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Asymptotic behavior of even-order damped differential equations with p-Laplacian like operators and deviating arguments

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Abstract
We study the asymptotic properties of the solutions of a class of even-order damped differential equations with p-Laplacian like operators, delayed and advanced arguments. We present new theorems that improve and complement related contributions reported in the literature. Several examples are provided to illustrate the practicability, maneuverability, and efficiency of the results obtained. An open problem is proposed.

MSC: 34C10; 34K11

Keywords: asymptotic behavior; functional differential equation; even-order; damping term; p-Laplacian equation

1 Introduction

In this paper, we study the asymptotic behavior of a class of even-order damped differential equations with p-Laplacian like operators and deviating arguments

$$\begin{aligned}&\bigl(a(t)\bigl|x^{(n-1)}(t)\bigr|^{p-2}x^{(n-1)}(t)\bigr)'+r(t)\bigl|x^{(n-1)}(t)\bigr|^{p-2}x^{(n-1)}(t)\\&\quad+q(t)\bigl|x\bigl(g(t)\bigr)\bigr|^{p-2}x\bigl(g(t)\bigr)=0,\end{aligned}\tag{1.1}$$

where $t\in\mathbb{I}:=[t_0,\infty)$, $t_0\in(0,\infty)$, $n\geq 2$ is an even integer, $p>1$ is a constant, $a\in\mathrm{C}^1(\mathbb{I},(0,\infty))$, $r,q,g\in\mathrm{C}(\mathbb{I},\mathbb{R})$, $r(t)\geq 0$, $a'(t)+r(t)\geq 0$, $q(t)>0$, and $\lim_{t\to\infty}g(t)=\infty$. As pointed out by Hale [1], differential equations have applications in the natural sciences, engineering technology, and automatic control. In particular, equation (1.1) has numerous applications in continuum mechanics as seen from Agarwal *et al.* [2] and Zhang *et al.* [3].

As usual, by a solution of (1.1) we mean a continuous function $x\in\mathrm{C}^{n-1}([T_x,\infty),\mathbb{R})$ which has the property that $a|x^{(n-1)}|^{p-2}x^{(n-1)}\in\mathrm{C}^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. We consider only those extendable solutions of (1.1) that satisfy condition $\sup\{|x(t)|:t\geq T\geq T_x\}>0$ and we tacitly assume that (1.1) possesses such solutions. A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is oscillatory if all its solutions oscillate.

There has been a growing interest in the study of the oscillatory and asymptotic behavior of various classes of differential equations during the past decades; we refer the reader to [2–21] and the references cited therein. In the following, we briefly review several related results that have motivated the work in this paper. Zhang *et al.* [3], Liu *et al.* [13], and Zhang *et al.* [21] considered the oscillation of (1.1) under the assumptions that

$$g(t) \leq t \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left[\frac{1}{a(s)} \exp\left(-\int_{t_0}^{s} \frac{r(\tau)}{a(\tau)} \, \mathrm{d}\tau \right) \right]^{1/(p-1)} \mathrm{d}s = \infty. \tag{1.3}$$

Assuming that $\gamma > 0$ is a quotient of odd positive integers, Erbe *et al.* [7] studied a second-order damped differential equation

$$\left(a(t)\left(x'(t)\right)^{\gamma}\right)' + r(t)\left(x'(t)\right)^{\gamma} + q(t)x^{\gamma}\left(g(t)\right) = 0, \tag{1.4}$$

and established some oscillation results in the case where (1.2) holds and

$$\begin{aligned} &\int_{t_0}^{\infty} \left[\frac{1}{a(t)\exp(\int_{t_0}^{t} \frac{r(s)}{a(s)} \, \mathrm{d}s)} \int_{t_0}^{t} q(s) \exp\left(\int_{t_0}^{s} \frac{r(u)}{a(u)} \, \mathrm{d}u \right) \right. \\ &\quad \left. \times \left(\int_{g(s)}^{\infty} \frac{\mathrm{d}u}{(a(u)\exp(\int_{t_0}^{u} \frac{r(v)}{a(v)} \, \mathrm{d}v))^{1/\gamma}} \right)^{\gamma} \mathrm{d}s \right]^{1/\gamma} \mathrm{d}t = \infty. \end{aligned}$$

Rogovchenko and Tuncay [16] and Saker *et al.* [17] investigated the oscillation of a second-order damped differential equation

$$\left(a(t)x'(t)\right)' + r(t)x'(t) + q(t)f\left(x(t)\right) = 0, \tag{1.5}$$

and they obtained several sufficient conditions which ensure that every solution x of (1.5) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Zhang [20] considered oscillatory behavior of (1.4) in the case when (1.2) is satisfied and

$$\int_{t_0}^{\infty} \left[\frac{1}{a(t)} \int_{t_0}^{t} \exp\left(-\int_{s}^{t} \frac{r(\tau)}{a(\tau)} \, \mathrm{d}\tau \right) \left(\int_{s}^{\infty} \left(\frac{1}{a(u)} \right)^{1/\gamma} \mathrm{d}u \right)^{\gamma} q(s) \, \mathrm{d}s \right]^{1/\gamma} \mathrm{d}t = \infty.$$

So far, the study of the asymptotic behavior of equation (1.1) when the integral in (1.3) is finite, *i.e.*,

$$\int_{t_0}^{\infty} \left[\frac{1}{a(s)} \exp\left(-\int_{t_0}^{s} \frac{r(\tau)}{a(\tau)} \, \mathrm{d}\tau \right) \right]^{1/(p-1)} \mathrm{d}s < \infty, \tag{1.6}$$

has received considerably less attention in the literature. Hence, our objective in this paper is not only to analyze the asymptotic properties of (1.1) in the case where (1.6) holds, but also to derive new asymptotic tests for (1.1) under the assumption that

$$g(t) \geq t. \tag{1.7}$$

The new theorems obtained improve and complement the relevant results reported in [3, 7, 13, 16, 17, 20, 21]. As is customary, all functional inequalities considered in the sequel are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1) since, if x is a solution, so is $-x$.

For a compact presentation of our results, we use the following notation:

$$\begin{aligned} &E(t_0,t) := \exp\biggl(\int_{t_0}^t \frac{r(\tau)}{a(\tau)}\,\mathrm{d}\tau\biggr), \qquad \delta(t) := \int_t^\infty \frac{\mathrm{d}s}{(a(s)E(t_0,s))^{1/(p-1)}}, \\ &h(t) := \frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}, \qquad h_+(t) := \max\bigl(0,h(t)\bigr), \\ &\varphi(t) := \frac{r(t)}{a(t)} + \frac{(p-1)^p}{p^p}\frac{\phi_+^p(t)E(t_0,t)}{\delta(t)a^{1/(p-1)}(t)}, \\ &\phi(t) := \frac{1}{E^{1/(p-1)}(t_0,t)} - \frac{\delta(t)r(t)a^{(2-p)/(p-1)}(t)}{p-1}, \\ &\phi_+(t) := \max\bigl(0,\phi(t)\bigr), \qquad \varrho_+(t,s) := \max\bigl(0,\varrho(t,s)\bigr), \end{aligned}$$

where the meaning of ρ and ϱ will be specified later.

