Gao et al. *Journal of Inequalities and Applications* (2016) 2016:310
DOI 10.1186/s13660-016-1254-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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The quadratic variation for mixed-fractional Brownian motion

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Abstract

Let $W = \lambda B + \nu B^H$ be a mixed-fractional Brownian motion with Hurst index $0 < H < \frac{1}{2}$ and $\lambda, \nu \neq 0$. In this paper we study the *quadratic covariation* $[f(W), W]^{(H)}$ defined by

$$[f(W), W]_t^{(H)} := \lim_{\varepsilon \downarrow 0} \frac{1}{\nu^2 \varepsilon^{2H}} \int_0^t \{f(W_{s+\varepsilon}) - f(W_s)\}(W_{s+\varepsilon} - W_s)\, d\eta_s$$

in probability, where f is a Borel function and $\eta_s = \lambda^2 s + \nu^2 s^{2H}$. For some suitable function f we show that the quadratic covariation exists in $L^2(\Omega)$ and the Itô formula

$$F(W_t) = F(0) + \int_0^t f(W_s)\, dW_s + \frac{1}{2}[f(W), W]_t^{(H)}$$

holds for all absolutely continuous function F with $F' = f$, where the integral is the Skorohod integral with respect to W.

MSC: Primary 60G15; 60H05; secondary 60H07

Keywords: mixed fractional Brownian motion; Malliavin calculus; local time; fractional Itô formula

1 Introduction

As is well known, in recent years, there has been considerable interest in studying fractional Brownian motion (in short, fBm) due to its simple properties and some applications in various scientific areas such as telecommunications, turbulence, image processing, and finance. For some surveys on fBm we refer to Biagini *et al.* [1], Hu [2], Mishura [3], Nourdin [4], Nualart [5], Tudor [6] and the references therein. On the other hand, in order to make some better applications of fBm in finance, many authors have proposed to use the mixed-fBm as stochastic models. For this purpose, we refer to Bender *et al.* [7], Cheridito [8, 9], El-Nouty [10], He and Chen [11], Mishura [3], Shokrollahi and Kiliiman [12], Prakasa Rao [13] and the references therein. The so-called mixed-fBm W with Hurst index $H \in (0,1)$ is a stationary Gaussian process with the following decomposition:

$$W_t = \lambda B_t + \nu B_t^H, \quad t \geq 0,$$

where B^H is a standard fBm with Hurst index $H \in (0,1)$, B is a standard Brownian motion independent of B^H and $\lambda, \nu \in \mathbb{R} \setminus \{0\}$. It is important to note that Brownian motion B in the mixed-fBm W can offset some irregularity of fBm such that the theoretical issues for them became relatively easy and the application issues became also relatively favorable. For example, the mixed-fBm is equivalent to a standard Brownian motion when $\frac{3}{4} < H < 1$. Therefore, it seems interesting to study the mixed-fBm. In this paper, we consider the quadratic variation of mixed-fBm W.

Recall that a fBm on $\mathbb{R}$ with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in [0,T]\}$ such that $B_0^H = 0$ and

$$EB_t^H = 0, \qquad E\bigl[B_t^H B_s^H\bigr] = \frac{1}{2}\bigl[t^{2H} + s^{2H} - |t-s|^{2H}\bigr]$$

for all $t, s \in [0,T]$. When $H = 1/2$, B^H coincides with the standard Brownian motion B, and when $H \neq \frac{1}{2}$ it is neither a semi-martingale nor a Markov process. We know also that the usual quadratic variation $[B^H, B^H]$ equals zero when $H > \frac{1}{2}$, and it does not exist when $H < \frac{1}{2}$. However, we can easily see that

$$[W,W]_t = \begin{cases} \lambda^2 t, & H > \frac{1}{2}, \\ (\lambda^2 + \nu^2)t, & H = \frac{1}{2}, \\ +\infty, & H < \frac{1}{2}, \end{cases}$$

for all $t > 0$. This simple result points out some irregularities of fBm cannot be offset by Brownian motion when $0 < H < \frac{1}{2}$. Motivated by the above fact, in this paper we consider the substitution of quadratic variation when $0 < H < \frac{1}{2}$. We shall introduce a substitution of quadratic variation of W and study some related questions, and the idea follows from Yan *et al.* [14]. For some continuous processes with infinite quadratic variation, Errami and Russo [15] and Russo and Vallois [16] introduced the α-variation and n-covariation.

Definition 1.1 Let $0 < H < 1$ and let f be a measurable function on $\mathbb{R}$. Denote

$$J_\varepsilon(f,t) := \frac{1}{\nu^2 \varepsilon^{(2H)\wedge 1}} \int_0^t \bigl\{f(W_{s+\varepsilon}) - f(W_s)\bigr\}(W_{s+\varepsilon} - W_s)\, d\eta_s \tag{1.1}$$

for all $t \in [0,T]$ and $\varepsilon > 0$, where $\eta_s = \lambda^2 s + \nu^2 s^{2H}$. The limit

$$\bigl[f(W), W\bigr]^{(H)}_{\cdot} := \lim_{\varepsilon \downarrow 0} J_\varepsilon(f, \cdot)$$

is called the *fractional quadratic covariation* of $f(W)$ and W, provided the limit, which exists in probability.

Clearly, when $H \geq \frac{1}{2}$ the fractional quadratic covariation coincides with the usual quadratic covariation. However, for the case $0 < H < \frac{1}{2}$, the fractional quadratic covariation is very different from the usual quadratic covariation. In the present paper our main object is to introduce the existence of the fractional quadratic covariation and it is organized as follows. In Section 2 we present some preliminaries, and in particular we give some technical estimates associated with mixed-fBm. In Section 3, we prove the existence of the fractional quadratic covariation for $0 < H < \frac{1}{2}$. To prove the existence of the fractional quadratic covariation, we consider the decomposition

$$\begin{aligned}&\frac{1}{\nu^2\varepsilon^{(2H)\wedge 1}}\int_0^t \big\{f(W_{s+\varepsilon})-f(W_s)\big\}(W_{s+\varepsilon}-W_s)\,d\eta_s\\&\quad=\frac{1}{\nu^2\varepsilon^{(2H)\wedge 1}}\int_0^t f(W_{s+\varepsilon})(W_{s+\varepsilon}-W_s)\,d\eta_s-\frac{1}{\nu^2\varepsilon^{(2H)\wedge 1}}\int_0^t f(W_s)(W_{s+\varepsilon}-W_s)\,d\eta_s \qquad (1.2)\end{aligned}$$

and by estimating the two terms of the right hand side in (1.2), respectively, we construct a Banach space $\mathbb{H}$ of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{\mathbb{H}} = \sqrt{\int_0^T \frac{d\eta_s}{\sqrt{2\pi\eta_s}}\int_{\mathbb{R}} \big|f(x)\big|^2 e^{-\frac{x^2}{2\eta_s}}\,dx} < \infty.$$

We show that the fractional quadratic covariation exists in $L^2(\Omega)$ for all $t \in [0, T]$ if $f \in \mathbb{H}$. In Section 4, we introduce an Itô formula including the fractional quadratic covariation and give an integral with respect to local time of mixed-fBm.

2 Stochastic calculus for mixed-fBm

In this section, we briefly recall some basic results of mixed-fBm and give some basic estimates.

2.1 Stochastic calculus for mixed-fBm

We refer to Alós *et al.* [17], Nualart [5] and the references therein for more details. Throughout this paper we assume that $0 < H < \frac{1}{2}$ is arbitrary but fixed and we let $W_t = \lambda B_t + \nu B_t^H, 0 \le t \le T$ be a one-dimensional mixed-fBm with Hurst index H and $\lambda, \nu \neq 0$. Then we have

$$R_H(t,s) := EW_tW_s = \frac{1}{2}\lambda^2(t\wedge s) + \frac{1}{2}\nu^2\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) \qquad (2.1)$$

for all $t, s \ge 0$.

