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Nonlinear problem with subcritical exponent in Sobolev space

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Abstract
Using Brouwer's fixed point theorem, we prove the existence of solutions for some nonlinear problem with subcritical Sobolev exponent in S_+^4.

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1 Introduction and the main result

The exponent Lebesgue space $L^p(\Omega)$ is defined by

$$L^p(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega |u(x)|^p \, dx < \infty \right\}.$$

This space is endowed with the norm

$$\|u\|_{L^p(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^p dx \le 1 \right\}.$$

The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in W^{1,1}_{\mathrm{loc}}(\Omega) : u \in L^p(\Omega) \text{ and } |\nabla u| \in L^p(\Omega) \right\}.$$

The corresponding norm for this space is

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Define $W^1_0(\Omega) = H^1_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ with respect to the $W^{1,p}(\Omega)$ norm which is a Hilbert space [1].

We consider the problem of the scalar curvature on the standard four dimensional half sphere under minimal boundary conditions (S):

$$(S) \quad \begin{cases} L_g u := -\Delta_g u + 2u = Ku^3, & u > 0 \quad \text{in } S^4_+, \\ \frac{\partial u}{\partial \nu} = 0 & \quad\quad\;\; \text{on } \partial S^4_+, \end{cases}$$

where $S^4_+ = \{x \in \mathbb{R}^5/|x| = 1, x_5 > 0\}$, g is the standard metric, and K is a C^3 positive Morse function on $\overline{S^4_+}$.

The scalar curvature problem on S^n and also on S^n_+ was the subject of several works in recent years, we can cite for example [2–12].

Recall that the embedding of $H^1(S^4_+)$ into $L^4(S^4_+)$ is noncompact. For this reason, we have focused our study on the family of subcritical problems (S_ε)

$$(S_\varepsilon) \quad \begin{cases} -\Delta_g u + 2u = Ku^{3-\varepsilon}, & u > 0 \quad \text{in } S^4_+, \\ \frac{\partial u}{\partial \nu} = 0 & \quad\text{on } \partial S^4_+, \end{cases}$$

where ε is a small positive parameter.

Note that the solutions of problem (S) can be the limit as $\varepsilon \to 0$ of some solutions (u_ε) for (S_ε).

Djadli *et al.* [13] studied this problem in the case of the three dimensional half sphere. Assuming that the critical points of K_1 verify $(\partial K/\partial \nu)(a_i) > 0$ they demonstrated that there exist solutions (u_ε) concentrated at the points $(a_1, \dots, a_p)$. Moreover, in [14], we established the existence of another type of solutions (u_ε) of (S_ε) such that is concentrated at two points $a_1 \in \partial S^4_+$ and $a_2 \in S^4_+$.

In this work, we aim to construct some positive solutions of (S_ε) which are concentrated at two different points of the boundary. To state our result, we will give the following notations. For $a \in \overline{S^4_+}$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda}{(\lambda^2 + 1 + (1 - \lambda^2)\cos d(a,x))}, \tag{1}$$

where d is the geodesic distance on $(\overline{S^4_+}, g)$ and c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem:

$$-\Delta u + 2u = u^3, \qquad u > 0, \quad \text{in } S^4.$$

The space $H^1(S^4_+)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $\langle \cdot, \cdot \rangle$:

$$\|f\|^2 = \int_{S^4_+} |\nabla f|^2 + 2\int_{S^4_+} f^2, \quad \text{and} \quad \langle f, g\rangle = \int_{S^4_+} \nabla f \nabla g + 2\int_{S^4_+} fg, \quad f, g \in H^1\big(S^4_+\big).$$

Theorem 1 *Let z_1 and z_2 be a nondegenerate critical points of $K_1 = K_{|\partial S^4_+}$ with $(\partial K/\partial \nu)(z_i) > 0$, $i = 1, 2$. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (S_ε) has a solution (u_ε) of the form*

$$u_\varepsilon = \alpha_1 \delta_{(x_1,\lambda_1)} + \alpha_2 \delta_{(x_2,\lambda_2)} + v,$$

where, as $\varepsilon \to 0$, $\alpha_i \to K(z_i)^{-1/2}$; $\|v\| \to 0$; $x_i \to z_i$; $x_i \in \partial S^4_+$; $\lambda_i \to +\infty$; $\lambda_1 = c\lambda_2(1 + o(1))$.

The rest of this work is summarized as follows. In Section 2, we present a classical preliminaries and we perform a useful estimations of functional (I_ε) associated to the problem (S_ε) for $(\varepsilon > 0)$ and its gradient. Section 3 is devoted to the construction of solutions and the proof of our result.