2 Lemmas

To establish our main results, we make use of the following auxiliary lemmas.

Lemma 2.1 (Philos [14]) *Let $u \in \mathrm{C}^n(\mathbb{I},(0,\infty))$. If $u^{(n-1)}(t)u^{(n)}(t) \le 0$ for $t \ge t_u$. Then, for every $\lambda \in (0,1)$, there exists a constant $M > 0$ such that, for all sufficiently large t,*

$$u(\lambda t) \ge Mt^{n-1}\bigl|u^{(n-1)}(t)\bigr|.$$

Lemma 2.2 (Agarwal *et al.* [5]) *Let $u \in \mathrm{C}^n(\mathbb{I},(0,\infty))$ and $u^{(n)}(t) \le 0$. If $\lim_{t\to\infty} u(t) \ne 0$, then, for every $\lambda \in (0,1)$, there exists a $t_\lambda \in \mathbb{I}$ such that, for all $t \in [t_\lambda,\infty)$,*

$$u(t) \ge \frac{\lambda}{(n-1)!}t^{n-1}\bigl|u^{(n-1)}(t)\bigr|.$$

3 Asymptotic results via the Riccati method

Theorem 3.1 *Let conditions* (1.2) *and* (1.6) *be satisfied and*

$$g \in \mathrm{C}^1(\mathbb{I},\mathbb{R}) \quad \text{and} \quad g'(t) > 0. \tag{3.1}$$

Assume that there exists a function $\rho \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ such that, for all constants $M > 0$,

$$\limsup_{t\to\infty}\int_{t_0}^t \biggl[\rho(s)q(s) - \frac{2^{p-1}}{p^p}\frac{\rho(s)a(s)(h_+(s))^p}{(Mg'(s)g^{n-2}(s))^{p-1}}\biggr]\,\mathrm{d}s = \infty. \tag{3.2}$$

If, for some constant $\lambda_0 \in (0,1)$,

$$\limsup_{t\to\infty}\int_{t_0}^t \biggl[q(s)\biggl(\frac{\lambda_0}{(n-2)!}g^{n-2}(s)\delta(s)\biggr)^{p-1}E(t_0,s) - \varphi(s)\biggr]\,\mathrm{d}s = \infty, \tag{3.3}$$

then every solution x of (1.1) *is either oscillatory or satisfies condition* $\lim_{t\to\infty} x(t) = 0$.

Proof Assume that (1.1) has a nonoscillatory solution x which is eventually positive and such that

$$\lim_{t\to\infty} x(t) \neq 0. \tag{3.4}$$

Modifying the proof in Zhang *et al.* ([3], Lemma 2.3), we can show that, for all $t \geq t_1$, there exist two possible cases:

(1) $x(t) > 0$, $x'(t) > 0$, $x^{(n-1)}(t) > 0$, $x^{(n)}(t) < 0$;
(2) $x(t) > 0$, $x^{(n-2)}(t) > 0$, $x^{(n-1)}(t) < 0$,

where $t_1 \geq t_0$ is sufficiently large. We consider each of the two cases separately.

Case I. Assume first that case (1) holds. For $t \geq t_1$, we define the function ω by

$$\omega(t) := \rho(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{x^{p-1}(\frac{g(t)}{2})}. \tag{3.5}$$

Then $\omega(t) > 0$ for all $t \geq t_1$ and

$$\begin{aligned}\omega'(t) = {}& \rho'(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{x^{p-1}(\frac{g(t)}{2})} + \rho(t)\frac{(a(t)(x^{(n-1)}(t))^{p-1})'}{x^{p-1}(\frac{g(t)}{2})} \\ & - \frac{p-1}{2}\rho(t)g'(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}x'(\frac{g(t)}{2})}{x^{p}(\frac{g(t)}{2})}.\end{aligned}$$

Let $u := x'$. It follows from Lemma 2.1 that, for some constant $M > 0$ and for all sufficiently large t,

$$x'\left(\frac{g(t)}{2}\right) \geq Mg^{n-2}(t)x^{(n-1)}\bigl(g(t)\bigr) \geq Mg^{n-2}(t)x^{(n-1)}(t).$$

Thus, we deduce that

$$\begin{aligned}\omega'(t) \leq {}& \rho'(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{x^{p-1}(\frac{g(t)}{2})} + \rho(t)\frac{(a(t)(x^{(n-1)}(t))^{p-1})'}{x^{p-1}(\frac{g(t)}{2})} \\ & - \frac{p-1}{2}M\rho(t)g'(t)g^{n-2}(t)\frac{a(t)(x^{(n-1)}(t))^{p}}{x^{p}(\frac{g(t)}{2})}.\end{aligned}$$

From (1.1) and (3.5), we obtain

$$\begin{aligned}\omega'(t) \leq {}& -\rho(t)q(t) + \left[\frac{\rho'(t)}{\rho(t)} - \frac{r(t)}{a(t)}\right]\omega(t) - \frac{p-1}{2}Mg'(t)g^{n-2}(t)\frac{\omega^{p/(p-1)}(t)}{(\rho(t)a(t))^{1/(p-1)}} \\ = {}& -\rho(t)q(t) + h(t)\omega(t) - \frac{p-1}{2}Mg'(t)g^{n-2}(t)\frac{\omega^{p/(p-1)}(t)}{(\rho(t)a(t))^{1/(p-1)}}.\end{aligned} \tag{3.6}$$

Hence, we have

$$\omega'(t) \leq -\rho(t)q(t) + h_{+}(t)\omega(t) - \frac{p-1}{2}Mg'(t)g^{n-2}(t)\frac{\omega^{p/(p-1)}(t)}{(\rho(t)a(t))^{1/(p-1)}}.$$

Let

$$y := \omega(t), \qquad D := h_+(t), \quad \text{and} \quad C := \frac{(p-1)Mg'(t)g^{n-2}(t)}{2(\rho(t)a(t))^{1/(p-1)}}.$$

Using the inequality

$$Dy - Cy^{p/(p-1)} \leq \frac{(p-1)^{p-1}}{p^p} \frac{D_+^p}{C^{p-1}}, \tag{3.7}$$

where $C > 0$, $y \geq 0$, and $D_+ := \max(0, D)$ (see Fišnarová and Mařík ([8], Lemma 1) for details), we get

$$\omega'(t) \leq -\rho(t)q(t) + \frac{2^{p-1}}{p^p} \frac{\rho(t)a(t)(h_+(t))^p}{(Mg'(t)g^{n-2}(t))^{p-1}}.$$

Integrating this inequality from t_1 to t, we obtain

$$\int_{t_1}^{t} \left[\rho(s)q(s) - \frac{2^{p-1}}{p^p} \frac{\rho(s)a(s)(h_+(s))^p}{(Mg'(s)g^{n-2}(s))^{p-1}} \right] \mathrm{d}s \leq \omega(t_1),$$

which contradicts (3.2).