Denote by $\mathcal{E}$ the linear space generated by the indicator functions $1_{[0,t]}, t \in [0,T]$. Let $\mathcal{H}$ and $\mathcal{H}_0$ be the completions of the linear space $\mathcal{E}$ with respect to the inner products

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathcal{H}} = R_H(t,s)$$

and

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathcal{H}_0} = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big),$$

respectively. Then $\mathcal{H} = \mathcal{H}_0 \cap L^2([0,T])$. For $\varphi \in \mathcal{E}$, by linearity and $1_{[0,t]} \to B_t^{a,b}$ for all $t \in [0,T]$, we can define the map

$$\varphi \mapsto W(\varphi) := \int_0^T \varphi(s)\,dW_s^H = \lambda\int_0^t \varphi(s)\,dB_s + \nu\int_0^t \varphi(s)\,dB_s^H.$$

Then the map is an isometry from $\mathcal{E}$ to the Gaussian space generated by mixed-fBm W, and, moreover, it can be extended to $\mathcal{H}$. The map

$$W(\varphi) = \int_0^T \varphi(s)\,dW_s$$

is called the Wiener integral of $\varphi \in \mathcal{H}$ with respect to the mixed-fBm W, and we have

$$E\left|\int_0^T \varphi(s)\,dW_s\right|^2 = \|\varphi\|^2_{\mathcal{H}}.$$

Consider now the set $\mathcal{S}_W$ of smooth functionals

$$F = f\bigl(W(\varphi_1), W(\varphi_2), \ldots, W(\varphi_n)\bigr), \tag{2.2}$$

where the function f and all its derivatives are bounded (denoted by $f \in C_b^\infty(\mathbb{R}^n)$) and $\varphi_i \in \mathcal{H}$. As usual, we can define the *Malliavin derivative* (operator) $\mathcal{D}_W$ and the *divergence operator* (the Skorohod integral) δ_W with respect to the mixed-fBm W. For the functional F of the form (2.2) we define

$$\mathcal{D}_W F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\bigl(W(\varphi_1), W(\varphi_2), \ldots, W(\varphi_n)\bigr)\varphi_j,$$

and we can show that the operator $\mathcal{D}_W$ is a closable operator from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$. Denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}_W$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|\mathcal{D}_W F\|^2_{\mathcal{H}}}.$$

The operator δ_W is the adjoint of derivative operator $\mathcal{D}_W$. A random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain $\mathrm{Dom}(\delta_W)$ of the divergence operator δ_W, if

$$E\bigl|\langle \mathcal{D}_W F, u\rangle_{\mathcal{H}}\bigr| \le c_u \|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}_W$, and we have $\mathbb{D}^{1,2} \subset \mathrm{Dom}(\delta_W)$. In this case, the operator $\delta_W(u)$ is determined by the so-called duality relationship

$$E\bigl[F\delta_W(u)\bigr] = E\langle \mathcal{D}_W F, u\rangle_{\mathcal{H}} \tag{2.3}$$

for any $u \in \mathbb{D}^{1,2}$. Moreover, we can localize the operators $\mathcal{D}_W$ and δ_W via their domains. That is, if $\{(\Omega_n, F^n), n = 1, 2, \ldots\}$ localizes F in $\mathbb{D}^{1,2}$, then $\mathcal{D}_W F$ is defined without ambiguity by $\mathcal{D}_W F = \mathcal{D}_W F^n$ on Ω_n, $n \ge 1$, and

$$\Omega_n \uparrow \Omega, \qquad F = F_n, \quad n \ge 1,$$

almost surely. Similarly, if $\{(\Omega_n, u^n), n = 1, 2, \ldots\}$ localizes u, then the divergence $\delta_W(u)$ is defined as a random variable determined by the conditions

$$\delta_W(u) = \delta_W\bigl(u^n\bigr)$$

on Ω_n for all $n \geq 1$. We will also use the following notations:

$$\delta_W(u) = \int_0^T u_s \, d_s = \lambda \int_0^t u_s \, dB_s + \nu \int_0^t u_s \, dB_s^H$$

and

$$\int_0^t u_s \, dW_s^H = \delta_W(u 1_{[0,t]})$$

for all $t \in [0, T]$. The following Itô formula holds (see Alós *et al.* [17]).

Theorem 2.1 *Let $F \in C^2(\mathbb{R})$ such that*

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \leq \kappa e^{\beta x^2}, \tag{2.4}$$

where κ and β are positive constants with $\beta < \frac{1}{4(\lambda^2 T + \nu^2 T^{2H})}$. Then we have

$$F(W_t) = F(0) + \int_0^t \frac{d}{dx} F(W_s) \, ds + \frac{1}{2} \int_0^t \frac{d^2}{dx^2} F(W_s) \, d\eta_s$$

for all $t \in [0, T]$, where $\eta_s = \lambda^2 s + \nu^2 s^{2H}$.

Finally, from Theorem 6.4 in Geman and Horowitz [18] we can easily see that the mixed-fBm W with Hurst index $H \in (0, 1)$ admits a bi-continuous local time L^H such that

$$L^H(x, t) = \int_0^t \delta(W_s - x) \, ds.$$

Thus, we can define its weighted local time as follows:

$$\mathcal{L}^H(x, t) = \int_0^t \left(\lambda + 2H\nu s^{2H-1}\right) dL^H(x, ds) = \int_0^t \delta(W_s - x) \, d\eta_s,$$

where δ is the Dirac delta function (for the local time of fractional Brownian motion, see, for example, Coutin *et al.* [19] and Hu *et al.* [20]).

2.2 Some basic estimates

In this subsection we will introduce some inequalities associated with mixed-fBm. For convenience, in this paper we assume that C is a positive constant and its value may be different in different positions, and, moreover, we use also the notation $F \asymp G$ to denote the following relationship:

$$c_1 F(x) \leq G(x) \leq c_2 F(x)$$

for some positive constants c_1 and c_2.

Lemma 2.1 *For all $s \geq r > 0$ and $0 < H < 1$ we have*

$$\eta_r \eta_s - \mu^2 \asymp r^{(2H)\wedge 1}(s - r)^{(2H)\wedge 1}, \tag{2.5}$$

where $\mu = E(W_r W_s^H)$ and $\eta_s = \lambda^2 s + \nu^2 s^{2H}$.

Proof Clearly, we have

$$\begin{aligned}\eta_r\eta_s-\mu^2&=\left(\lambda^2 s+\nu^2 s^{2H}\right)\left(\lambda^2 r+\nu^2 r^{2H}\right)-\left(\lambda^2 r+\frac{1}{2}\nu^2\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)\right)^2\\&=\lambda^4 r(s-r)+\lambda^2\nu^2 r^{2H}(s-r)+\lambda^2\nu^2 r(s-r)^{2H}\\&\quad+\nu^4\left(s^{2H}r^{2H}-\frac{1}{4}\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)^2\right)\end{aligned}$$

for all $s\geq r>0$ and $0<H<1$. Thus, (2.5) follows from the estimates:

$$s^{2H}r^{2H}-\frac{1}{4}\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)^2\asymp r^{2H}(s-r)^{2H}$$

for all $s>r\geq 0$. But this is introduced in Yan et al. [14] and the lemma follows. □

Lemma 2.2 *Let* $0<H<\frac{1}{2}$. *For all* $0<r\leq s\leq T$ *we have*

$$\eta_s-\mu\asymp(s-r)^{2H} \tag{2.6}$$

and

$$0\leq\eta_r-\mu\asymp\left(\frac{r}{s}\right)^{2H}(s-r)^{2H}. \tag{2.7}$$

Proof For (2.6) we have

$$\begin{aligned}\eta_s-\mu&=\lambda^2 s+\nu^2 s^{2H}-\lambda^2 r-\frac{1}{2}\nu^2\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)\\&=\lambda^2(s-r)+\nu^2\left(s^{2H}-\frac{1}{2}\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)\right)\\&=\lambda^2(s-r)+\frac{1}{2}\nu^2\left(s^{2H}-r^{2H}\right)+\frac{1}{2}\nu^2(s-r)^{2H}\asymp(s-r)^{(2H)\wedge 1}.\end{aligned}$$

