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RESEARCH

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A certain (p,q)-derivative operator and associated divided differences

Serkan Araci[1*], Uğur Duran[2], Mehmet Acikgoz[2] and Hari M Srivastava[3,4]

*Correspondence: mtsrkn@hotmail.com
[1]Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, Gaziantep, 27410, Turkey
Full list of author information is available at the end of the article

Abstract

Recently, Sofonea (Gen. Math. 16:47-54, 2008) considered some relations in the context of *quantum calculus* associated with the q-derivative operator D_q and divided difference. As applications of the *post-quantum calculus* known as the (p,q)-calculus, we derive several relations involving the (p,q)-derivative operator and divided differences.

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1 Introduction

The quantum calculus has many applications in the fields of special functions and many other areas (see [1–7]). Further there is possibility of extension of the q-calculus to post-quantum calculus denoted by the (p,q)-calculus. Actually such an extension of quantum calculus cannot be obtained directly by substitution of q by q/p in q-calculus. When the case $p = 1$ in (p,q)-calculus, the q-calculus may be obtained (see [6, 7]). Recently, Chakrabarti and Jagannathan [8] introduced a consideration of the (p,q)-integer in order to generalize or unify several forms of q-oscillator algebras well known in the physics literature related to the representation theory of single-paramater quantum algebras (see also [3–5] and [9]). They also considered the necessary elements of the (p,q)-calculus involving (p,q)-exponential, (p,q)-integration and the (p,q)-differentiation. Corcino [10] developed the theory of a (p,q)-extension of the binomial coefficients and also established some properties parallel to those of the ordinary and q-binomial coefficients, which is comprised horizontal generating function, the triangular, vertical, and the horizontal recurrence relations and the inverse and the orthogonality relations. Sadjang [11] investigated some properties of the (p,q)-derivatives and the (p,q)-integrations. Sadjang [11] also provided two suitable polynomial bases for the (p,q)-derivative and gave various properties of these bases.

The (p,q)-number is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad (p \neq q),$$

which is a natural generalization of the q-number: that is, we have (*cf.* [10] and [11])

$$\lim_{p\to 1}[n]_{p,q} := [n]_q.$$

It is clear that the notation $[n]_{p,q}$ is symmetric, that is,

$$[n]_{p,q} = [n]_{q,p}.$$

The (p,q)-Gauss binomial coefficients given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} \quad (n \geqq k)$$

and the (p,q)-factorial given by

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q} \quad (n \in \mathbb{N})$$

are also known from [10] and [11]. Further, the (p,q)-analogs of Pascal's identity are given by

$$\begin{aligned}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q},\end{aligned}$$

where $k \in \{0,1,2,\dots,n\}$ (*cf.* [10] and [11]).

Let p and q be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$. Here, in this investigation, we give the following two definitions which involve a post-quantum generalization of Sofonea's work [1].

Definition 1 Let $0 < |q| < |p| \leqq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called (p,q)-differentiable under the restriction that, if $0 \in D_{p,q}$, then $f'(0)$ exists.

Definition 2 Let $0 < |q| < |p| \leqq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called (p,q)-differentiable of order n, if and only if $0 \in D_{p,q}$ implies that $f^{(n)}(0)$ exists.

The (p,q)-derivative operator of a function f is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x} \quad (x \neq 0) \tag{1.1}$$

and

$$(D_{p,q}f)(0) = f'(0),$$

provided that the function f is differentiable at 0. We note that

$$D_{p,q} = D_{q,p}.$$

Furthermore,

$$(D_{p,q}fg)(x) = g(px)(D_{p,q}f)(x) + f(qx)(D_{p,q}g)(x) \tag{1.2}$$

and

$$\left(D_{p,q}\frac{f}{g}\right)(x) = \frac{g(px)(D_{p,q}f)(x) - f(px)(D_{p,q}g)(x)}{g(px)g(qx)} \quad \bigl(g(px)g(qx) \neq 0\bigr) \tag{1.3}$$

hold true for the linear operator $D_{p,q}$ (*cf.* [11]).

The divided differences at a system of distinct points $x_0, x_1, \ldots, x_n$ are denoted by $[x_0, x_1, \ldots, x_n; f]$. In fact, we have (see [1] and [2])

$$[x_0, x_1, \ldots, x_n; f] = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{\substack{i=0 \\ (i\neq k)}}^{n}(x_k - x_i)}. \tag{1.4}$$

In the next part of the paper, we obtain some potentially useful results and relations between the (p,q)-derivative operator and divided differences. The results presented here provide a good generalization of the above-mentioned Sofonea results.

2 Main results

Let us consider the points

$$x_k = p^k q^{n-k} x \quad (k = 0, 1, \ldots, n)$$

as follows:

$$x_0 = q^n x, \qquad x_1 = q^{n-1}px, \qquad \ldots, \qquad x_{n-1} = qp^{n-1}x, \qquad x_n = p^n x.$$

We now state the following theorem.