Case II. Assume now that case (2) is satisfied. For $t \geq t_1$, we define another function ν as follows:

$$\nu(t) := -\frac{a(t)(-x^{(n-1)}(t))^{p-1}}{(x^{(n-2)}(t))^{p-1}}. \tag{3.8}$$

Then $\nu(t) < 0$ for all $t \geq t_1$. Since

$$\left(-a(t)\left(-x^{(n-1)}(t)\right)^{p-1} E(t_0, t)\right)' = -q(t)x^{p-1}\left(g(t)\right)E(t_0, t) < 0, \tag{3.9}$$

we conclude that $-a(t)(-x^{(n-1)}(t))^{p-1}E(t_0, t)$ is decreasing. Thus, for all $s \geq t \geq t_1$,

$$-a(s)\left(-x^{(n-1)}(s)\right)^{p-1} E(t_0, s) \leq -a(t)\left(-x^{(n-1)}(t)\right)^{p-1} E(t_0, t).$$

Hence, for all $s \geq t \geq t_1$,

$$x^{(n-1)}(s) \leq \frac{(a(t)E(t_0, t))^{1/(p-1)}}{(a(s)E(t_0, s))^{1/(p-1)}} x^{(n-1)}(t).$$

Integrating this inequality from t to ι, we obtain

$$x^{(n-2)}(\iota) \leq x^{(n-2)}(t) + \left(a(t)E(t_0, t)\right)^{1/(p-1)} x^{(n-1)}(t) \int_t^{\iota} \frac{\mathrm{d}s}{(a(s)E(t_0, s))^{1/(p-1)}}.$$

Taking $\iota \to \infty$ and using the fact that $\lim_{\iota \to \infty} x^{(n-2)}(\iota) \geq 0$ and the definition of δ, we have

$$0 \leq x^{(n-2)}(t) + \left(a(t)E(t_0, t)\right)^{1/(p-1)} x^{(n-1)}(t)\delta(t). \tag{3.10}$$

Inequality (3.10) implies that

$$-\frac{x^{(n-1)}(t)}{x^{(n-2)}(t)}\bigl(a(t)E(t_0,t)\bigr)^{1/(p-1)}\delta(t)\leq 1. \tag{3.11}$$

Hence, by (3.8) and (3.11), we get

$$-v(t)\delta^{p-1}(t)E(t_0,t)\leq 1. \tag{3.12}$$

Differentiation of (3.8) yields

$$v'(t)=\frac{(-a(t)(-x^{(n-1)}(t))^{p-1})'}{(x^{(n-2)}(t))^{p-1}}-(p-1)\frac{a(t)(-x^{(n-1)}(t))^{p}}{(x^{(n-2)}(t))^{p}}.$$

From (1.1) and (3.8), it follows that

$$v'(t)=-r(t)\frac{v(t)}{a(t)}-q(t)\frac{x^{p-1}(g(t))}{(x^{(n-2)}(t))^{p-1}}-(p-1)\frac{(-v(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}. \tag{3.13}$$

On the other hand, by Lemma 2.2, we have, for every $\lambda\in(0,1)$ and for all sufficiently large t,

$$x(t)\geq\frac{\lambda}{(n-2)!}t^{n-2}x^{(n-2)}(t). \tag{3.14}$$

Using (3.12) in (3.13), we have

$$\begin{aligned}v'(t)\leq{}&\frac{r(t)}{a(t)\delta^{p-1}(t)E(t_0,t)}-q(t)\frac{x^{p-1}(g(t))}{(x^{(n-2)}(g(t)))^{p-1}}\frac{(x^{(n-2)}(g(t)))^{p-1}}{(x^{(n-2)}(t))^{p-1}}\\&-(p-1)\frac{(-v(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}.\end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$v'(t)\leq\frac{r(t)}{a(t)\delta^{p-1}(t)E(t_0,t)}-q(t)\left(\frac{\lambda}{(n-2)!}g^{n-2}(t)\right)^{p-1}-(p-1)\frac{(-v(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}. \tag{3.16}$$

Multiplying (3.16) by $\delta^{p-1}(t)E(t_0,t)$ and integrating the resulting inequality from t_1 to t, we obtain

$$\begin{aligned}&\delta^{p-1}(t)E(t_0,t)v(t)-\delta^{p-1}(t_1)E(t_0,t_1)v(t_1)-\int_{t_1}^{t}\frac{r(s)}{a(s)}\,\mathrm{d}s\\&\quad+(p-1)\int_{t_1}^{t}a^{-1/(p-1)}(s)\delta^{p-2}(s)E(t_0,s)\phi_+(s)v(s)\,\mathrm{d}s\\&\quad+\int_{t_1}^{t}q(s)\left(\frac{\lambda}{(n-2)!}g^{n-2}(s)\right)^{p-1}\delta^{p-1}(s)E(t_0,s)\,\mathrm{d}s\\&\quad+(p-1)\int_{t_1}^{t}\frac{(-v(s))^{p/(p-1)}}{a^{1/(p-1)}(s)}\delta^{p-1}(s)E(t_0,s)\,\mathrm{d}s\leq 0.\end{aligned}$$

Let

$$y := -v(s), \qquad D := a^{-1/(p-1)}(s)\delta^{p-2}(s)E(t_0,s)\phi_+(s),$$

and

$$C := \delta^{p-1}(s)E(t_0,s)/a^{1/(p-1)}(s).$$

Using inequalities (3.7), (3.12), and the definition of φ, we have

$$\int_{t_1}^{t}\left[q(s)\left(\frac{\lambda}{(n-2)!}g^{n-2}(s)\right)^{p-1}\delta^{p-1}(s)E(t_0,s)-\varphi(s)\right]\mathrm{d}s \le \delta^{p-1}(t_1)E(t_0,t_1)v(t_1)+1,$$

which contradicts (3.3). This completes the proof. □

Assume $n = 2$ and let the definition of ω in (3.5) be replaced by

$$\omega(t) := \rho(t)\frac{a(t)(x'(t))^{p-1}}{x^{p-1}(g(t))}, \quad t \ge t_1.$$

Then we have the following result.

Theorem 3.2 *Let conditions* (1.2), (1.6), *and* (3.1) *be satisfied and* $n = 2$. *Suppose that there exists a function* $\rho \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s)-\frac{1}{p^p}\frac{\rho(s)a(s)(h_+(s))^p}{(g'(s))^{p-1}}\right]\mathrm{d}s = \infty. \tag{3.17}$$

If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[q(s)\delta^{p-1}(s)E(t_0,s)-\varphi(s)\right]\mathrm{d}s = \infty, \tag{3.18}$$

then (1.1) *is oscillatory.*

Example 3.3 For $t \ge 1$, consider a second-order delay differential equation with damping

$$\left(t^2x'(t)\right)' + \frac{t}{2}x'(t) + q_0 x\left(\frac{t}{2}\right) = 0, \tag{3.19}$$

where $q_0 > 0$ is a constant. Let $t_0 = 1$, $p = 2$, $a(t) = t^2$, $r(t) = t/2$, $q(t) = q_0$, $g(t) = t/2$, and $\rho(t) = 1$. Then $h_+(t) = 0$ and thus condition (3.17) is satisfied. It is easy to see that $E(t_0,t) = t^{1/2}$, $\delta(t) = 2t^{-3/2}/3$, $\phi(t) = 2t^{-1/2}/3$, and $\varphi(t) = 2t^{-1}/3$. Then condition (3.18) holds for $q_0 > 1$. Therefore, by Theorem 3.2, equation (3.19) is oscillatory provided that $q_0 > 1$.