For (2.7) we also have

$$\begin{aligned}\eta_r-\mu&=\lambda^2 r+\nu^2 r^{2H}-\left(\lambda^2 r+\frac{1}{2}\nu^2\left(s^{2H}+r^{2H}-(s-r)^{2H}\right)\right)\\&=\frac{1}{2}\nu^2\left((s-r)^{2H}-s^{2H}+r^{2H}\right)\\&=\frac{1}{2}\nu^2 s^{2H}\left((1-x)^{2H}-\left(1-x^{2H}\right)\right)\end{aligned}$$

with $x=\frac{r}{s}$. Thus, (2.7) follows from the estimates

$$(1-x)^{2H}-\left(1-x^{2H}\right)\asymp x^{2H}(1-x)^{2H} \tag{2.8}$$

for all $0\leq x\leq 1$. But (2.8) can be introduced by the convergence

$$\lim_{x\downarrow 0}\frac{(1-x)^{2H}-(1-x^{2H})}{x^{2H}(1-x)^{2H}}=1,\qquad\lim_{x\uparrow 1}\frac{(1-x)^{2H}-(1-x^{2H})}{x^{2H}(1-x)^{2H}}=1,$$

and the continuity of the functions

$$x \mapsto \frac{(1-x)^{2H} - (1-x^{2H})}{x^{2H}(1-x)^{2H}}, \qquad x \mapsto \frac{x^{2H}(1-x)^{2H}}{(1-x^{2H}) - (1-x)^{2H}},$$

with $x \in [0,1]$. This completes the proof. □

Lemma 2.3 *Let* $0 < r' < s' < r < s$ *and* $0 < H < \frac{1}{2}$. *We have*

$$\left|E\big[(W_s - W_r)(W_{s'} - W_{r'})\big]\right| \le C\frac{(s-r)^{(1-\alpha)H+\alpha}(s'-r')^{(1-\alpha)H+\alpha}}{(r-s')^{\alpha(2-2H)}} \tag{2.9}$$

for all $\alpha \in [0,1]$.

Proof Clearly, we have

$$E\big[(W_s - W_r)(W_{s'} - W_{r'})\big] = \nu^2 E\big[\big(B_s^H - B_r^H\big)\big(B_{s'}^H - B_{r'}^H\big)\big]$$

by the independence. Thus, the lemma follows from Yan *et al.* [14]. □

Lemma 2.4 *For* $t > s > r > 0$ *and* $0 < H < \frac{1}{2}$ *we have*

$$\begin{aligned}
&\left|E\big[W_t(W_t - W_s)\big]\right| \le \nu^2(t-s)^{2H},\\
&\left|E\big[W_t(W_s - W_r)\big]\right| \le \nu^2(s-r)^{2H},\\
&\left|E\big[W_r(W_t - W_s)\big]\right| \le \nu^2(t-s)^{2H}.
\end{aligned}$$

Proof The lemma is a simple exercise. □

Let $\varphi(x,y)$ denote the density function of (W_s, W_r). That is,

$$\varphi(x,y) = \frac{1}{2\pi\rho}\exp\left\{-\frac{1}{2\rho^2}\big(\eta_r x^2 - 2\mu xy + \eta_s y^2\big)\right\},$$

where $\mu = E(W_s W_r)$ and $\rho^2 = \eta_r\eta_s - \mu^2$.

Lemma 2.5 *Let* $0 < H < \frac{1}{2}$ *and let* $f \in C^1(\mathbb{R})$ *have compact support. Then the estimates*

$$\left|E\big[f'(W_s)f'(W_r)\big]\right| \le \frac{\sqrt{\eta_r\eta_s}}{\rho^2}E\big[f(W_s)f(W_r)\big]$$

and

$$\left|E\big[f''(W_s)f(W_r)\big]\right| \le \frac{\eta_r}{\rho^2}E\big[f(W_s)f(W_r)\big]$$

hold for all $0 < r < s \le T$.

Proof This a simple exercise. In fact, we have

$$\begin{aligned}
E\big[f'(W_s)f'(W_r)\big] &= \int_{\mathbb{R}^2} f(x)f(y)\frac{\partial^2}{\partial x\partial y}\varphi(x,y)\,dx\,dy\\
&= \int_{\mathbb{R}^2} f(x)f(y)\left\{\frac{1}{\rho^4}(\eta_s y - \mu x)(\eta_r x - \mu y) + \frac{\mu}{\rho^2}\right\}\varphi(x,y)\,dx\,dy
\end{aligned}$$

by integration by parts. On the other hand, an elementary calculation can show that

$$\begin{aligned}&\int_{\mathbb{R}^2} f^2(y)\left(x-\frac{\mu}{\eta_r}y\right)^2\varphi(x,y)\,dx\,dy\\&\quad=\frac{\rho^2}{\eta_r}\int_{\mathbb{R}} f^2(y)\frac{1}{\sqrt{2\pi\eta_r}}e^{-\frac{y^2}{2\eta_r}}\,dy=\frac{\rho^2}{\eta_r}E\bigl[\bigl|f(W_r)\bigr|^2\bigr]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\rho^4}\int_{\mathbb{R}^2}\bigl|f(x)f(y)(\eta_s y-\mu x)(\eta_r x-\mu y)\bigr|\varphi(x,y)\,dx\,dy\\&\quad\le\frac{\sqrt{\eta_r\eta_s}}{\rho^2}\bigl(E\bigl[\bigl|f(W_s)\bigr|^2\bigr]E\bigl[\bigl|f(W_r)\bigr|^2\bigr]\bigr)^{1/2}\end{aligned}$$

for all $s > r > 0$. This completes the first estimate and similarly, one can also obtain the second estimate. □

3 Existence of the fractional quadratic covariation

In this section, we study the existence of the fractional quadratic covariation. Recall that

$$J_\varepsilon(f,t)=\frac{1}{\nu^2\varepsilon^{2H}}\int_0^t\bigl\{f(W_{s+\varepsilon})-f(W_s)\bigr\}(W_{s+\varepsilon}-W_s)\,d\eta_s$$

for $\varepsilon > 0$ and $t \ge 0$, and

$$\bigl[f(W),W\bigr]_t^{(H)}=\lim_{\varepsilon\downarrow 0}J_\varepsilon(f,t), \tag{3.1}$$

provided the limit exists in probability, where $\eta_t = \lambda^2 t + \nu^2 t^{2H}$.

In order to prove the main theorem we need some preliminaries.

Lemma 3.1 (Gradinaru and Nourdin [21]) *Let g be a continuous function on $\mathbb{R}$ satisfying the condition*

$$\bigl|g(x)-g(y)\bigr|\le C\bigl(1+x^2+y^2\bigr)^\beta|x-y|^\alpha,\quad\forall x,y\in\mathbb{R} \tag{3.2}$$

for some $\beta > 0, 0 < \alpha \le 1$ and let X be a Hölder continuous paths process with index $\gamma \in (0,1)$. Suppose that V is a bounded variation continuous process such that

$$\bigl\|X_\varepsilon^g(t)-V_t\bigr\|_{L^2}^2=O\bigl(\varepsilon^\alpha\bigr)\quad(\varepsilon\to 0) \tag{3.3}$$

for some $\alpha > 0$ and all $t \ge 0$, where

$$X_\varepsilon^g(t)=\int_0^t g\left(\frac{X_{s+\varepsilon}-X_s}{\varepsilon^\gamma}\right)dV_s$$

for $t \ge 0$, $\varepsilon > 0$, then $\lim_{\varepsilon\to 0}X_\varepsilon^g(t)=V_t$ a.s., for any $t \ge 0$, and

$$\lim_{\varepsilon\to 0}\int_0^t Y_s g\left(\frac{X_{s+\varepsilon}-X_s}{\varepsilon^\gamma}\right)dV_s\longrightarrow\int_0^t Y_s\,dV_s\quad a.s., \tag{3.4}$$

uniformly in t on each compact interval for any continuous stochastic process $\{Y_t : t \geq 0\}$, provided g is non-negative.