2 Useful estimations

We introduce the structure variational associated to the problem (S_ε) for $\varepsilon > 0$

$$I_\varepsilon(u) = \frac{1}{2}\int_{S_+^4} |\nabla u|^2 + \int_{S_+^4} u^2 - \frac{1}{4-\varepsilon}\int_{S_+^4} K|u|^{4-\varepsilon}, \quad u \in H^1(S_+^4). \tag{2}$$

It is well known that there is an equivalence between the existence of solutions for (S_ε) and the positive critical point of I_ε. Moreover, in order to reduce our problem to $\mathbb{R}_+^4$ we will perform some stereographic projection. We denote $D^{1,2}(\mathbb{R}_+^4)$ for the completion of $C_c^\infty(\overline{\mathbb{R}_+^4})$ with respect to the Dirichlet norm. Recall that an isometry $\imath : H^1(S_+^4) \to D^{1,2}(\mathbb{R}_+^4)$ is induced by the stereographic projection π_a about a point $a \in \partial S_+^4$ following the formula

$$(\imath\phi)(y) = \left(\frac{2}{1+|x|^2}\right)\phi\left(\pi_a^{-1}(y)\right), \quad \phi \in H^1(S_+^4), y \in \mathbb{R}_+^4. \tag{3}$$

For every $\phi \in H^1(S_+^4)$, one can check that the following holds true:

$$\int_{S_+^4}\left(|\nabla\phi|^2 + 2\phi^2\right) = \int_{\mathbb{R}_+^4}\left|\nabla(\imath\phi)\right|^2 \quad \text{and} \quad \int_{S_+^4}|\phi|^4 = \int_{\mathbb{R}_+^4}|\imath\phi|^4.$$

Furthermore, using (3) with π_{-a}, it is easy to see that $\imath\delta_{(a,\lambda)}$ is given by

$$\imath\delta_{(a,\lambda)} = \frac{c_0\lambda}{1+\lambda^2|x-a|^2}.$$

$\delta_{(a,\lambda)}$ will be written instead of $\imath\delta_{(a,\lambda)}$ in the sequel.

Let

$$\begin{aligned} M_\varepsilon = \Bigg\{ & m = (\alpha,\lambda,x,v) \in \mathbb{R}^2 \times \left(\mathbb{R}_+^*\right)^2 \times \left(\partial S_+^4\right)^2 \times H^1\left(S_+^4\right) : v \in E_{(x,\lambda)}, \|v\| < v_0; \\ & \left|\frac{\alpha_i^2 K(x_i)}{\alpha_j^2 K(x_j)} - 1\right| < v_0, \lambda_i > \frac{1}{v_0}, \varepsilon\log\lambda_i < v_0, \forall i; c_0 < \frac{\lambda_1}{\lambda_2} < c_0^{-1}; |x_1 - x_2| > d_0; \\ & \left|-2c_3\frac{\partial K}{\partial\nu}(x_i)\frac{1}{\lambda_i} + \frac{\varepsilon K(x_i)S_4}{8}\right| < \varepsilon^{1+\frac{\sigma}{2}}\Bigg\}, \end{aligned}$$

where v_0 is a small positive constant, σ, c_0, d_0 are some suitable positive constants, and

$$E_{(x,\lambda)} = \left\{ w \in H^1\left(S_+^4\right) / \langle w,\varphi\rangle = 0\ \forall\varphi \in \operatorname{Span}\left\{\delta_i, \frac{\partial\delta_i}{\partial\lambda_i}, \frac{\partial\delta_i}{\partial x_i^j}, i = 1,2; j \le 4\right\}\right\}.$$

Here, x_i^j denotes the jth component of x_i. Also

$$\Psi_\varepsilon : M_\varepsilon \to \mathbb{R}; \quad m = (\alpha,\lambda,x,v) \mapsto I_\varepsilon(\alpha_1\delta_{(x_1,\lambda_1)} + \alpha_2\delta_{(x_2,\lambda_2)} + v). \tag{4}$$

In the sequel, we will write δ_i instead of $\delta_{(x_i,\lambda_i)}$ and $u = \alpha_1\delta_1 + \alpha_2\delta_2 + v$ for the sake of simplicity.

In the remainder of this section, we will give expansions of the gradient of the functional I_ε associated to (S_ε) for $\varepsilon > 0$. Thus estimations are needed in Section 3. We need to recall that [15] proved the following remark when $n = 3$, but the same argument is available for the dimension 4.

Remark 2 For $\varepsilon > 0$ and $\delta_{(a,\lambda)}$ defined in (1), we have

$$\delta_{(a,\lambda)}^{-\varepsilon}(x) = 1 - \varepsilon \log \delta_{(a,\lambda)} + O\bigl(\varepsilon^2 \log^2 \lambda\bigr) \quad \text{in } S_+^4.$$

Now, explicit computations, using Remark 2, yield the following propositions.

Proposition 3 *Let $(\alpha, \lambda, x, v) \in M_\varepsilon$. Then, for $u = \alpha_1 \delta_{(x_1,\lambda_1)} + \alpha_2 \delta_{(x_2,\lambda_2)} + v$, we have the following expansion*:

$$\bigl\langle \nabla I_\varepsilon(u), \delta_i \bigr\rangle = \frac{\alpha_i S_4}{2}\bigl(1 - \alpha_i^{2-\varepsilon} K(x_i)\bigr) + O\biggl(\varepsilon \log \lambda_i + \frac{1}{\lambda_i} + \varepsilon_{12} + \|v\|^2\biggr),$$

where

$$\varepsilon_{ij} = \frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2},$$

$$S_4 = 64 \int_{\mathbb{R}^4} \frac{dx}{(1 + |x|^2)^4}.$$

Proof We have

$$\bigl\langle \nabla I_\varepsilon(u), h \bigr\rangle = \int_{S_+^4} \nabla u \nabla h + 2 \int_{S_+^4} uh - \int_{S_+^4} K u^{3-\varepsilon} h. \tag{5}$$