Observe, however, that if $\gamma = 1$, then

$$\begin{aligned}
&\int_{t_0}^{\infty}\Bigg[\frac{1}{a(t)\exp(\int_{t_0}^{t}\frac{r(s)}{a(s)}\,\mathrm{d}s)}\int_{t_0}^{t}q(s)\exp\bigg(\int_{t_0}^{s}\frac{r(u)}{a(u)}\,\mathrm{d}u\bigg)\\
&\qquad\times\bigg(\int_{g(s)}^{\infty}\frac{\mathrm{d}u}{(a(u)\exp(\int_{t_0}^{u}\frac{r(v)}{a(v)}\,\mathrm{d}v))^{1/\gamma}}\bigg)^{\gamma}\,\mathrm{d}s\Bigg]^{1/\gamma}\,\mathrm{d}t\\
&\quad=\frac{2^{5/2}}{3}q_0\int_{1}^{\infty}\frac{\ln t}{t^{5/2}}\,\mathrm{d}t<\infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{\infty}\Bigg[\frac{1}{a(t)}\int_{t_0}^{t}\exp\bigg(-\int_{s}^{t}\frac{r(\tau)}{a(\tau)}\,\mathrm{d}\tau\bigg)\bigg(\int_{s}^{\infty}\bigg(\frac{1}{a(u)}\bigg)^{1/\gamma}\,\mathrm{d}u\bigg)^{\gamma}q(s)\,\mathrm{d}s\Bigg]^{1/\gamma}\,\mathrm{d}t\\
&\quad\le q_0\int_{1}^{\infty}\frac{\ln t}{t^{2}}\,\mathrm{d}t<\infty,
\end{aligned}$$

which mean that the results obtained in [7, 20] fail to apply in equation (3.19).

Theorem 3.4 *Let conditions* (1.6) *and* (1.7) *hold. Assume that there exists a function* $\rho \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that, for all constants* $M > 0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\bigg[\rho(s)q(s)-\frac{2^{p-1}}{p^p}\frac{\rho(s)a(s)(h_+(s))^p}{(Ms^{n-2})^{p-1}}\bigg]\,\mathrm{d}s=\infty. \tag{3.20}$$

If there exists a function $m \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that*

$$\frac{m(t)}{(a(t)E(t_0,t))^{1/(p-1)}\delta(t)}+m'(t)\le 0 \tag{3.21}$$

and, for some constant $\lambda_0 \in (0,1)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\bigg[q(s)\bigg(\frac{\lambda_0}{(n-2)!}g^{n-2}(s)\frac{m(g(s))}{m(s)}\delta(s)\bigg)^{p-1}E(t_0,s)-\varphi(s)\bigg]\,\mathrm{d}s=\infty, \tag{3.22}$$

then the conclusion of Theorem 3.1 *remains intact.*

Proof Assume that x is an eventually positive solution of (1.1) that satisfies (3.4). Similar analysis to that in Zhang *et al.* ([3], Lemma 2.3) leads to the conclusion that, for all $t \ge t_1$, there exist two possible cases (1) and (2) (as those in the proof of Theorem 3.1), where $t_1 \ge t_0$ is sufficiently large. Assume first that case (1) holds. We define the function ω by

$$\omega(t):=\rho(t)\frac{a(t)(x^{(n-1)}(t))^{p-1}}{x^{p-1}(\frac{t}{2})},\quad t\ge t_1.$$

With a similar proof as that of Case I in Theorem 3.1, one arrives at a contradiction with condition (3.20). Assume, instead, that case (2) holds. Define the function ν as in (3.8). As in the proof of Theorem 3.1, we obtain (3.11), (3.12), (3.14), and (3.15). On the other hand, we derive from (3.11) that

$$\frac{x^{(n-1)}(t)}{x^{(n-2)}(t)} \geq -\frac{1}{(a(t)E(t_0,t))^{1/(p-1)}\delta(t)}.$$

Using the latter inequality and (3.21), we have

$$\begin{aligned}\left(\frac{x^{(n-2)}(t)}{m(t)}\right)' &= \frac{x^{(n-1)}(t)m(t) - x^{(n-2)}(t)m'(t)}{m^2(t)} \\ &\geq -\frac{x^{(n-2)}(t)}{m^2(t)}\left[\frac{m(t)}{(a(t)E(t_0,t))^{1/(p-1)}\delta(t)} + m'(t)\right] \geq 0,\end{aligned}$$

which implies that $x^{(n-2)}/m$ is nondecreasing. Hence, it follows from (1.7) that

$$\frac{x^{(n-2)}(g(t))}{x^{(n-2)}(t)} \geq \frac{m(g(t))}{m(t)}.$$

Thus, by (3.14) and (3.15), we have

$$v'(t) \leq \frac{r(t)}{a(t)\delta^{p-1}(t)E(t_0,t)} - q(t)\left(\frac{\lambda g^{n-2}(t)}{(n-2)!}\right)^{p-1}\left(\frac{m(g(t))}{m(t)}\right)^{p-1} - (p-1)\frac{(-v(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}.$$

The remaining proof is similar to that of Case II in Theorem 3.1, and hence is omitted. □

Remark 3.5 The optional function m satisfying condition (3.21) exists and can be constructed by taking $m(t) := \delta(t)$.

Assume $n = 2$ and let ω be as follows:

$$\omega(t) := \rho(t)\frac{a(t)(x'(t))^{p-1}}{x^{p-1}(t)}, \quad t \geq t_1.$$

Then we obtain the following result that leads to the conclusion of Theorem 3.2.

Theorem 3.6 *Let conditions* (1.6) *and* (1.7) *be satisfied and* $n = 2$. *Assume that there exists a function* $\rho \in \mathrm{C}^1(\mathbb{I}, (0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)q(s) - \frac{\rho(s)a(s)(h_+(s))^p}{p^p}\right] \mathrm{d}s = \infty. \tag{3.23}$$

If there exists a function $m \in \mathrm{C}^1(\mathbb{I}, (0,\infty))$ *such that* (3.21) *is satisfied and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s)\left(\frac{m(g(s))}{m(s)}\delta(s)\right)^{p-1} E(t_0,s) - \varphi(s)\right] \mathrm{d}s = \infty, \tag{3.24}$$

then the conclusion of Theorem 3.2 *remains intact.*

Example 3.7 For $t \geq 1$, consider a second-order advanced differential equation with damping

$$\left(t^2x'(t)\right)' + \frac{t}{2}x'(t) + q_0x(2t) = 0, \tag{3.25}$$

where $q_0 > 0$ is a constant. Let $t_0 = 1$, $p = 2$, $a(t) = t^2$, $r(t) = t/2$, $q(t) = q_0$, $g(t) = 2t$, $\rho(t) = 1$, and $m(t) = \delta(t) = 2t^{-3/2}/3$. Similar analysis to that in Example 3.3 implies that condition (3.23) holds and condition (3.24) is satisfied for $q_0 > 2\sqrt{2}$. Thus, by Theorem 3.6, equation (3.25) is oscillatory if $q_0 > 2\sqrt{2}$. Observe that the results reported in [7, 20] cannot be applied to equation (3.25) since $g(t) > t$.