As an immediate consequence of the above lemma, one can get the next corollary.

Corollary 3.1 *Let $f \in C^1(\mathbb{R})$. We have*

$$[f(W), W]_t^{(H)} = \int_0^t f'(W_s)\, d\eta_s, \tag{3.5}$$

and, in particular, we have

$$[W, W]_t^{(H)} = \eta_t$$

for all $t \geq 0$, where $\eta_t = \lambda^2 t + \nu^2 t^{2H}$.

Proof Denote

$$Z(\varepsilon, t) = \frac{1}{\nu^2 \varepsilon^{2H}} \int_0^t (W_{s+\varepsilon} - W_s)^2\, d\eta_s$$

for all $0 < \varepsilon < t$. By Lemma 3.1 it is enough to prove the next convergence

$$E\big(Z(\varepsilon, t) - \eta_t\big)^2 = O\big(\varepsilon^\beta\big) \quad (\varepsilon \to 0) \tag{3.6}$$

for some $\beta > 0$. In fact, if the convergence (3.6) holds, we then have

$$\begin{aligned} J_\varepsilon(f, t) &= \frac{1}{\nu^2 \varepsilon^{2H}} \int_0^t \big\{f(W_{s+\varepsilon}) - f(W_s)\big\}(W_{s+\varepsilon} - W_s)\, d\eta_s \\ &\sim \frac{1}{\nu^2 \varepsilon^{2H}} \int_0^t f'(W_s)(W_{s+\varepsilon} - W_s)^2\, d\eta_s \longrightarrow \int_0^t f'(W_s)\, d\eta_s \end{aligned}$$

almost surely, as ε tends to zero, by taking $Y_s = f'(W_s)$. This gives (3.5).

Now, let us prove the convergence (3.6). Denote

$$\begin{aligned} G_\varepsilon(s, r) &= E\big((W_{s+\varepsilon} - W_s)^2 - \nu^2 \varepsilon^{2H}\big)\big((W_{r+\varepsilon} - W_r)^2 - \nu^2 \varepsilon^{2H}\big) \\ &= E\big[(W_{s+\varepsilon} - W_s)^2 (W_{r+\varepsilon} - W_r)^2\big] \\ &\quad - \nu^2 \varepsilon^{2H} E(W_{s+\varepsilon} - W_s)^2 - \nu^2 \varepsilon^{2H} E(W_{r+\varepsilon} - W_r)^2 + \nu^4 \varepsilon^{4H} \end{aligned}$$

for all $s, r > 0$ and $\varepsilon > 0$. Notice that

$$E\big[(W_{r+\varepsilon} - W_r)^2\big] = \lambda^2 \varepsilon + \nu^2 \varepsilon^{2H}$$

and

$$\begin{aligned} E\big[(W_{s+\varepsilon} - W_s)^2 (W_{r+\varepsilon} - W_r)^2\big] &= E\big[(W_{s+\varepsilon} - W_s)^2\big] E\big[(W_{r+\varepsilon} - W_r)^2\big] \\ &\quad + 2\big(E\big[(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\big]\big)^2 \\ &= \big(\lambda^2 \varepsilon + \nu^2 \varepsilon^{2H}\big)^2 + 2\big(E\big[(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\big]\big)^2 \end{aligned}$$

for all $s, r, \varepsilon > 0$. We get

$$G_\varepsilon(s,r) = \lambda^4\varepsilon^2 + 2\big(E\big[(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\big]\big)^2$$

for all $s, r > 0$ and $\varepsilon > 0$ and

$$\begin{aligned} &E\big(Z(\varepsilon,t) - \eta_t\big)^2 \\ &= \frac{1}{\nu^4\varepsilon^{4H}}\int_0^t\int_0^t E\big((W_{s+\varepsilon} - W_s)^2 - \nu^2\varepsilon^{2H}\big)\big((W_{r+\varepsilon} - W_r)^2 - \nu^2\varepsilon^{2H}\big)\,d\eta_s\,d\eta_r \\ &= \frac{1}{\nu^4\varepsilon^{4H}}\int_0^t\int_0^t G_\varepsilon(s,r)\,d\eta_s\,d\eta_r \\ &= \frac{\lambda^4}{\nu^4}(\eta_t)^2\varepsilon^{2-4H} + \frac{2}{\nu^4\varepsilon^{4H}}\int_0^t\int_0^t E\big(E\big[(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\big]\big)^2\,d\eta_s\,d\eta_r \\ &= O\big(\varepsilon^\beta\big) \quad (\varepsilon \to 0) \end{aligned}$$

by the inequality (2.9), which gives the convergence (3.6) and the corollary follows. □

Now, we assume that $f \notin C^1(\mathbb{R})$ and discuss the existence of the fractional quadratic covariation $[f(W), W]^{(H)}$ when $0 < H < \frac{1}{2}$. Consider the set

$$\mathbb{H} = \left\{f:\ \text{Borel functions on } \mathbb{R} \text{ such that } \|f\|_{\mathbb{H}}^2 := \int_0^T\int_{\mathbb{R}}\big|f(x)\big|^2 e^{-\frac{x^2}{2\eta_s}}\frac{dx\,d\eta_s}{\sqrt{2\pi\eta_s}} < \infty\right\}.$$

Lemma 3.2 *For $0 < H < \frac{1}{2}$, the set $\mathbb{H}$ is a Banach space $L^2(\mathbb{R}, \mu(dx))$ with*

$$\mu(dx) = dx\int_0^T e^{-\frac{x^2}{2\eta_s}}\frac{d\eta_s}{\sqrt{2\pi\eta_s}},$$

and the set $\mathcal{E}$ of elementary functions on $\mathbb{R}$ is dense in $\mathbb{H}$.

Our main theorem is expounded as follows.

Theorem 3.1 *Let $0 < H < \frac{1}{2}$ and $f \in \mathbb{H}$. Then the fractional quadratic covariation $[f(W), W]^{(H)}$ exists in $L^2(\Omega)$ and*

$$E\big|\big[f(W), W\big]_t^{(H)}\big|^2 \le C\|f\|_{\mathbb{H}}^2 \tag{3.7}$$

for all $t \in [0, T]$.

To show that the theorem holds we consider the following integrals:

$$J_\varepsilon(1,f,t) := \frac{1}{\nu^2\varepsilon^{2H}}\int_0^t f(W_{s+\varepsilon})(W_{s+\varepsilon} - W_s)\,d\eta_s$$

and

$$J_\varepsilon(2,f,t) := -\frac{1}{\nu^2\varepsilon^{2H}}\int_0^t f(W_s)(W_{s+\varepsilon} - W_s)\,d\eta_s$$

for $\varepsilon > 0$ and $t \in [0, T]$. Then we have

$$J_\varepsilon(f,t) = \frac{1}{\nu^2\varepsilon^{2H}}\int_0^t \bigl\{f(W_{s+\varepsilon}) - f(W_s)\bigr\}(W_{s+\varepsilon} - W_s)\,d\eta_s$$
$$= J_\varepsilon(1,f,t) - J_\varepsilon(2,f,t) \tag{3.8}$$

for $\varepsilon > 0$. For simplicity we let $T = 1$ and it is enough to show that the next statements hold:

(a) for all $f \in \mathbb{H}$, $t \in [0,1]$ and $k = 1,2$, we have

$$E\bigl|J_\varepsilon(k,f,t)\bigr|^2 \le C\|f\|^2_{\mathbb{H}}; \tag{3.9}$$

(b) for all $f \in \mathbb{H}$, $t \in [0,1]$ and $k = 1,2$, $\{J_\varepsilon(k,f,t), \varepsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega)$.