A computation similar to the one performed in [16] shows that, for $i = 1, 2$,

$$\|\delta_i\|^2 = \int_{\mathbb{R}_+^4} |\nabla \delta_i|^2 = \frac{S_4}{2} \tag{6}$$

and

$$\int_{S_+^4} \nabla \delta_i \nabla \delta_j + 2 \int_{S_+^4} \delta_i \delta_j = \int_{\mathbb{R}_+^4} \nabla \delta_i \nabla \delta_j = \int_{\mathbb{R}_+^4} \delta_i^3 \delta_j = O(\varepsilon_{12}). \tag{7}$$

For the other integral, we write

$$\int_{S_+^4} K u^{3-\varepsilon} \delta_i = \int_{S_+^4} K(\alpha_1 \delta_1 + \alpha_2 \delta_2)^{3-\varepsilon} \delta_i + O\bigl(\varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + |v|^2\bigr). \tag{8}$$

We also write

$$\begin{aligned} \int_{S_+^4} K(\alpha_1 \delta_1 + \alpha_2 \delta_2)^{3-\varepsilon} \delta_i &= \alpha_i^{3-\varepsilon} \int_{S_+^4} K \delta_i^{4-\varepsilon} + \alpha_j^{3-\varepsilon} \int_{S_+^4} K \delta_j^{3-\varepsilon} \delta_i \\ &\quad + (3 - \varepsilon) \alpha_i^{2-\varepsilon} \alpha_j \int_{S_+^4} K \delta_i^{3-\varepsilon} \delta_j + O\bigl(\varepsilon_{12}^2 \log \varepsilon_{12}^{-1}\bigr). \end{aligned} \tag{9}$$

Expansions of K around x_i and x_j give

$$\int_{S_+^4} K\delta_i^{4-\varepsilon} = \int_{\mathbb{R}_+^4} K\delta_i^{4-\varepsilon} = K(x_i)\frac{S_4}{2} + O\left(\varepsilon \log \lambda_i + \frac{1}{\lambda_i}\right), \tag{10}$$

$$\int_{S_+^4} K\delta_j^{3-\varepsilon}\delta_i = \int_{\mathbb{R}_+^4} K\delta_j^{3-\varepsilon}\delta_i = O(\varepsilon \log \lambda_i + \varepsilon_{12}), \tag{11}$$

$$\int_{S_+^4} K\delta_i^{3-\varepsilon}\delta_j = \int_{\mathbb{R}_+^4} K\delta_i^{3-\varepsilon}\delta_j = O(\varepsilon \log \lambda_i + \varepsilon_{12}). \tag{12}$$

Combining (5)-(12), we derive our proposition. □

Proposition 4 *Let $(\alpha, \lambda, x, v) \in M_\varepsilon$. Then, for $u = \alpha_1\delta_{(x_1,\lambda_1)} + \alpha_2\delta_{(x_2,\lambda_2)} + v$, we have*

$$\begin{aligned}\left\langle \nabla I_\varepsilon(u), \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right\rangle &= \alpha_j\left(1 - \alpha_j^{2-\varepsilon}K(x_j) - \alpha_i^{2-\varepsilon}K(x_i)\right)c_2\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \alpha_i^{3-\varepsilon}\frac{\varepsilon S_4 K(x_i)}{8} \\ &\quad + \alpha_i^{3-\varepsilon}\frac{2c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(x_i) + O\left(\|v\|^2 + \frac{1}{\lambda_i^2} + \varepsilon^2\log\lambda_i + \frac{\varepsilon\log\lambda_i}{\lambda_i}\right) \\ &\quad + O\left(\varepsilon\varepsilon_{12}\left(\log\varepsilon_{12}^{-1}\right)^{1/2} + \varepsilon_{12}^2\log\varepsilon_{12}^{-1} + \frac{\varepsilon_{12}}{\lambda_j}\left(\log\varepsilon_{12}^{-1}\right)^{1/2}\right),\end{aligned}$$

where

$$S_4 = 64\int_{\mathbb{R}^4}\frac{dx}{(1+|x|^2)^4}, \qquad c_2 = 64\int_{\mathbb{R}^4}\frac{dx}{(1+|x|^2)^3}, \qquad c_3 = 64\int_{\mathbb{R}_+^4}\frac{x_4(|x|^2-1)}{(1+|x|^2)^5}\,dx.$$

Proof Observe that (see [16])

$$\int_{\mathbb{R}_+^4}\nabla\delta_i\nabla\left(\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right) = \int_{\mathbb{R}_+^4}\delta_i^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = 0, \tag{13}$$

$$\int_{\mathbb{R}_+^4}\nabla\delta_j\nabla\left(\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right) = \int_{\mathbb{R}_+^4}\delta_j^3\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = \frac{1}{2}c_2\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + O\left(\varepsilon_{12}^2\log\left(\varepsilon_{12}^{-1}\right)\right). \tag{14}$$