In the next theorem, we consider equation (1.1) under the assumptions that (1.7) holds and

$$A(t) := \int_t^{\infty} \frac{\mathrm{d}s}{a^{1/(p-1)}(s)} \quad \text{and} \quad A(t_0) < \infty. \tag{3.26}$$

Note that condition (1.6) is also satisfied in this case.

Theorem 3.8 *Let conditions* (1.7) *and* (3.26) *hold. Assume that there exists a function* $\rho \in \mathrm{C}^1(\mathbb{I}, (0,\infty))$ *such that* (3.20) *holds for all constants* $M > 0$. *If there exists a function* $\xi \in \mathrm{C}^1(\mathbb{I}, (0,\infty))$ *such that*

$$\frac{\xi(t)}{a^{1/(p-1)}(t)A(t)} + \xi'(t) \le 0 \tag{3.27}$$

and, for some constant $\lambda_0 \in (0,1)$,

$$r(t) \le q(t)\left(\frac{\lambda_0 g^{n-2}(t)\delta(g(t))}{(n-2)!}\right)^{p-1} a(t)E(t_0,t) \tag{3.28}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[q(s)\left(\frac{\lambda_0 g^{n-2}(s)\xi(g(s))A(s)}{(n-2)!\xi(s)}\right)^{p-1} - \frac{r(s)}{a(s)} - \frac{(p-1)^p}{p^p}\frac{1}{A(s)a^{1/(p-1)}(s)}\right]\mathrm{d}s = \infty, \tag{3.29}$$

then the conclusion of Theorem 3.1 *remains intact.*

Proof Assuming again that x is an eventually positive solution of (1.1) that satisfies (3.4) and proceeding as in the proof of Theorem 3.4, we end up having to show case (2) (as the corresponding case in Theorem 3.1). As in the proof of Case II in Theorem 3.1, one arrives at the inequalities (3.9), (3.10), and (3.14) which holds for all $\lambda \in (0,1)$. Inequalities (3.10) and (3.14) yield, for all $\lambda_0 \in (0,1)$ and for all sufficiently large t,

$$x(t) \ge -\frac{\lambda_0}{(n-2)!}t^{n-2}\big(a(t)E(t_0,t)\big)^{1/(p-1)}x^{(n-1)}(t)\delta(t). \tag{3.30}$$

From (1.1) and (3.30), we obtain

$$\begin{aligned}&\big(-a(t)\big(-x^{(n-1)}(t)\big)^{p-1}\big)' \\ &\quad \le r(t)\big(-x^{(n-1)}(t)\big)^{p-1} - a\big(g(t)\big)E\big(t_0,g(t)\big)\big(-x^{(n-1)}\big(g(t)\big)\big)^{p-1}q(t)\left(\frac{\lambda_0 g^{n-2}(t)\delta(g(t))}{(n-2)!}\right)^{p-1}.\end{aligned}$$

Using (3.9) and condition (1.7), we have

$$-a\big(g(t)\big)E\big(t_0,g(t)\big)\big(-x^{(n-1)}\big(g(t)\big)\big)^{p-1} \le -a(t)E(t_0,t)\big(-x^{(n-1)}(t)\big)^{p-1}.$$

Thus, by (3.28), we get

$$\begin{aligned}&\big(-a(t)\big(-x^{(n-1)}(t)\big)^{p-1}\big)'\\&\quad\le \big(-x^{(n-1)}(t)\big)^{p-1}\left[r(t)-q(t)\left(\frac{\lambda_0 g^{n-2}(t)\delta(g(t))}{(n-2)!}\right)^{p-1}a(t)E(t_0,t)\right]\le 0,\end{aligned}$$

which implies that, for all $s \ge t \ge t_1$,

$$x^{(n-1)}(s) \le \frac{a^{1/(p-1)}(t)}{a^{1/(p-1)}(s)}x^{(n-1)}(t).$$

Integrating this inequality from t to ι, we obtain

$$x^{(n-2)}(\iota) \le x^{(n-2)}(t) + a^{1/(p-1)}(t)x^{(n-1)}(t)\int_t^{\iota}\frac{\mathrm{d}s}{a^{1/(p-1)}(s)}.$$

Letting $\iota \to \infty$ and using the definition of A, we get

$$0 \le x^{(n-2)}(t) + a^{1/(p-1)}(t)x^{(n-1)}(t)A(t),$$

which yields

$$-\frac{x^{(n-1)}(t)}{x^{(n-2)}(t)}a^{1/(p-1)}(t)A(t) \le 1. \tag{3.31}$$

Now, we define the function ν by (3.8). From (3.8) and (3.31), we see that

$$-\nu(t)A^{p-1}(t) \le 1. \tag{3.32}$$

Differentiating (3.8) and using (1.1), we have (3.13). On the other hand, by (3.27) and (3.31), we obtain

$$\left(\frac{x^{(n-2)}(t)}{\xi(t)}\right)' = \frac{x^{(n-1)}(t)\xi(t)-x^{(n-2)}(t)\xi'(t)}{\xi^2(t)} \ge -\frac{x^{(n-2)}(t)}{\xi^2(t)}\left[\frac{\xi(t)}{a^{1/(p-1)}(t)A(t)}+\xi'(t)\right] \ge 0,$$

which shows that $x^{(n-2)}/\xi$ is nondecreasing. Hence, using condition (1.7), we get

$$\frac{x^{(n-2)}(g(t))}{x^{(n-2)}(t)} \ge \frac{\xi(g(t))}{\xi(t)}. \tag{3.33}$$

Thus, from (3.13), (3.14), (3.32), and (3.33), it follows that

$$\begin{aligned}\nu'(t) &\le \frac{r(t)}{a(t)A^{p-1}(t)} - q(t)\frac{x^{p-1}(g(t))}{(x^{(n-2)}(g(t)))^{p-1}}\frac{(x^{(n-2)}(g(t)))^{p-1}}{(x^{(n-2)}(t))^{p-1}} - (p-1)\frac{(-\nu(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}\\&\le \frac{r(t)}{a(t)A^{p-1}(t)} - q(t)\left(\frac{\lambda_0 g^{n-2}(t)\xi(g(t))}{(n-2)!\xi(t)}\right)^{p-1} - (p-1)\frac{(-\nu(t))^{p/(p-1)}}{a^{1/(p-1)}(t)}.\end{aligned} \tag{3.34}$$

Multiplying (3.34) by $A^{p-1}(t)$ and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned}&A^{p-1}(t)v(t) - A^{p-1}(t_1)v(t_1) - \int_{t_1}^{t} \frac{r(s)}{a(s)}\,ds + (p-1)\int_{t_1}^{t} a^{-1/(p-1)}(s)A^{p-2}(s)v(s)\,ds \\ &\quad + \int_{t_1}^{t} q(s)\left(\frac{\lambda_0 g^{n-2}(s)\xi(g(s))}{(n-2)!\xi(s)}\right)^{p-1} A^{p-1}(s)\,ds \\ &\quad + (p-1)\int_{t_1}^{t} \frac{(-v(s))^{p/(p-1)}}{a^{1/(p-1)}(s)} A^{p-1}(s)\,ds \le 0.\end{aligned}$$

Let

$$y := -v(s), \qquad D := a^{-1/(p-1)}(s)A^{p-2}(s), \quad \text{and} \quad C := A^{p-1}(s)/a^{1/(p-1)}(s).$$

Using inequality (3.7), we derive from (3.32) that

$$\begin{aligned}&\int_{t_1}^{t}\left[q(s)\left(\frac{\lambda_0 g^{n-2}(s)\xi(g(s))A(s)}{(n-2)!\xi(s)}\right)^{p-1} - \frac{r(s)}{a(s)} - \frac{(p-1)^p}{p^p}\frac{1}{A(s)a^{1/(p-1)}(s)}\right]ds \\ &\quad \le A^{p-1}(t_1)v(t_1) + 1,\end{aligned}$$

which contradicts (3.29). This completes the proof. □

Remark 3.9 The optional function ξ satisfying assumption (3.27) can reasonably be constructed by taking $\xi(t) := A(t)$.