Proof of Statement (a) Given $f \in \mathbb{H}$. We have

$$E\bigl|J_\varepsilon(1,f,t)\bigr|^2$$
$$= \frac{1}{\nu^2\varepsilon^{2H}}\int_0^t\int_0^t E\bigl[f(W_s)f(W_r)(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\bigr]\,d\eta_s\,d\eta_r$$

for all $\varepsilon > 0$ and $t \ge 0$. We need to estimate

$$\Delta_\varepsilon(s,r) := \bigl[f(W_s)f(W_r)(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\bigr]$$

for all $\varepsilon > 0$ and $s, r > 0$. By approximating we may assume that $f \in C_0^\infty(\mathbb{R})$ and denote

$$\begin{aligned}
\Lambda_\varepsilon(s,r,1) &:= E\bigl[f(W_s)f(W_r)\bigr]E\bigl[(W_{r+\varepsilon} - W_r)(W_{s+\varepsilon} - W_s)\bigr],\\
\Lambda_\varepsilon(s,r,2) &:= E\bigl[f'(W_s)f'(W_r)\bigr]E\bigl[W_s(W_{r+\varepsilon} - W_r)\bigr]E\bigl[W_r(W_{s+\varepsilon} - W_s)\bigr],\\
\Lambda_\varepsilon(s,r,3) &:= E\bigl[f''(W_s)f(W_r)\bigr]E\bigl[W_s(W_{r+\varepsilon} - W_r)\bigr]E\bigl[W_s(W_{s+\varepsilon} - W_s)\bigr],\\
\Lambda_\varepsilon(s,r,4) &:= E\bigl[f'(W_s)f'(W_r)\bigr]E\bigl[W_r(W_{r+\varepsilon} - W_r)\bigr]E\bigl[W_s(W_{s+\varepsilon} - W_s)\bigr],\\
\Lambda_\varepsilon(s,r,5) &:= E\bigl[f(W_s)f''(W_r)\bigr]E\bigl[W_r(W_{r+\varepsilon} - W_r)\bigr]E\bigl[W_r(W_{s+\varepsilon} - W_s)\bigr],
\end{aligned}$$

for all $\varepsilon > 0$ and $s, r > 0$. It follows that

$$\begin{aligned}
\Delta_\varepsilon(s,r) &= E\bigl[f(W_s)f(W_r)(W_{s+\varepsilon} - W_s)\delta_W(1_{[r,r+\varepsilon]})\bigr]\\
&= E\bigl[f(W_s)f(W_r)\bigr]E\bigl[(W_{r+\varepsilon} - W_r)(W_{s+\varepsilon} - W_s)\bigr]\\
&\quad + E\bigl[f'(W_s)f(W_r)(W_{s+\varepsilon} - W_s)\bigr]E\bigl[W_s(W_{r+\varepsilon} - W_r)\bigr]\\
&\quad + E\bigl[f(W_s)f'(W_r)(W_{s+\varepsilon} - W_s)\bigr]E\bigl[W_r(W_{r+\varepsilon} - W_r)\bigr]\\
&= \sum_{j=1}^{5}\Lambda_\varepsilon(s,r,j)
\end{aligned}$$

by (2.3), which gives

$$E\bigl|J_\varepsilon(1,f,t)\bigr|^2 = \sum_{j=1}^{5}\frac{2}{\nu^2\varepsilon^{2H}}\int_0^t\int_0^s \Lambda_\varepsilon(s,r,j)\,d\eta_s\,d\eta_r.$$

For the first term $\Lambda_\varepsilon(s,r,1)$ we have

$$\left|E\left[(W_{r+\varepsilon}-W_r)(W_{s+\varepsilon}-W_s)\right]\right| \le \varepsilon 1_{\{0<s-r<\varepsilon\}} + \nu^2\frac{\varepsilon^{4H}}{(s-r)^{2H}}1_{\{s-r>0\}}$$

for $s>r>0$ by Lemma 2.3 and Cauchy's inequality. Moreover, by the fact

$$\begin{aligned} E\left[f^2(W_r)\right] &= \int_{\mathbb{R}} f^2(x)\frac{1}{\sqrt{2\pi\eta_r}}e^{-\frac{x^2}{2\eta_r}}\,dx \\ &\le \frac{\sqrt{\eta_s}}{\sqrt{\eta_r}}\int_{\mathbb{R}} f^2(x)\frac{1}{\sqrt{2\pi\eta_s}}e^{-\frac{x^2}{2\eta_s}}\,dx = \frac{\sqrt{\eta_s}}{\sqrt{\eta_r}}E\left[f^2(W_s)\right] \end{aligned} \tag{3.10}$$

with $s\ge r>0$, we have also

$$\begin{aligned} E\left|f(W_s)f(W_r)\right| &\le \sqrt{E\left[f^2(W_s)\right]E\left[f^2(W_r)\right]} \\ &\le E\left[f^2(W_s)\right]\sqrt[4]{\frac{\eta_s}{\eta_r}} \end{aligned} \tag{3.11}$$

for all $s\ge r>0$. It follows that

$$\left|\Lambda_\varepsilon(s,r,1)\right| \le \left(\varepsilon 1_{\{0<s-r<\varepsilon\}} + \frac{\varepsilon^{4H}}{(s-r)^{2H}}1_{\{s-r>0\}}\right)E\left[f^2(W_s)\right]\sqrt[4]{\frac{\eta_s}{\eta_r}}$$

for all $s\ge r>0$ and

$$\begin{aligned} &\frac{1}{\varepsilon^{4H}}\int_0^t\int_0^s\left|\Lambda_\varepsilon(s,r,1)\right|d\eta_s\,d\eta_r \\ &\quad= \frac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^s\left|\Lambda_\varepsilon(s,r,1)\right|d\eta_s\,d\eta_r \\ &\qquad+ \frac{1}{\varepsilon^2}\int_\varepsilon^t\int_0^{s-\varepsilon}\left|\Lambda_\varepsilon(s,r,1)\right|d\eta_s\,d\eta_r + \frac{1}{\varepsilon^2}\int_\varepsilon^t\int_{s-\varepsilon}^s\left|\Lambda_\varepsilon(s,r,1)\right|d\eta_s\,d\eta_r \le C\|f\|_{\mathbb{H}}^2 \end{aligned}$$

for all $\varepsilon>0$ and $0\le t\le 1$.

Now for the second term $\Lambda_\varepsilon(s,r,2)$. By Lemma 2.4, Lemma 2.5, and the fact (3.11) we have

$$\begin{aligned} \left|\Lambda_\varepsilon(s,r,2)\right| &= \left|E\left[W_s(W_{r+\varepsilon}-W_r)\right]E\left[W_r(W_{s+\varepsilon}-W_s)\right]E\left[f'(W_s)f'(W_r)\right]\right| \\ &\le C\varepsilon^{4H}\frac{\sqrt{\eta_r\eta_s}}{\rho^2}\left(E\left[\left|f(W_s)\right|^2\right]E\left[\left|f(W_r)\right|^2\right]\right)^{1/2}, \end{aligned}$$

which implies that

$$\frac{1}{\varepsilon^{4H}}\int_0^t\int_0^s\left|\Lambda_\varepsilon(s,r,2)\right|d\eta_s\,d\eta_r \le C\|f\|_{\mathbb{H}}^2$$

for all $0<\varepsilon\le 1$ by (3.10) and Lemma 2.1.