For the other part, we have the expansions of K around x_i and using Remark 2,

$$\int_{\mathbb{R}_+^4}K\delta_i^{3-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = -\frac{\varepsilon S_4K(x_i)}{8} - \frac{2c_3}{\lambda_i}\nabla K(x_i)e_4 + O\left(\varepsilon^2\log\lambda_i + \frac{1}{\lambda_i^2} + \frac{\varepsilon}{\lambda_i}\right), \tag{15}$$

$$\begin{aligned}\int_{\mathbb{R}_+^4}KP\delta_j^{3-\varepsilon}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= K(x_j)\frac{1}{2}c_2\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + O\left(\varepsilon\varepsilon_{12}\left(\log\left(\varepsilon_{12}^{-1}\right)\right)^{\frac{1}{2}} + \frac{1}{\lambda_j^2}\right) \\ &\quad + O\left(\varepsilon_{12}^2\log\left(\varepsilon_{12}^{-1}\right)\right),\end{aligned} \tag{16}$$

$$\begin{aligned}(3-\varepsilon)\int_{\mathbb{R}_+^4}K\delta_i^{2-\varepsilon}\delta_j\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= K(x_i)\frac{1}{2}c_2\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + O\left(\varepsilon\varepsilon_{12}\left(\log\left(\varepsilon_{12}^{-1}\right)\right)^{\frac{1}{2}}\right) \\ &\quad + O\left(\varepsilon_{12}^2\log\left(\varepsilon_{12}^{-1}\right) + \frac{\varepsilon_{12}}{\lambda_j}\left(\log\left(\varepsilon_{12}^{-1}\right)\right)^{\frac{1}{2}}\right).\end{aligned} \tag{17}$$

Combining (5), (13), (14), (15), (16), and (17), the proof follows. □

Proposition 5 *Let* $(\alpha, \lambda, x, v) \in M_\varepsilon$. *Then, for* $u = \alpha_1\delta_{(x_1,\lambda_1)} + \alpha_2\delta_{(x_2,\lambda_2)} + v$, *we have the following expansion*:

$$\begin{aligned}\left\langle \nabla I_\varepsilon(u), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\right\rangle = {} & \biggl(\alpha_i c_4\bigl(1-\alpha_i^{2-\varepsilon}K(x_i)\bigr) + \alpha_i^{3-\varepsilon}K(x_i)\varepsilon(c_4\log\lambda_i + c_7) \\ & + 2\alpha_i^{3-\varepsilon}\frac{c_5}{\lambda_i}\frac{\partial K}{\partial\nu}(x_i)\biggr)e_4 + \alpha_j\Bigl(1-\sum\alpha_i^{2-\varepsilon}K(x_i)\Bigr)\frac{c_2}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial x_i} \\ & - 2\alpha_i^{3-\varepsilon}c_5\frac{\nabla_T K(x_i)}{\lambda_i} + O\biggl(\|v\|^2 + \lambda_j|x_1-x_2|\varepsilon_{12}^{\frac{5}{2}} + \frac{\varepsilon\log\lambda_i}{\lambda_i}\bigl|\nabla_T K(x_i)\bigr|\biggr) \\ & + O\biggl(\varepsilon\varepsilon_{12}\bigl(\log\varepsilon_{12}^{-1}\bigr)^{\frac{1}{2}} + \varepsilon_{12}^2\log\varepsilon_{12}^{-1} + \frac{\varepsilon_{12}}{\lambda_j}\bigl(\log\varepsilon_{12}^{-1}\bigr)^{\frac{1}{2}} + \frac{1}{\lambda_i^2} + \varepsilon^2\log^2\lambda_i\biggr),\end{aligned}$$

where

$$c_4 = 132\int_{\mathbb{R}^4_+}\frac{x_4}{(1+|x|^2)^5}\,dx, \qquad c_5 = 16\int_{\mathbb{R}^4}\frac{x_4^2}{(1+|x|^2)^5}\,dx.$$

Proof We have

$$\int_{\mathbb{R}^4_+}\nabla\delta_i\nabla\biggl(\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\biggr) = \int_{\mathbb{R}^4_+}\delta_i^3\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i} = c_4e_4, \tag{18}$$

$$\int_{\mathbb{R}^4_+}\nabla\delta_j\nabla\biggl(\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\biggr) = \int_{\mathbb{R}^4_+}\delta_j^3\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i} = \frac{1}{2}\frac{c_2}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial x_i} + O\bigl(\varepsilon_{12}^2\log\bigl(\varepsilon_{12}^{-1}\bigr) + \varepsilon_{12}^{\frac{5}{2}}\lambda_j|x_1-x_2|\bigr). \tag{19}$$

For the other part

$$\begin{aligned}\int_{\mathbb{R}^4_+}K\delta_i^{3-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i} = {} & K(x_i)c_4e_4 + 2\frac{c_5}{\lambda_i}\nabla K(x_i) - \varepsilon\log\lambda_i K(x_i)c_4e_4 \\ & - \varepsilon K(x_i)c_7e_4 + O\biggl(\frac{1}{\lambda_i^2} + \varepsilon^2\log^2\lambda_i\biggr),\end{aligned} \tag{20}$$

$$\begin{aligned}\int_{\mathbb{R}^4_+}K\delta_j^{3-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i} = {} & K(x_j)\frac{1}{2}c_2\frac{1}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial a_i} + O\bigl(\varepsilon_{12}^{\frac{5}{2}}\lambda_j|x_1-x_2|\bigr) \\ & + O\biggl(\varepsilon_{12}^2\log\bigl(\varepsilon_{12}^{-1}\bigr) + \frac{1}{\lambda_j}\varepsilon_{12}\bigl(\log\bigl(\varepsilon_{12}^{-1}\bigr)\bigr)^{\frac{1}{2}}\biggr),\end{aligned} \tag{21}$$

$$\begin{aligned}(3-\varepsilon)\int_{\mathbb{R}^4_+}K\delta_i^{2-\varepsilon}\delta_j\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i} = {} & K(x_i)\frac{1}{2}c_2\frac{1}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial x_i} + O\bigl(\varepsilon_{12}^{\frac{5}{2}}\lambda_j|x_1-x_2|\bigr) \\ & + O\biggl(\varepsilon_{12}^2\log\bigl(\varepsilon_{12}^{-1}\bigr) + \frac{1}{\lambda_i}\varepsilon_{12}\bigl(\log\bigl(\varepsilon_{12}^{-1}\bigr)\bigr)^{\frac{1}{2}}\biggr).\end{aligned} \tag{22}$$