Similarly, we have the following criterion for (1.1) in the case when $n = 2$.

Theorem 3.10 *Let* (1.7) *and* (3.26) *be satisfied and* $n = 2$. *Suppose that there exists a function* $\rho \in C^1(\mathbb{I}, (0,\infty))$ *such that* (3.23) *holds. If*

$$r(t) \le q(t)\delta^{p-1}\big(g(t)\big)a(t)E(t_0, t) \tag{3.35}$$

and there exists a function $\xi \in C^1(\mathbb{I}, (0,\infty))$ *such that* (3.27) *holds and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[q(s)\left(\frac{\xi(g(s))A(s)}{\xi(s)}\right)^{p-1} - \frac{r(s)}{a(s)} - \frac{(p-1)^p}{p^p}\frac{1}{A(s)a^{1/(p-1)}(s)}\right]ds = \infty, \tag{3.36}$$

then the conclusion of Theorem 3.2 *remains intact.*

Example 3.11 For $t \ge 1$ and $q_0 > 0$, consider the second-order advanced differential equation (3.25). Let $t_0 = 1$, $p = 2$, $a(t) = t^2$, $r(t) = t/2$, $q(t) = q_0$, $g(t) = 2t$, $\rho(t) = 1$, and $\xi(t) = A(t) = t^{-1}$. Then $E(t_0, t) = t^{1/2}$, $\delta(t) = 2t^{-3/2}/3$, and $h_+(t) = 0$. It is not difficult to verify that all conditions of Theorem 3.10 are satisfied for $q_0 \ge 3\sqrt{2}/2$. Therefore, using Theorem 3.10, equation (3.25) is oscillatory provided that $q_0 \ge 3\sqrt{2}/2$, whereas Theorem 3.6 implies that equation (3.25) is oscillatory if $q_0 > 2\sqrt{2}$. Hence, Theorem 3.10 improves Theorem 3.6 in some cases. However, to achieve such improvement, an additional assumption (3.35) is required. Therefore, we observe that Theorems 3.4, 3.6, 3.8, and 3.10 are of independent interest.

The following example is provided to show that our results are sharp for the second-order Euler differential equation $(t^2x'(t))' + q_0x(t) = 0$, $q_0 > 0$.

Example 3.12 For $t \geq 1$, consider a second-order differential equation with damping

$$\big(t^2x'(t)\big)' + r_0x'(t) + q_0x(t) = 0, \tag{3.37}$$

where $r_0 \geq 0$ and $q_0 > 0$ are constants. Let $t_0 = 1, p = 2, a(t) = t^2, r(t) = r_0, q(t) = q_0, g(t) = t$, and $\rho(t) = 1$. Then $h_+(t) = 0$ and so condition (3.23) is satisfied. It is not hard to verify that $1 \leq E(t_0, t) \leq \mathrm{e}^{r_0}$, $\mathrm{e}^{-r_0}t^{-1} \leq \delta(t) \leq t^{-1}$, and $A(t) = 1/t$. Then condition (3.35) is satisfied for all sufficiently large t and, for $q_0 > 1/4$,

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^t \left[q(s)\left(\frac{\xi(g(s))A(s)}{\xi(s)}\right)^{p-1} - \frac{r(s)}{a(s)} - \frac{(p-1)^p}{p^p}\frac{1}{A(s)a^{1/(p-1)}(s)}\right] \mathrm{d}s \\
&\quad = \limsup_{t\to\infty} \int_1^t \left[\frac{q_0}{s} - \frac{r_0}{s^2} - \frac{1}{4s}\right] \mathrm{d}s = \infty.
\end{aligned}$$

Hence, by Theorem 3.10, equation (3.37) is oscillatory provided that $q_0 > 1/4$ (it is well known that $q_0 > 1/4$ is the best possible for the oscillation of equation (3.37) when $r_0 = 0$). Observe, however, that if $\gamma = 1$, then

$$\begin{aligned}
&\int_{t_0}^\infty \left[\frac{1}{a(t)\exp(\int_{t_0}^t \frac{r(s)}{a(s)}\,\mathrm{d}s)} \int_{t_0}^t q(s)\exp\left(\int_{t_0}^s \frac{r(u)}{a(u)}\,\mathrm{d}u\right)\right. \\
&\qquad \times \left.\left(\int_{g(s)}^\infty \frac{\mathrm{d}u}{(a(u)\exp(\int_{t_0}^u \frac{r(v)}{a(v)}\,\mathrm{d}v))^{1/\gamma}}\right)^\gamma \mathrm{d}s\right]^{1/\gamma} \mathrm{d}t \\
&\quad \leq q_0\mathrm{e}^{r_0}\int_1^\infty \frac{\ln t}{t^2}\,\mathrm{d}t < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^\infty \left[\frac{1}{a(t)}\int_{t_0}^t \exp\left(-\int_s^t \frac{r(\tau)}{a(\tau)}\,\mathrm{d}\tau\right)\left(\int_s^\infty \left(\frac{1}{a(u)}\right)^{1/\gamma} \mathrm{d}u\right)^\gamma q(s)\,\mathrm{d}s\right]^{1/\gamma} \mathrm{d}t \\
&\quad \leq q_0 \int_1^\infty \frac{\ln t}{t^2}\,\mathrm{d}t < \infty,
\end{aligned}$$

which mean that the results reported in [7, 20] cannot be applied to equation (3.37).

Finally, the following example is given to present an open problem of this paper.

Example 3.13 For $t \geq 1$, consider the second-order Euler differential equation

$$\big(t^2x'(t)\big)' + \frac{t}{2}x'(t) + q_0x(t) = 0, \tag{3.38}$$

where $q_0 > 0$ is a constant. Let $t_0 = 1, p = 2, a(t) = t^2, r(t) = t/2, q(t) = q_0, g(t) = t/2, \rho(t) = 1$, $m(t) = \delta(t) = 2t^{-3/2}/3$, and $\xi(t) = A(t) = t^{-1}$. It is easy to see that $h_+(t) = 0$, $E(t_0, t) = t^{1/2}$, $\phi(t) = 2t^{-1/2}/3$, and $\varphi(t) = 2t^{-1}/3$. Applications of Theorems 3.2 and 3.6 imply that equation (3.38) is oscillatory if $q_0 > 1$, whereas Theorem 3.10 yields oscillation of equation (3.38) for $q_0 > 3/4$. Similar analysis to that in Example 3.3 shows that the results obtained in [7, 20] fail to apply in equation (3.38). However, it is well known that equation (3.38) is oscillatory if and only if $q_0 > 9/16$. How to extend this sharp result to equation (1.1) remains open at the moment.