Similarly, we can also obtain the estimates

$$\frac{1}{\varepsilon^{4H}}\left|\int_0^t\int_0^s\Lambda_\varepsilon(s,r,j)\,d\eta_s\,d\eta_r\right| \le C\|f\|_{\mathbb{H}}^2,\quad j=3,4,5,$$

for all $0 < \varepsilon \le 1$ and $t \in [0,1]$. Thus, we have obtained the estimate (3.9) for $k = 1$. In the same way one can give (3.9) for $k = 2$. □

Proof of Statement (b) We need to prove

$$\mathcal{B}_k(\varepsilon_1, \varepsilon_2) := E\big|J_{\varepsilon_1}(k,f,t) - J_{\varepsilon_2}(k,f,t)\big|^2 \longrightarrow 0, \tag{3.12}$$

for all and $t \ge 0$ and $k = 1, 2$, as $\varepsilon_1, \varepsilon_2 \downarrow 0$. Recall that

$$\Delta_\varepsilon(s,r) = E\big[f(W_s)f(W_r)(W_{s+\varepsilon} - W_s)(W_{r+\varepsilon} - W_r)\big],$$

and denote

$$\Delta_{\varepsilon_1,\varepsilon_2}(s,r) = E\big[f(W_s)f(W_r)(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_2} - W_r)\big]$$

for all $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ and $s, r \ge 0$. Then we have

$$\begin{aligned}
\mathcal{B}_1(\varepsilon_1, \varepsilon_2) &= \frac{1}{\varepsilon_1^{4H}} \int_0^t \int_0^t Ef(W_s)f(W_r)(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_1} - W_r)\, d\eta_r\, d\eta_s \\
&\quad - \frac{2}{\varepsilon_1^{2H}\varepsilon_2^{2H}} \int_0^t \int_0^t Ef(W_s)f(W_r)(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_2} - W_r)\, d\eta_r\, d\eta_s \\
&\quad + \frac{1}{\varepsilon_2^{4H}} \int_0^t \int_0^t Ef(W_r)f(W_r)(W_{s+\varepsilon_2} - W_s)(W_{r+\varepsilon_2} - W_r)\, d\eta_r\, d\eta_s \\
&= \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \int_0^t \int_0^t \big\{\Delta_{\varepsilon_1}(s,r)\varepsilon_2^{2H} - \Delta_{\varepsilon_1,\varepsilon_2}(s,r)\varepsilon_1^{2H}\big\}\, d\eta_r\, d\eta_s \\
&\quad + \frac{1}{\varepsilon_1^{2H}\varepsilon_2^{4H}} \int_0^t \int_0^t \big\{\Delta_{\varepsilon_2}(s,r)\varepsilon_1^{2H} - \Delta_{\varepsilon_1,\varepsilon_2}(s,r)\varepsilon_2^{2H}\big\}\, d\eta_r\, d\eta_s
\end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 > 0$ and $t \ge 0$. Thus, to show that $\{J_\varepsilon(1,f,t), \varepsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega)$ we need to prove

$$\lim_{\varepsilon_i,\varepsilon_j \to 0} \frac{1}{\varepsilon_i^{4H}\varepsilon_j^{2H}} \int_0^t \int_0^t \big\{\Delta_{\varepsilon_i}(s,r)\varepsilon_j^{2H} - \Delta_{\varepsilon_1,\varepsilon_2}(s,r)\varepsilon_i^{2H}\big\}\, d\eta_r\, d\eta_s = 0 \tag{3.13}$$

for all $i, j \in \{1, 2\}$ and $i \ne j$. Without loss of generality one may assume that $\varepsilon_1 > \varepsilon_2$ and by approximating we may also assume that $f \in C_0^\infty(\mathbb{R})$. Denote

$$\begin{aligned}
Q_j(1,s,r,\varepsilon) &:= \varepsilon_j^{2H} E\big[(W_{r+\varepsilon} - W_r)(W_{s+\varepsilon} - W_s)\big] \\
&\quad - \varepsilon^{2H} E\big[(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_2} - W_r)\big], \\
Q_j(2,s,r,\varepsilon) &:= \varepsilon_j^{2H} E\big[W_r(W_{r+\varepsilon} - W_r)\big] E\big[W_r(W_{s+\varepsilon} - W_s)\big] \\
&\quad - \varepsilon^{2H} E\big[B_r^H(B_{r+\varepsilon_2}^H - B_r^H)\big] E\big[W_r(W_{s+\varepsilon_1} - W_s)\big], \\
Q_j(3,s,r,\varepsilon) &:= \varepsilon_j^{2H} E\big[W_s(W_{r+\varepsilon} - W_r)\big] E\big[W_s(W_{s+\varepsilon} - W_s)\big] \\
&\quad - \varepsilon^{2H} E\big[W_s(W_{r+\varepsilon_2} - W_r)\big] E\big[W_s(W_{s+\varepsilon_1} - W_s)\big], \\
Q_j(4,s,r,\varepsilon) &:= \varepsilon_j^{2H} E\big[W_r(W_{r+\varepsilon} - W_r)\big] E\big[W_s(W_{s+\varepsilon} - W_s)\big]
\end{aligned}$$

$$\begin{aligned}&\quad - \varepsilon^{2H} E\big[W_r(W_{r+\varepsilon_2} - W_r)\big]E\big[W_s(W_{s+\varepsilon_1} - W_s)\big],\\
Q_j(5,s,r,\varepsilon) &:= \varepsilon_j^{2H} E\big[W_s(W_{r+\varepsilon} - W_r)\big]E\big[W_r(W_{s+\varepsilon} - W_s)\big]\\
&\quad - \varepsilon^{2H} E\big[W_s(W_{r+\varepsilon_2} - W_r)\big]E\big[W_r(W_{s+\varepsilon_1} - W_s)\big],\end{aligned}$$

for $j \in \{1,2\}$ and $\varepsilon, \varepsilon_1, \varepsilon_2, s, r > 0$. From the proof of Statement (a) it follows that

$$\begin{aligned}\Delta_{\varepsilon_1,\varepsilon_2}(s,r) = {}& E\big[f(W_s)f(W_r)\big]E\big[(W_{r+\varepsilon_2} - W_r)(W_{s+\varepsilon_1} - W_s)\big]\\
&+ E\big[f'(W_s)f'(W_r)\big]E\big[W_r(W_{s+\varepsilon_1} - W_s)\big]E\big[W_s(W_{r+\varepsilon_2} - W_r)\big]\\
&+ E\big[f''(W_s)f\big(B_r^H\big)\big]E\big[W_s(W_{s+\varepsilon_1} - W_s)\big]E\big[W_s(W_{r+\varepsilon_2} - W_r)\big]\\
&+ E\big[f'(W_s)f'(W_r)\big]E\big[W_s(W_{s+\varepsilon_1} - W_s)\big]E\big[W_r(W_{r+\varepsilon_2} - W_r)\big]\\
&+ E\big[f(W_s)f''(W_r)\big]E\big[W_r(W_{s+\varepsilon_1} - W_s)\big]E\big[W_r(W_{r+\varepsilon_2} - W_r)\big]\end{aligned}$$

and

$$\begin{aligned}&\varepsilon_j^{2H}\Delta_{\varepsilon_i}(s,r) - \varepsilon_i^{2H}\Delta_{\varepsilon_1,\varepsilon_2}(s,r)\\
&\quad = E\big[f(W_s)f(W_r)\big]Q_j(1,s,r,\varepsilon_i)\\
&\qquad + E\big[f(W_s)f''(W_r)\big]Q_j(2,s,r,\varepsilon_i) + E\big[f''(W_s)f(W_r)\big]Q_j(3,s,r,\varepsilon_i)\\
&\qquad + E\big[f'(W_s)f'(W_r)\big]\big(Q_j(4,s,r,\varepsilon_i) + Q_j(5,s,r,\varepsilon_i)\big)\end{aligned}$$

with $i \neq j$ and $i,j \in \{1,2\}$. Now, let us prove the convergence (3.13) in three steps. We only need to show that (3.13) holds with $j = 2$ and $i = 1$ by symmetry.