Using (5), (18)-(22), our proposition follows. □

3 Construction of the solution

The method of this type of theorem was followed first by Bahri, Li and Rey [17] when they studied an approximation problem of the Yamabe-type problem on domains. Many authors used this idea to construct some solutions to other problems. The method becomes standard. Here we will follow the idea of [17] and take account of the new estimates since we have an equation different from the one studied in [17]. From the idea of [17], using the coefficients of Euler-Lagrange, we obtain

Proposition 6 *Let A point $m = (\alpha, \lambda, x, v) \in M_\varepsilon$ is a critical point of the function Ψ_ε if and only if $u = \alpha_1\delta_1 + \alpha_2\delta_2 + v$ is a critical point of functional I_ε, which means the existence of some $(A, B, C) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^4)^2$ with the following*:

$$(E_{\alpha_i}) \frac{\partial \Psi_\varepsilon}{\partial \alpha_i} = 0, \quad \forall i = 1, 2, \tag{23}$$

$$(E_{\lambda_i}) \frac{\partial \Psi_\varepsilon}{\partial \lambda_i} = B_i \left\langle \frac{\partial^2 \delta_i}{\partial \lambda_i^2}, v \right\rangle + \sum_{j=1}^{4} C_{ij} \left\langle \frac{\partial^2 \delta_i}{\partial x_i^j \partial \lambda_i}, v \right\rangle, \quad \forall i = 1, 2, \tag{24}$$

$$(E_{x_i}) \frac{\partial \Psi_\varepsilon}{\partial x_i} = B_i \left\langle \frac{\partial^2 \delta_i}{\partial \lambda_i \partial x_i}, v \right\rangle + \sum_{j=1}^{4} C_{ij} \left\langle \frac{\partial^2 \delta_i}{\partial x_i^j \partial x_i}, v \right\rangle, \quad \forall i = 1, 2, \tag{25}$$

$$(E_v) \frac{\partial \Psi_\varepsilon}{\partial v} = \sum_{i=1,2} \left(A_i \delta_i + B_i \frac{\partial \delta_i}{\partial \lambda_i} + \sum_{j=1}^{4} C_{ij} \frac{\partial \delta_i}{\partial x_i^j} \right). \tag{26}$$

Now, by a careful study of equation (E_v), we get the following.

Proposition 7 [12] *For any $(\varepsilon, \alpha, \lambda, x)$ with $(\alpha, \lambda, x, 0) \in M_\varepsilon$, there exists a smooth map which associates $\overline{v} \in E_{(x,\lambda)}$ with $\|\overline{v}\| < \nu_0$ and equation (26) in the previous proposition is verified for some $(A, B, C) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^4)^2$. Such a $\overline{v}$ is unique, minimizes $\Psi_\varepsilon(\alpha, \lambda, x, v)$ with respect to v in $\{v \in E_{(x,\lambda)} / \|v\| < \nu_0\}$, and*

$$\|\overline{v}\| = O\left(\varepsilon + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \varepsilon_{12} \left(\log \varepsilon_{12}^{-1} \right)^{1/2} \right). \tag{27}$$

Proof of Theorem 1 Once $\overline{v}$ is defined by Proposition 7, we estimate the corresponding numbers A, B, C by taking the scalar product in $H^1(S_+^4)$ of (E_v) with δ_i, $\partial\delta_i/\partial\lambda_i$, $\partial\delta_i/\partial x_i$ for $i = 1, 2$, respectively. So we get the following coefficients of a quasi-diagonal system:

$$\int_{\mathbb{R}_+^4} |\nabla \delta_i|^2 = \frac{S_4}{2}; \qquad \int_{\mathbb{R}_+^4} \nabla\delta_1 \nabla\delta_2 = O\left(\frac{1}{\lambda_2\lambda_1}\right); \qquad \int_{\mathbb{R}_+^4} \nabla\delta_i \nabla \frac{\partial \delta_i}{\partial \lambda_i} = 0;$$

$$\int_{\mathbb{R}_+^4} \nabla\delta_1 \nabla \frac{\partial \delta_2}{\partial \lambda_2} = O\left(\frac{1}{\lambda_1\lambda_2^2}\right), \qquad \int_{\mathbb{R}_+^4} \nabla\delta_2 \nabla \frac{\partial \delta_1}{\partial \lambda_1} = O\left(\frac{1}{\lambda_1^2\lambda_2}\right); \qquad \int_{\mathbb{R}_+^4} \left|\nabla \frac{\partial \delta_i}{\partial \lambda_i}\right|^2 = \frac{\Gamma_1}{2\lambda_i^2};$$