4 Asymptotic results via the integral averaging technique

In this section, we employ the integral averaging technique to establish Philos-type (see Philos [15]) asymptotic tests for (1.1). In the following, we use the notation $\mathbb{D} := \{(t,s) : t \geq s \geq t_0\}$. We say that a continuous function $H : \mathbb{D} \to [0,\infty)$ belongs to the class $\mathfrak{H}$ if

(i) $H(t,t) = 0$ for $t \geq t_0$, and $H(t,s) > 0$ for $t > s \geq t_0$;

(ii) H has a nonpositive continuous partial derivative $\partial H/\partial s$ with respect to the second variable satisfying, for some locally integrable function $\varrho \in L_{\mathrm{loc}}(\mathbb{D},\mathbb{R})$ and for some function $\rho \in \mathrm{C}^1(\mathbb{I},(0,\infty))$,

$$\frac{\partial}{\partial s}H(t,s) + h(s)H(t,s) = \frac{\varrho(t,s)}{\rho(s)}\bigl(H(t,s)\bigr)^{(p-1)/p}.$$

We say that a continuous function $K : \mathbb{D} \to [0,\infty)$ belongs to the class $\mathfrak{K}$ if

(j) $K(t,t) = 0$ for $t \geq t_0$, and $K(t,s) > 0$ for $t > s \geq t_0$;

(jj) K has a nonpositive continuous partial derivative $\partial K/\partial s$ with respect to the second variable satisfying, for some locally integrable function $\zeta \in L_{\mathrm{loc}}(\mathbb{D},\mathbb{R})$,

$$-\frac{\partial}{\partial s}K(t,s) = \zeta(t,s)\bigl(K(t,s)\bigr)^{(p-1)/p}.$$

Theorem 4.1 *Let conditions* (1.2), (1.6), *and* (3.1) *be satisfied. Assume that there exists a function $H \in \mathfrak{H}$ such that, for all constants $M > 0$ and for all $t_1 \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \biggl[H(t,s)\rho(s)q(s) - \frac{2^{p-1}}{p^p} \frac{a(s)(\varrho_+(t,s))^p}{(Mg'(s)g^{n-2}(s)\rho(s))^{p-1}}\biggr]\,\mathrm{d}s = \infty. \tag{4.1}$$

If there exists a function $K \in \mathfrak{K}$ such that, for some constant $\lambda_0 \in (0,1)$ and for all $t_1 \geq t_0$,

$$\begin{aligned}\limsup_{t\to\infty} \int_{t_1}^{t} \biggl[&K(t,s)q(s)\biggl(\frac{\lambda_0}{(n-2)!}g^{n-2}(s)\biggr)^{p-1} \\ &- \frac{K(t,s)r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)} - \frac{a(s)(\zeta(t,s))^p}{p^p}\biggr]\,\mathrm{d}s > 0,\end{aligned} \tag{4.2}$$

then the conclusion of Theorem 3.1 *remains intact.*

Proof Assume that x is an eventually positive solution of (1.1) that satisfies (3.4). Similar analysis to that in Zhang *et al.* ([3], Lemma 2.3) leads to the conclusion that, for all $t \geq t_1$, there exist two possible cases (1) and (2) (as those in the proof of Theorem 3.1), where $t_1 \geq t_0$ is sufficiently large. First, assume that case (1) holds. Defining the function ω by (3.5) and proceeding as in the proof of Theorem 3.1, we arrive at inequality (3.6). Multiplying (3.6) by $H(t,s)$ and integrating the resulting inequality from t_1 to t, we obtain

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)\rho(s)q(s)\,\mathrm{d}s \\ &\quad\leq H(t,t_1)\omega(t_1) + \int_{t_1}^{t}\biggl[\frac{\partial H(t,s)}{\partial s} + h(s)H(t,s)\biggr]\omega(s)\,\mathrm{d}s\end{aligned}$$

$$\begin{aligned} &- \int_{t_1}^{t} \frac{p-1}{2} MH(t,s)g'(s)g^{n-2}(s)\frac{\omega^{p/(p-1)}(s)}{(\rho(s)a(s))^{1/(p-1)}}\,\mathrm{d}s \\ &\leq H(t,t_1)\omega(t_1) + \int_{t_1}^{t} \frac{\varrho_+(t,s)}{\rho(s)}\bigl(H(t,s)\bigr)^{(p-1)/p}\omega(s)\,\mathrm{d}s \\ &- \int_{t_1}^{t} \frac{p-1}{2} MH(t,s)g'(s)g^{n-2}(s)\frac{\omega^{p/(p-1)}(s)}{(\rho(s)a(s))^{1/(p-1)}}\,\mathrm{d}s. \end{aligned}$$

Let

$$C^{p/(p-1)} := \frac{p-1}{2} MH(t,s)g'(s)g^{n-2}(s)\frac{\omega^{p/(p-1)}(s)}{(\rho(s)a(s))^{1/(p-1)}}$$

and

$$D^{1/(p-1)} := \frac{2^{(p-1)/p}(p-1)^{1/p}\varrho_+(t,s)(a(s)\rho(s))^{1/p}}{pM^{(p-1)/p}\rho(s)(g'(s)g^{n-2}(s))^{(p-1)/p}}.$$

Using the following inequality (a variation of the well-known Young inequality)

$$\frac{p}{p-1} CD^{1/(p-1)} - C^{p/(p-1)} \leq \frac{1}{p-1} D^{p/(p-1)}, \tag{4.3}$$

where $p > 1$, $C \geq 0$, and $D \geq 0$, we conclude that

$$\begin{aligned} &\frac{\varrho_+(t,s)}{\rho(s)}\bigl(H(t,s)\bigr)^{(p-1)/p}\omega(s) - \frac{p-1}{2} MH(t,s)g'(s)g^{n-2}(s)\frac{\omega^{p/(p-1)}(s)}{(\rho(s)a(s))^{1/(p-1)}} \\ &\leq \frac{2^{p-1}}{p^p}\frac{a(s)(\varrho_+(t,s))^p}{(Mg'(s)g^{n-2}(s)\rho(s))^{p-1}}. \end{aligned}$$

Hence, we have

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\biggl[H(t,s)\rho(s)q(s) - \frac{2^{p-1}}{p^p}\frac{a(s)(\varrho_+(t,s))^p}{(Mg'(s)g^{n-2}(s)\rho(s))^{p-1}}\biggr]\,\mathrm{d}s \leq \omega(t_1),$$

which contradicts (4.1). Assume now that case (2) holds and define the function ν as in (3.8). As in the proof of Theorem 3.1, we obtain (3.16). Multiplying (3.16) by $K(t,s)$ and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned} &\int_{t_1}^{t} K(t,s)\biggl[q(s)\biggl(\frac{\lambda g^{n-2}(s)}{(n-2)!}\biggr)^{p-1} - \frac{r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)}\biggr]\,\mathrm{d}s \\ &\leq K(t,t_1)\nu(t_1) + \int_{t_1}^{t}\frac{\partial K(t,s)}{\partial s}\nu(s)\,\mathrm{d}s - \int_{t_1}^{t}(p-1)K(t,s)\frac{(-\nu(s))^{p/(p-1)}}{a^{1/(p-1)}(s)}\,\mathrm{d}s \\ &= K(t,t_1)\nu(t_1) - \int_{t_1}^{t}\zeta(t,s)\bigl(K(t,s)\bigr)^{(p-1)/p}\nu(s)\,\mathrm{d}s - \int_{t_1}^{t}(p-1)K(t,s)\frac{(-\nu(s))^{p/(p-1)}}{a^{1/(p-1)}(s)}\,\mathrm{d}s. \end{aligned}$$