Step I. The convergence

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \int_0^t \int_0^t E\big[f(W_s)f(W_r)\big]Q_2(1,s,r,\varepsilon_1)\,d\eta_r\,d\eta_s = 0 \tag{3.14}$$

holds. Clearly, by Cauchy's inequality we have

$$\begin{aligned}\big|E\big[(W_{s+\varepsilon_i} - W_s)(W_{r+\varepsilon_j} - W_r)\big]\big| &\le \sqrt{E\big[(W_{s+\varepsilon_i} - W_s)^2 E(W_{r+\varepsilon_j} - W_r)^2\big]}\\
&\le \big(\lambda^2\varepsilon_i^{1-2H} + \nu^2\big)\big(\lambda^2\varepsilon_j^{1-2H} + \nu^2\big)\varepsilon_i^H\varepsilon_j^H \le C\frac{\varepsilon_i^{2H+\theta}\varepsilon_j^{2H}}{|s-r|^{2H+\theta}}\end{aligned}$$

for $0 < |s-r| < \varepsilon_i \wedge \varepsilon_j \le 1$ and $0 < \theta < 1 - 2H$, where $i,j \in \{1,2\}$. It follows from (2.9) with $\alpha = \frac{2H+\theta}{2-2H}$ that

$$\big|E\big[(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_1} - W_r)\big]\big| \le \frac{C\varepsilon_1^{4H+\theta}}{|s-r|^{2H+\theta}}$$

and

$$\big|E\big[(W_{s+\varepsilon_1} - W_s)(W_{r+\varepsilon_2} - W_r)\big]\big| \le \frac{C\varepsilon_1^{2H+\theta}\varepsilon_2^{2H}}{|s-r|^{2H+\theta}}$$

for all $|s-r|>0$ and $0<\theta<1-2H$, which gives

$$\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|Q_2(1,s,r,\varepsilon_1)\right| \le \frac{C\varepsilon_1^{\theta}}{|s-r|^{2H+\theta}} \longrightarrow 0 \quad (\varepsilon_1,\varepsilon_2\to 0)$$

for all $r,s>0$ and $0<\theta<1-2H$.

On the other hand, from the above proof we have also

$$\begin{aligned}\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|Q_2(1,s,r,\varepsilon_1)\right| &\le \frac{1}{\varepsilon_1^{4H}}\left|E\left[(W_{s+\varepsilon_1}-W_s)(W_{r+\varepsilon_1}-W_r)\right]\right| \\ &\quad + \frac{1}{\varepsilon_1^{2H}\varepsilon_2^{2H}}\left|E\left[(W_{s+\varepsilon_1}-W_s)(W_{r+\varepsilon_2}-W_r)\right]\right| \\ &\le \frac{C}{|s-r|^{2H}}\end{aligned}$$

for all $|s-r|>0$ and $\varepsilon_1,\varepsilon_2>0$, and

$$\int_0^t\int_0^t \frac{1}{|s-r|^{2H}}\left|E\left[f(W_s)f(W_r)\right]\right|\,d\eta_r\,d\eta_s \le C\|f\|_{\mathbb{H}}^2$$

for any $0<\varepsilon_1,\varepsilon_2<1$. Thus, Lebesgue's dominated convergence theorem implies that the convergence (3.14) holds.

Step II. The convergence

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\int_0^t\int_0^t E\left[f(W_s)f''(W_r)\right]Q_2(2,s,r,\varepsilon_1)\,d\eta_r\,d\eta_s = 0 \tag{3.15}$$

holds. By Lemma 2.4, we have

$$\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|Q_2(2,s,r,\varepsilon_1)\right| \le 2$$

and

$$\int_0^t\int_0^t \left|E\left[f(W_s)f''(W_r)\right]\right|\,d\eta_r\,d\eta_s \le C\|f\|_{\mathbb{H}}^2$$

for $\varepsilon_1,\varepsilon_2>0$. On the other hand, by Lemma 2.4 and the fact

$$b^{\gamma}-a^{\gamma} \le b^{\gamma-\theta}(b-a)^{\theta} \tag{3.16}$$

with $b>a>0$ and $1\ge\theta\ge\gamma\ge 0$, we have

$$\begin{aligned}\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|Q_2(2,s,r,\varepsilon_1)\right| &= \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|E\left[W_r(W_{s+\varepsilon_1}-W_s)\right]\right| \\ &\quad \times\left|\varepsilon_2^{2H}E\left[W_r(W_{r+\varepsilon_1}-W_r)\right]-\varepsilon_1^{2H}E\left[W_r(W_{r+\varepsilon_2}-W_r)\right]\right| \\ &= \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}}\left|E\left[W_r(W_{s+\varepsilon_1}-W_s)\right]\right| \\ &\quad \times C\left|\varepsilon_2^{2H}\left((r+\varepsilon_1)^{2H}-r^{2H}\right)-\varepsilon_1^{2H}\left((r+\varepsilon_2)^{2H}-r^{2H}\right)\right| \\ &\le Cr^{2H-\theta}\varepsilon_1^{\theta-2H} \longrightarrow 0 \quad (\varepsilon_1,\varepsilon_2\to 0)\end{aligned} \tag{3.17}$$

for all $2H < \theta \le 1$ and $r > 0$. Thus, the convergence (3.15) follows from the Lebesgue dominated convergence theorem. Similarly, we can introduce the convergence

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \int_0^t \int_0^t E\big[f''(W_s)f(W_r)\big] Q_2(3,s,r,\varepsilon_1)\,d\eta_r\,d\eta_s = 0. \tag{3.18}$$

Step III. The convergence

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \int_0^t \int_0^t \big(Q_2(4,s,r,\varepsilon_1) + Q_2(5,s,r,\varepsilon_1)\big) E\big[f'(W_s)f'(W_r)\big]\,d\eta_r\,d\eta_s = 0 \tag{3.19}$$

holds. From Step II we have

$$\begin{aligned}\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \big|Q_2(4,s,r,\varepsilon_1)\big| &\le \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \big|E\big[W_s(W_{s+\varepsilon_1} - W_s)\big]\big| \\ &\quad \times \big|\varepsilon_2^{2H} E\big[W_r(W_{r+\varepsilon_1} - W_r)\big] - \varepsilon_1^{2H} E\big[W_r(W_{r+\varepsilon_2} - W_r)\big]\big| \\ &\le Cr^{2H-\theta}\varepsilon_1^{\theta-2H} \longrightarrow 0 \quad (\varepsilon_1,\varepsilon_2 \to 0)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \big|Q_2(5,s,r,\varepsilon_1)\big| &= \frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \big|E\big[W_r(W_{s+\varepsilon_1} - W_s)\big]\big| \\ &\quad \times \big|\varepsilon_2^{2H} E\big[W_s(W_{r+\varepsilon_1} - W_r)\big] - \varepsilon_1^{2H} E\big[W_s(W_{r+\varepsilon_2} - W_r)\big]\big| \\ &\le \big(r^{2H-\theta} + |s-r|^{2H-\theta}\big)\varepsilon^{\theta-2H} \longrightarrow 0\end{aligned}$$

for all $2H < \theta \le 1$ and $|s-r| > 0$, as $\varepsilon_1, \varepsilon_2 \to 0$. On the other hand, we also have

$$\begin{aligned}&\frac{1}{\varepsilon_1^{4H}\varepsilon_2^{2H}} \int_0^t \int_0^t \big|E\big[f'(W_s)f'(W_r)\big]\big| \big|Q_2(4,s,r,\varepsilon_1) + Q_2(5,s,r,\varepsilon_1)\big|\,d\eta_r\,d\eta_s \\ &\quad \le 4\int_0^t \int_0^t \big|E\big[f'(W_s)f'(W_r)\big]\big|\,d\eta_r\,d\eta_s \le C\|f\|_{\mathbb{H}}^2\end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 > 0$. Thus, Lebesgue dominated convergence theorem implies that the convergence (3.19) holds.

Consequently, we have found the desired convergence (3.12) for $k = 1$. In the same way one can also introduce the convergence (3.12), which with $k = 2$ holds, and Statement (b) follows. □

4 Itô's formula

In this section we introduce an Itô formula and study the integral

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t), \tag{4.1}$$

where f is a Borel function and

$$\mathcal{L}^H(x,t) = \int_0^t \delta(W_s - x)\,d\eta_s$$

is the weighted local time of mixed-fBm W. By using the result given in Section 3, we can immediately get an extension of Itô formula stated as follows, which is an analog of Föllmer-Protter-Shiryayev's equation (some more work can be found in Eisenbaum [22], Föllmer *et al.* [23], Moret-Nualart [24], Russo-Vallois [16, 25], and the references therein).