$$\int_{\mathbb{R}_+^4} \nabla \frac{\partial \delta_1}{\partial \lambda_1} \nabla \frac{\partial \delta_2}{\partial \lambda_2} = O\left(\frac{1}{\lambda_1^2\lambda_2^2}\right), \qquad \int_{\mathbb{R}_+^4} \left|\nabla \frac{\partial \delta_i}{\partial x_i}\right|^2 = \frac{\Gamma_2}{2}\lambda_i^2; \qquad \int_{\mathbb{R}_+^4} \nabla\delta_i \nabla \frac{\partial \delta_i}{\partial x_i} = O(\lambda_1);$$

$$\int_{\mathbb{R}_+^4} \nabla\delta_1 \nabla \frac{\partial \delta_2}{\partial x_2} = O\left(\frac{1}{\lambda_1}\right), \qquad \int_{\mathbb{R}_+^4} \nabla\delta_2 \nabla \frac{\partial \delta_1}{\partial x_1} = O\left(\frac{1}{\lambda_2}\right);$$

$$\int_{\mathbb{R}_+^4} \nabla \frac{\partial \delta_1}{\partial x_1} \nabla \frac{\partial \delta_2}{\partial x_2} = \frac{n+2}{n-2} \int_{\mathbb{R}_+^4} \delta_2^{\frac{4}{n-2}} \nabla \frac{\partial \delta_2}{\partial x_2} \frac{\partial \delta_1}{\partial x_1} = O\left(\frac{1}{\lambda_1}\right),$$

with $|x_1 - x_2| \geq c > 0$ and Γ_1, Γ_2 are positive constants.

We have also

$$\left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, \delta_i \right\rangle = \frac{\partial \Psi_\varepsilon}{\partial \alpha_i}; \qquad \left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = \frac{1}{\alpha_i} \frac{\partial \Psi_\varepsilon}{\partial \lambda_i}; \qquad \left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, \frac{\partial \delta_i}{\partial x_i} \right\rangle = \frac{1}{\alpha_i} \frac{\partial \Psi_\varepsilon}{\partial x_i}.$$

Using Propositions 3, some computations yield

$$\frac{\partial \Psi_\varepsilon}{\partial \alpha_i} = -S_4\beta_i + V_{\alpha_i}(\varepsilon,\alpha,\lambda,x), \tag{28}$$

with $\beta_i = \alpha_i - 1/K(z_i)^{\frac{1}{2}}$ and

$$V_{\alpha_i} = O\left(\beta_i^2 + \varepsilon\log\lambda_i + \frac{1}{\lambda_i} + |x_i - z_i|^2\right). \tag{29}$$

In the same way, using Propositions 4, we get

$$\frac{\partial \Psi_\varepsilon}{\partial \lambda_i} = \frac{1}{K(z_i)}\left(\frac{2c_3}{\lambda_i^2}\frac{\partial K}{\partial \nu}(x_i) + \frac{\varepsilon K(x_i)S_4}{8\lambda_i}\right) + V_{\lambda_i}(\varepsilon,\alpha,\lambda,x), \tag{30}$$

where c_2 and c_3 are defined in Proposition 4 and

$$V_{\lambda_i} = O\left[\frac{1}{\lambda_i}\left(\frac{1}{\lambda_i^2} + \varepsilon^2\log\lambda_i + \frac{\varepsilon\log\lambda_i}{\lambda_i}\right) + \left(|\beta| + \varepsilon + |x_i - z_i|^2\right)\left(\frac{\varepsilon}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right]. \tag{31}$$

Lastly, using Propositions 5, we have

$$\frac{\partial \Psi_\varepsilon}{\partial x_i} = -2c_5\nabla_T K(x_i) + V_{x_i}(\varepsilon,\alpha,\lambda,x), \tag{32}$$

where

$$V_{x_i} = O\left(\frac{1}{\lambda_i} + \left(|\beta| + \varepsilon\log\lambda_i + |x_i - z_i|^2\right)|x_i - z_i|\right). \tag{33}$$

From these estimates, we deduce

$$\frac{\partial \Psi_\varepsilon}{\partial \alpha_i} = O\left(|\beta| + \varepsilon\log\lambda_i + \frac{1}{\lambda_i} + |x_i - z_i|^2\right),$$
$$\frac{\partial \Psi_\varepsilon}{\partial \lambda_i} = O\left(\frac{\varepsilon^{1+\sigma/2}}{\lambda_i}\right); \qquad \frac{\partial \Psi_\varepsilon}{\partial x_i} = O\left(|x_i - z_i| + \frac{1}{\lambda_i}\right).$$

By solving the system in A, B, and C, we find

$$\begin{cases} A_i = O(|\beta| + \varepsilon\log\lambda_i + \frac{1}{\lambda_i} + |x_i - z_i|^2), \\ B_i = O(\varepsilon^{1+\sigma/2}\lambda_i); \qquad C_i = O(\frac{|x_i - z_i|}{\lambda_i^2} + \frac{1}{\lambda_i^3}). \end{cases} \tag{34}$$