Let

$$C^{p/(p-1)} := (p-1)K(t,s)\frac{(-\nu(s))^{p/(p-1)}}{a^{1/(p-1)}(s)} \quad\text{and}\quad D^{1/(p-1)} := \frac{(p-1)^{1/p}\zeta(t,s)a^{1/p}(s)}{p}.$$

Using inequality (4.3), we obtain

$$\int_{t_1}^{t}\left[K(t,s)q(s)\left(\frac{\lambda}{(n-2)!}g^{n-2}(s)\right)^{p-1}-\frac{K(t,s)r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)}-\frac{a(s)(\zeta(t,s))^p}{p^p}\right]\mathrm{d}s$$
$$\leq K(t,t_1)v(t_1)<0,$$

which contradicts (4.2). This completes the proof. □

The validity of the following five propositions can be established in a similar manner as in the proof of Theorem 4.1. Therefore, to avoid unnecessary repetition, we only formulate the contents of the following theorems.

Theorem 4.2 *Let conditions* (1.2), (1.6), *and* (3.1) *be satisfied and* $n = 2$. *Assume that there exists a function* $H \in \mathfrak{H}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)q(s)-\frac{1}{p^p}\frac{a(s)(\varrho_+(t,s))^p}{(g'(s)\rho(s))^{p-1}}\right]\mathrm{d}s=\infty. \quad (4.4)$$

If there exists a function $K \in \mathfrak{K}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[K(t,s)q(s)-\frac{K(t,s)r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)}-\frac{a(s)(\zeta(t,s))^p}{p^p}\right]\mathrm{d}s>0, \quad (4.5)$$

then the conclusion of Theorem 3.2 *remains intact.*

Theorem 4.3 *Let conditions* (1.6) *and* (1.7) *hold. Assume that there exists a function* $H \in \mathfrak{H}$ *such that, for all constants* $M > 0$ *and for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)q(s)-\frac{2^{p-1}}{p^p}\frac{a(s)(\varrho_+(t,s))^p}{(Ms^{n-2}\rho(s))^{p-1}}\right]\mathrm{d}s=\infty. \quad (4.6)$$

Furthermore, suppose that there exists a function $m \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that* (3.21) *is satisfied. If there exists a function* $K \in \mathfrak{K}$ *such that, for some constant* $\lambda_0 \in (0,1)$ *and for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[K(t,s)q(s)\left(\frac{\lambda_0 g^{n-2}(s)m(g(s))}{(n-2)!m(s)}\right)^{p-1}\right.$$
$$\left.-\frac{K(t,s)r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)}-\frac{a(s)(\zeta(t,s))^p}{p^p}\right]\mathrm{d}s>0, \quad (4.7)$$

then the conclusion of Theorem 3.1 *remains intact.*

Theorem 4.4 *Let conditions* (1.6) *and* (1.7) *be satisfied and* $n = 2$. *Assume that there exists a function* $H \in \mathfrak{H}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\rho(s)q(s)-\frac{1}{p^p}\frac{a(s)(\varrho_+(t,s))^p}{\rho^{p-1}(s)}\right]\mathrm{d}s=\infty. \quad (4.8)$$

Furthermore, suppose that there exists a function $m \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that* (3.21) *is satisfied. If there exists a function* $K \in \mathfrak{K}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\biggl[K(t,s)q(s)\biggl(\frac{m(g(s))}{m(s)}\biggr)^{p-1} - \frac{K(t,s)r(s)}{a(s)\delta^{p-1}(s)E(t_0,s)} - \frac{a(s)(\zeta(t,s))^p}{p^p}\biggr]\,\mathrm{d}s > 0, \tag{4.9}$$

then the conclusion of Theorem 3.2 *remains intact.*

Theorem 4.5 *Let conditions* (1.7) *and* (3.26) *hold and let condition* (3.28) *be satisfied for some constant* $\lambda_0 \in (0,1)$. *Assume that there exists a function* $H \in \mathfrak{H}$ *such that* (4.6) *holds for all constants* $M > 0$ *and for all* $t_1 \geq t_0$. *Furthermore, suppose that there exists a function* $\xi \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that* (3.27) *holds. If there exists a function* $K \in \mathfrak{K}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\biggl[K(t,s)q(s)\biggl(\frac{\lambda_0 g^{n-2}(s)\xi(g(s))}{(n-2)!\xi(s)}\biggr)^{p-1} - \frac{K(t,s)r(s)}{a(s)A^{p-1}(s)} - \frac{a(s)(\zeta(t,s))^p}{p^p}\biggr]\,\mathrm{d}s > 0, \tag{4.10}$$

then the conclusion of Theorem 3.1 *remains intact.*

Theorem 4.6 *Let conditions* (1.7), (3.26), *and* (3.35) *be satisfied and* $n = 2$. *Assume that there exists a function* $H \in \mathfrak{H}$ *such that* (4.8) *holds for all* $t_1 \geq t_0$. *Furthermore, suppose that there exists a function* $\xi \in \mathrm{C}^1(\mathbb{I},(0,\infty))$ *such that* (3.27) *holds. If there exists a function* $K \in \mathfrak{K}$ *such that, for all* $t_1 \geq t_0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\biggl[K(t,s)q(s)\biggl(\frac{\xi(g(s))}{\xi(s)}\biggr)^{p-1} - \frac{K(t,s)r(s)}{a(s)A^{p-1}(s)} - \frac{a(s)(\zeta(t,s))^p}{p^p}\biggr]\,\mathrm{d}s > 0, \tag{4.11}$$

then the conclusion of Theorem 3.2 *remains intact.*

5 Conclusions

In this paper, we have established new asymptotic criteria for even-order damped differential equations with p-Laplacian like operators (1.1) assuming that (1.6) holds. Note that condition (1.6) brings about additional difficulties in the study of the asymptotic behavior of (1.1). One of the principal difficulties arises from the sign of $x^{(n-1)} < 0$ (which is simply eliminated if condition (1.3) holds; *cf.* [3]). Since the sign of the derivative $x^{(n-1)}$ is not known, our theorems for the asymptotic properties of (1.1) include a pair of assumptions, as for instance, (3.2) and (3.3).

Most asymptotic results reported in the literature for equation (1.1) and its particular cases have been obtained under the assumption (1.2); see, for instance, the papers [3, 7, 13, 16, 17, 20, 21]. Examples 3.3, 3.12, and 3.13 show that the results obtained in this paper improve those reported in [7, 20]. Furthermore, our theorems complement the related results in the cited papers since these criteria can be applied to the case (1.7).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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