Theorem 4.1 *Let $0 < H < \frac{1}{2}$ and let $f \in \mathbb{H}$ be left continuous with right limits. If F is an absolutely continuous function such that the derivative $F' = f$, then the Itô formula*

$$F(W) = F(0) + \lambda \int_0^t f(W_s)\,dB_s + \nu \int_0^t f(W_s)\,dB_s^H + \frac{1}{2}\bigl[f(W), W\bigr]_t^{(H)} \tag{4.2}$$

holds for all $t \geq 0$.

Proof When $f \in C^1(\mathbb{R})$, this is an Itô formula since

$$\begin{aligned}\bigl[f(W), W\bigr]_t^{(H)} &= \int_0^t f'(W_s)\,d\eta_s \\ &= \lambda^2 \int_0^t f'(W_s)\,ds + 2\nu^2 H \int_0^t f'(W_s) s^{2H-1}\,ds\end{aligned}$$

by Corollary 3.1.

When $f \notin C^1(\mathbb{R})$, by a localization argument we may assume that the function f is uniformly bounded. Let now $F' = f \in \mathbb{H}$ be uniformly bounded and left continuous, and define the function ζ on $\mathbb{R}$ by

$$\zeta(x) := \begin{cases} ce^{\frac{1}{(x-1)^2-1}}, & x \in (0,2), \\ 0, & \text{otherwise}, \end{cases} \tag{4.3}$$

where c is a normalizing constant such that $\int_{\mathbb{R}} \zeta(x)\,dx = 1$. Consider the sequence of functions

$$F_n(x) := n \int_{\mathbb{R}} F(x-y)\zeta(ny)\,dy, \quad n = 1, 2, \ldots,$$

with $x \in \mathbb{R}$. Then $F_n \in C^\infty(\mathbb{R})$,

$$F_n'(x) := n \int_{\mathbb{R}} f(x-y)\zeta(ny)\,dy, \quad n = 1, 2, \ldots,$$

with $x \in \mathbb{R}$ and the Itô formula

$$F_n\bigl(W_t^H\bigr) = F_n(0) + \lambda \int_0^t F_n'(W_s)\,dB_s + \nu \int_0^t F_n'(W_s)\,dB_s^H + \frac{1}{2}\int_0^t F_n''(W_s)\,d\eta_s \tag{4.4}$$

holds for all $n = 1, 2, \ldots$. Moreover, Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} F_n(x) = F(x), \qquad \lim_{n\to\infty} F_n'(x) = f(x)$$

for each x, and $\{F_n'\} \subset \mathbb{H}$, $\lim_{n\to\infty} F_n' = f$ in $\mathbb{H}$. It follows that

$$\int_0^t F_n''(W_s)\,d\eta_s = \left[F_n'(W), W\right]_t^{(H)} \longrightarrow \left[f(W), W\right]_t^{(H)}$$

and

$$F_n'(W_t) \longrightarrow f(W_t)$$

in $L^2(\Omega)$, as n tends to infinity, which gives

$$\begin{aligned}\int_0^t F_n'(W_s)\,dW_s &= F_n(W_t) - F_n(0) - \frac{1}{2}\left[F_n'(W), W\right]_t^{(H)} \\ &\longrightarrow F(W_t) - F(0) - \frac{1}{2}\left[f(W), W\right]_t^{(H)}\end{aligned}$$

in $L^2(\Omega)$, as n tends to infinity. This completes the proof. □

At the end of this paper, we use the Itô formula above to obtain the integral (4.1) and give the related Bouleau-Yor identity. Such an identity wais first studied by Bouleau and Yor [26], who characterized the relationship between the quadratic covariation of Brownian motion and the integral with respect to the local time of Brownian motion. Let B be a standard Brownian motion and let $\mathcal{L}^B(x,t)$ be the local time of B. Then Bouleau and Yor [26] showed that the identity

$$\left[f(B), B\right]_t = -\int_{\mathbb{R}} f(x)\mathcal{L}^B(dx, t)$$

holds for all locally square integrable functions f. The identity is called the *Bouleau-Yor identity*. For more work we refer to Eisenbaum [22, 27], Föllmer *et al.* [23], Feng and Zhao [28], Peskir [29], Rogers and Walsh [30], Yan *et al.* [14, 31, 32] and the references therein. Let $F(x) = (x-a)^+ - (x-b)^+$. Then F is absolutely continuous with $F' = 1_{(a,b]}$, and Itô's formula (4.2) implies that

$$\begin{aligned}\left[1_{(a,b]}(W), W\right]_t^{(H)} &= 2F(W_t) - 2F(0) - 2\int_0^t 1_{(a,b]}(W_s)\,dW_s \\ &= \mathcal{L}^H(a,t) - \mathcal{L}^H(b,t)\end{aligned}$$

holds for all $t \geq 0$. Thus, from the linear property of fractional quadratic covariation one deduces the following result.

Lemma 4.1 *For any $f = \sum_j \beta_j 1_{(a_{j-1},a_j]} \in \mathcal{E}$, the integral*

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t) := \sum_j \beta_j\left[\mathcal{L}^H(a_j, t) - \mathcal{L}^H(a_{j-1}, t)\right] \tag{4.5}$$

exists and

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t) = -\left[f(W), W\right]_t^{(H)} \tag{4.6}$$

for all $t \geq 0$.

Since $\mathcal{E}$ is dense in $\mathbb{H}$, we can extend the definition (4.5) to the elements of $\mathbb{H}$ by setting

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t) := \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\mathcal{L}^H(dx,t) \tag{4.7}$$

in L^2 for $f \in \mathbb{H}$ provided $\lim_{n\to\infty} f_n = f$ in $\mathbb{H}$, where $f_n \in \mathcal{E}$ for all $n \geq 1$. We can show that the limit does not depend on the choice of the sequences $\{f_{\Delta,n}\}$ and the following theorem holds.

Theorem 4.2 (Bouleau-Yor identity) *Let $0 < H < \frac{1}{2}$ and $f \in \mathbb{H}$. Then the integral (4.7) is well defined and*

$$[f(W), W]_t^{(H)} = -\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)$$

holds for all $t \geq 0$.

Corollary 4.1 (Tanaka formula) *Let $0 < H < \frac{1}{2}$. For any $x \in \mathbb{R}$ we have*

$$\begin{aligned}
(W_t - a)^+ &= (-a)^+ + \int_0^t 1_{\{W_s > a\}}\, dW_s + \frac{1}{2}\mathcal{L}^H(a,t),\\
(W_t - a)^- &= (-a)^- - \int_0^t 1_{\{W_s < a\}}\, dW_s + \frac{1}{2}\mathcal{L}^H(a,t),\\
|W_t - a| &= |x| + \int_0^t \operatorname{sign}(W_s - a)\, dW_s + \mathcal{L}^H(a,t).
\end{aligned}$$

Proof Let $F(x) = (x-a)^+$. Then

$$F(x) = \int_{-\infty}^x 1_{(a,\infty)}(y)\, dy,$$

Itô's formula (4.2), and the above theorem imply that

$$\begin{aligned}
\mathcal{L}^H(a,t) &= [1_{(a,+\infty)}(W), W]_t^{(H)}\\
&= 2(W_t - a)^+ - 2(-a)^+ - 2\int_0^t 1_{\{W_s > a\}}\, dW_s
\end{aligned}$$

for all $t \in [0,T]$, which gives the first identity. In the same way one can obtain the second identity and the corollary follows. □

5 Results, discussion, and conclusions

Since the quadratic variation of a mixed-fractional Brownian motion does not exist when $0 < H < \frac{1}{2}$, we need to find a substitution tool. In this paper, we give a new substitution tool, and by using some *precise estimations and inequalities* we show that this substitution tool is well defined, and, moreover, we also discuss some related questions. It is important to note that the method used here is also applicative to many similar processes.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

LTY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. HG participated in the design of the study and performed the proofs of some inequalities. All authors read and approved the final manuscript.

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Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11571071, 11301068) and Innovation Program of Shanghai Municipal Education Commission (12ZZ063).

Received: 8 September 2016 Accepted: 18 November 2016 Published online: 29 November 2016

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