Now, we can evaluate the right hand side in (E_{λ_i}) and (E_{x_i}),

$$B_i\left\langle \frac{\partial^2\delta_i}{\partial\lambda_i^2}, \overline{v}\right\rangle + \sum_{j=1}^{4} C_{ij}\left\langle \frac{\partial^2\delta_i}{\partial x_i^j\partial\lambda_i}, \overline{v}\right\rangle = O\left(\left(\frac{\varepsilon^{1+\sigma/2}}{\lambda_i} + \frac{|x_i - z_i|}{\lambda_i^2} + \frac{1}{\lambda_i^3}\right)\|\overline{v}\|\right), \tag{35}$$

$$B_i\left\langle \frac{\partial^2\delta_i}{\partial\lambda_i\partial x_i}, \overline{v}\right\rangle + \sum_{j=1}^{4} C_{ij}\left\langle \frac{\partial^2\delta_i}{\partial x_i^j\partial x_i}, \overline{v}\right\rangle = O\left(\left(\varepsilon^{1+\sigma/2}\lambda_i + |x_i - z_i| + \frac{1}{\lambda_i}\right)\|\overline{v}\|\right), \tag{36}$$

where

$$\left\| \frac{\partial^2 P\delta_i}{\partial \lambda_i^2} \right\| = O\left(\frac{1}{\lambda_i^2}\right); \qquad \left\| \frac{\partial^2 P\delta_i}{\partial x_i \partial \lambda_i} \right\| = O(1); \qquad \left\| \frac{\partial^2 P\delta_i}{\partial x_i^2} \right\| = O\left(\lambda_i^2\right).$$

Now, we consider a point $(z_1, z_2) \in \partial S_+^4 \times \partial S_+^4$ such that z_1 and z_2 are nondegenerate critical points of K_1. We set

$$\frac{1}{\lambda_i} = \varepsilon \frac{S_4}{16 c_3} K(z_i) \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-1} (1 + \zeta_i); \qquad x_i = z_i + \xi_i,$$

where $\zeta_i \in \mathbb{R}$ and $(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ are assumed to be small.

Using (28) and these changes of variables, (E_{α_i}) becomes

$$\beta_i = V_{\alpha_i}(\varepsilon, \beta, \zeta, \xi) = O\left(\beta^2 + \varepsilon |\log \varepsilon| + |\xi|^2\right). \tag{37}$$

Also, we use (30), we have

$$\begin{aligned}
&\frac{2c_3}{\lambda_i^2} \frac{\partial K}{\partial \nu}(z_i + \xi_i) + \frac{\varepsilon K(z_i + \xi_i) S_4}{8 \lambda_i} \\
&\quad = \frac{\varepsilon^2 S_4^2 K(z_i)^2}{128 c_3} \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-2} (1 + 2\zeta_i) \left(-\frac{\partial K}{\partial \nu}(z_i) + D^2 K(z_i)(e_4, \xi_i)\right) \\
&\qquad + \frac{\varepsilon^2 S_4^2 K(z_i)^2}{128 c_3} \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-1} (1 + \zeta_i) + O\left(\varepsilon^2 \left(\zeta_i^2 + |\xi_i|^2\right)\right) \\
&\quad = -\frac{\varepsilon^2 S_4^2 K(z_i)^2}{128 c_3} \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-1} \zeta_i \\
&\qquad + \frac{\varepsilon^2 S_4^2 K(z_i)^2}{128 c_3} \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-2} D^2 K(z_i)(e_4, \xi_i) \\
&\qquad + O\left(\varepsilon^2 \left(\zeta_i^2 + |\xi_i|^2\right)\right).
\end{aligned}$$

Combining this with (31), then (E_{λ_i}) becomes

$$\begin{aligned}
-\zeta_i + \left(\frac{\partial K}{\partial \nu}(z_i)\right)^{-1} D^2 K_1(z_i)(e_4, \xi_i) &= V_{\lambda_i}(\varepsilon, \beta, \zeta, \xi) \\
&= O\left(\varepsilon |\log \varepsilon| + |\beta|^2 + \zeta_i^2 + |\xi|^2\right).
\end{aligned} \tag{38}$$

Using (32), (33), and (36), (E_{x_i}) is equivalent to

$$D^2 K_1(z_i) \xi_i = V_{x_i}(\varepsilon, \beta, \zeta, \xi) = O\left(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2\right). \tag{39}$$

Observe that the functions V_{α_i}, V_{λ_i}, and V_{x_i} are smooth.

We can also write the system as

$$\begin{cases} \beta = V(\varepsilon, \beta, \zeta, \xi), \\ L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi), \end{cases} \tag{40}$$

where L is a fixed linear operator on $\mathbb{R}^8$ defined by

$$L(\zeta,\xi) = \left(-\zeta_1 + \left(\frac{\partial K}{\partial \nu}(z_1)\right)^{-1} D^2K_1(z_1)(e_4,\xi_1); -\zeta_2 + \left(\frac{\partial K}{\partial \nu}(z_2)\right)^{-1} D^2K_1(z_2)(e_4,\xi_2); D^2K_1(z_1)\xi_1; D^2K_1(z_2)\xi_2\right),$$

and V, W are smooth functions satisfying

$$\begin{cases} V(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\xi|^2), \\ W(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{\frac{1}{2}} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \end{cases}$$

Now, by an easy computation, we see that the determinant of the linear operator L is not 0. Hence L is invertible, and according to Brouwer's fixed point theorem, there exists a solution $(\beta^\varepsilon,\zeta^\varepsilon,\xi^\varepsilon)$ of (40) for ε small enough, such that

$$|\beta^\varepsilon| = O(\varepsilon^{1/2}); \qquad |\zeta^\varepsilon| = O(\varepsilon^{1/2}); \qquad |\xi^\varepsilon| = O(\varepsilon^{1/2}).$$

