n a_{jk}$$

for all $n, k \in \mathbb{N}$. Then the necessary and sufficient conditions in order that A belongs to any of the classes $(b_c^{r,s} : X_\infty)$, $(b_c^{r,s} : X_p)$, $(b_c^{r,s} : \tilde{c})$, and $(b_c^{r,s} : \tilde{c})_m$ are obtained by taking $C = (c_{nk})$ instead of $A = (a_{nk})$ in the required ones in Theorems 4.3, 4.4 and Corollary 4.5, where X_p , X_∞ , and \tilde{c} are defined in [4] and [5], respectively.

Corollary 4.10 *Given an infinite matrix $A = (a_{nk})$ with complex entries, we define two matrices $C = (c_{nk})$ and $E = (e_{nk})$ as follows:*

$$c_{nk} = a_{nk} - a_{n+1,k} \quad \text{and} \quad e_{nk} = a_{nk} - a_{n-1,k}$$

for all $n, k \in \mathbb{N}$. Then the necessary and sufficient conditions in order that A belongs to any of the classes $(b_c^{r,s} : \ell_\infty(\Delta))$, $(b_c^{r,s} : c(\Delta))$, $(b_c^{r,s} : \ell_p(\Delta))$, and $(b_c^{r,s} : c(\Delta))_m$ are obtained by taking $C = (c_{nk})$ or $E = (e_{nk})$ instead of $A = (a_{nk})$ in the required ones in Theorems 4.3, 4.4 and Corollary 4.5, where $\ell_\infty(\Delta)$ and $c(\Delta)$ are defined in [19] and $\ell_p(\Delta)$ is defined in [17].

Corollary 4.11 *Given an infinite matrix $A = (a_{nk})$ with complex entries, we define a matrix $E = (e_{nk})$ as follows:*

$$e_{nk} = \frac{1}{n+1} \sum_{j=0}^n (1+t^j) a_{jk}$$

for all $n, k \in \mathbb{N}$, where $0 < t < 1$. Then the necessary and sufficient conditions in order that A belongs to any of the classes $(b_c^{r,s} : a_\infty^t)$, $(b_c^{r,s} : a_p^t)$, $(b_c^{r,s} : a_c^t)$, and $(b_c^{r,s} : a_c^t)_m$ are obtained by taking $E = (e_{nk})$ instead of $A = (a_{nk})$ in the required ones in Theorems 4.3, 4.4 and Corollary 4.5, where a_∞^t , a_p^t , and a_c^t are defined in [20] and [6], respectively.

Corollary 4.12 *Given an infinite matrix $A = (a_{nk})$ with complex entries, we define a matrix $E = (e_{nk})$ as follows:*

$$e_{nk} = \sum_{j=0}^n a_{jk}$$

for all $n, k \in \mathbb{N}$. Then the necessary and sufficient conditions in order that A belongs to any of the classes $(b_c^{r,s} : bs)$, $(b_c^{r,s} : cs)$, and $(b_c^{r,s} : cs)_m$ are obtained by taking $E = (e_{nk})$ instead of $A = (a_{nk})$ in the required ones in Theorems 4.3, 4.4 and Corollary 4.5.

5 Conclusion

By considering the definition of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$, we deduce that $B^{r,s} = (b_{nk}^{r,s})$ reduces in the case $r + s = 1$ to the $E^r = (e_{nk}^r)$, which is called the method of Euler means of order r . So, our results obtained from the matrix domain of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ are more general and more extensive than the results on the matrix domain of the Euler

means of order r . Moreover, the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is not a special case of the weighed mean matrices. So, this paper filled up a gap in the existent literature.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

The authors read and approved the final manuscript.

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