Hence, we have constructed $m^\varepsilon = (\alpha_1^\varepsilon,\alpha_2^\varepsilon,\lambda_1^\varepsilon,\lambda_2^\varepsilon,x_1^\varepsilon,x_2^\varepsilon)$ such that $u_\varepsilon := \sum \alpha_i^\varepsilon \delta_{(x_i^\varepsilon,\lambda_i^\varepsilon)} + \overline{v_\varepsilon}$, verifies (23)-(27). From Proposition 6, u_ε is a critical point of I_ε, which implies that u_ε verify

$$-\Delta u_\varepsilon + 2u_\varepsilon = K|u_\varepsilon|^{2-\varepsilon}u_\varepsilon \quad \text{in } S^4_+, \qquad \partial u_\varepsilon/\partial\nu = 0 \quad \text{on } \partial S^4_+. \tag{41}$$

We multiply equation (41) by $u_\varepsilon^- = \max(0,-u_\varepsilon)$ and we integrate on S^4_+, we get

$$\int_{S^4_+} |\nabla u_\varepsilon^-|^2 + 2\int_{S^4_+} (u_\varepsilon^-)^2 = \int_{S^4_+} K(u_\varepsilon^-)^{4-\varepsilon}. \tag{42}$$

We know also from the Sobolev embedding theorem that

$$|u_\varepsilon^-|^2_{4-\varepsilon} := \left(\int_{S^4_+} K(u_\varepsilon^-)^{4-\varepsilon}\right)^{\frac{2}{4-\varepsilon}} \le C\|u_\varepsilon^-\|^2. \tag{43}$$

Equations (42) and (43) imply that either $u_\varepsilon^- \equiv 0$, or $|u_\varepsilon^-|_{4-\varepsilon}$ is far away from zero. Since $m^\varepsilon \in M^\varepsilon$, we have $\|\overline{v_\varepsilon}\| < \nu_0$, where ν_0 is a small positive constant (see the definition of M_ε). This implies that $|u_\varepsilon^-|_{4-\varepsilon}$ is very small. Thus, $u_\varepsilon^- \equiv 0$ for ε small enough. Then u_ε is a non-negative function which satisfies (41). Finally, the maximum principle completes the proof of our theorem. □

4 Conclusion

Thus it has been concluded that under some assumptions on the function K, there exist solutions of the nonlinear problem (S_ε) which are concentrated at two different points of the boundary.

Competing interests

The author declares to have no competing interests.

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References

1. Diening, L, Harjulehto, P, Hasto, P, Ruzicka, M: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2011. Springer, Heidelberg (2011) MR2790542
2. Ambrosetti, A, Garcia Azorero, J, Peral, A: Perturbation of $\Delta u + u^{\frac{n+2}{n-2}} = 0$, the scalar curvature problem in $\mathbb{R}^n$ and related topics. J. Funct. Anal. **165**, 117-149 (1999)
3. Bahri, A, Coron, JM: The scalar curvature problem on the standard three dimensional spheres. J. Funct. Anal. **95**, 106-172 (1991)
4. Bianchi, G, Pan, XB: Yamabe equations on half spheres. Nonlinear Anal. **37**, 161-186 (1999)
5. Chang, SA, Yang, P: A perturbation result in prescribing scalar curvature on S^n. Duke Math. J. **64**, 27-69 (1991)
6. Cherrier, P: Problèmes de Neumann non linéaires sur les variétés riemaniennes. J. Funct. Anal. **57**, 154-207 (1984)
7. Escobar, J: Conformal deformation of Riemannian metric to scalar flat metric with constant mean curvature on the boundary. Ann. Math. **136**, 1-50 (1992)
8. Escobar, J, Schoen, R: Conformal metrics with prescribed scalar curvature. Invent. Math. **86**, 243-254 (1986)
9. Han, ZC, Li, YY: The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature. Commun. Anal. Geom. **8**, 809-869 (2000)
10. Hebey, E: The isometry concentration method in the case of a nonlinear problem with Sobolev critical exponent on compact manifolds with boundary. Bull. Sci. Math. **116**, 35-51 (1992)
11. Li, YY: Prescribing scalar curvature on S^n and related topics, Part I. J. Differ. Equ. **120**, 319-410 (1995); Part II. Existence and compactness. Comm. Pure Appl. Math. **49** 437-477 (1996).
12. Ould Bouh, K: Blowing up of sign-changing solutions to a subcritical problem. Manuscr. Math. **146**, 265-279 (2015)
13. Djadli, Z, Malchiodi, A, Ould Ahmedou, M: Prescribing the scalar and the boundary mean curvature on the three dimensional half sphere. J. Geom. Anal. **13**, 233-267 (2003)
14. Ben Ayed, M, Ghoudi, R, Ould Bouh, K: Existence of conformal metrics with prescribed scalar curvature on the four dimensional half sphere. Nonlinear Differ. Equ. Appl. **19**, 629-662 (2012)
15. Rey, O: The topological impact of critical points at infinity in a variational problem with lack of compactness: the dimension 3. Adv. Differ. Equ. **4**, 581-616 (1999)
16. Bahri, A: An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimension. A celebration of J. F. Nash jr. Duke Math. J. **81**, 323-466 (1996)
17. Bahri, A, Li, YY, Rey, O: On a variational problem with lack of compactness: The topological effect of the critical points at infinity. Calc. Var. Partial Differ. Equ. **3**, 67-94 (1995)