Theorem 1 *Let p and q be complex numbers with*

$$0 < |q| < |p| \leqq 1 \quad \text{and} \quad f : D_{p,q} \to \mathbb{C}.$$

Then, by taking the knots $x_k = p^k q^{n-k} x$,

$$\begin{aligned}&\bigl[q^n x, q^{n-1}px, \ldots, qp^{n-1}x, p^n x; f\bigr] \\ &\quad = \frac{1}{q^{\binom{n}{2}}[n]_{p,q}! x^n (p-q)^n} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{-k(2n-k-1)}{2}} q^{\binom{k}{2}} f\bigl(xp^k q^{n-k}\bigr).\end{aligned} \tag{2.1}$$

Proof For $0 \leqq l < k$, we have

$$x_k - x_l = xp^l q^{n-k}(p-q)[k-l]_{p,q}$$

and, for $k < l \leqq n$, we find that

$$x_k - x_l = xp^k q^{n-l}(q-p)[l-k]_{p,q}.$$

Since

$$\prod_{\substack{l=0\\l\neq k}}^{n}(x_k - x_l) = \prod_{l=0}^{k-1}(x_k - x_l)\prod_{l=k+1}^{n}(x_k - x_l)$$

$$= x^n p^{(n-k)k}(-1)^{n-k}(p-q)^n q^{k(n-k)+\binom{n-k}{2}}[k]_{p,q}!p^{k(n-k)+\binom{k}{2}}[n-k]_{p,q}!$$

$$= (-1)^{n-k}(p-q)^n x^n p^{k(2n-k-1)/2} q^{\binom{n}{2}-\binom{k}{2}}[k]_{p,q}![n-k]_{p,q}!,$$

we have the following consequence from (1.4):

$$[x_0, x_1, \ldots, x_n; f] = \frac{q^{-\binom{n}{2}}}{[n]_{p,q}!x^n(p-q)^n}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-k(2n-k-1)/2}q^{\binom{k}{2}}f\bigl(xp^k q^{n-k}\bigr).$$

Therefore, the proof of Theorem 1 is completed. □

By using the following expressions:

$$D_{p,q}^0 = I, \qquad D_{p,q}^1 = D_{p,q} \quad \text{and} \quad D_{p,q}^k = D_{p,q}D_{p,q}^{k-1},$$

we now give a representation of the operator $D_{p,q}^n$ as in Theorem 2 below.

Theorem 2 *Let the function* $f : D_{p,q} \to \mathbb{C}$ *be* (p,q)*-differentiable of order* n*. Then*

$$\bigl(D_{p,q}^n f\bigr)(x) = \frac{q^{-\binom{n}{2}}}{x^n(p-q)^n}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{q^{\binom{k}{2}}f(xp^k q^{n-k})}{p^{k(2n-k-1)/2}}. \tag{2.2}$$

Proof Theorem 2 is proved by making use of the following results:

$$(D_{p,q}f)(x) = \frac{f(qx) - f(px)}{(q-p)x} = \frac{f(qx)}{qx - px} + \frac{f(px)}{px - qx} = [1]_{p,q}![qx, px; f]$$

and

$$\begin{aligned}
&\bigl(D_{p,q}^2 f\bigr)(x) \\
&\quad = \frac{(D_{p,q}f)(qx) - (D_{p,q}f)(px)}{(q-p)x} \\
&\quad = \frac{\frac{f(q^2x)-f(pqx)}{(q-p)qx} - \frac{f(pqx)-f(p^2x)}{(q-p)px}}{(p-q)x} \\
&\quad = (p+q)\left[\frac{f(q^2x)}{(q^2-p^2)(q-p)x^2q} - \frac{f(pqx)}{(q-p)^2x^2pq} + \frac{f(p^2x)}{(q^2-p^2)(q-p)x^2p}\right] \\
&\quad = [2]_{p,q}!\bigl[q^2x, pqx, p^2x; f\bigr].
\end{aligned}$$

Continuing this process, we deduce

$$\bigl(D_{p,q}^n f\bigr)(x) = [n]_{p,q}!\bigl[q^n x, q^{n-1}px, \ldots, qp^{n-1}x, p^n x; f\bigr] \tag{2.3}$$

by using the following formula:

$$[x_0,x_1,\dots,x_n;\cdot] = \frac{[x_1,x_2,\dots,x_n;\cdot]-[x_0,x_1,\dots,x_{n-1};\cdot]}{x_n-x_0}.$$

It follows from Theorem 1 that

$$\left(D^n_{p,q}f\right)(x) = q^{-\binom{n}{2}}x^{-n}(p-q)^{-n}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-k(2n-k-1)/2}q^{\binom{k}{2}}f\left(xp^kq^{n-k}\right),$$

which completes the proof of Theorem 2. □

In the case when

$$f(x) = x^n$$

in Theorem 2, we get the following corollary.

Corollary 1 *The following result holds true*:

$$(p-q)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k+1}{2}}q^{\binom{n-k+1}{2}}\frac{(-1)^{n-k}}{[n]_{p,q}!}.$$

We now consider the (p,q)-analog of the Leibniz rule to represent it by means of the divided differences. First of all, we need to get the (p,q)-analog of the Leibniz rule by the following lemma.

Lemma *Let the functions $f: D_{p,q} \to \mathbb{C}$ and $g: D_{p,q} \to \mathbb{C}$ be (p,q)-differentiable of order n. Then*

$$D^n_{p,q}(fg)(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D^k_{p,q}(f)\left(xp^{n-k}\right)D^{n-k}_{p,q}(g)\left(xq^k\right).$$

Proof The lemma can easily be proved by applying the principle of mathematical induction. We, therefore, omit the proof of the lemma. □

We now state the (p,q)-Leibniz rule by using divided differences as follows.

Theorem 3 *Let the functions $f: D_{p,q} \to \mathbb{C}$ and $g: D_{p,q} \to \mathbb{C}$ be (p,q)-differentiable of order n. Then $(fg)(x)$ is also (p,q)-differentiable of order n and*

$$\begin{aligned} D^n_{p,q}(fg)(x) = [n]_{p,q}!\sum_{k=0}^{n}&\left[q^nx,q^{n-1}px,\dots,q^{n-k+1}p^{k-1}x,q^{n-k}p^kx;f\right] \\ &\cdot\left[q^{n-k}p^kx,q^{n-k-1}p^{k+1}x,\dots,qp^{n-1}x,p^nx;g\right]. \end{aligned}$$

Proof Our assertion in Theorem 3 follows from equation (2.3) and the above lemma. The details involved are being omitted here. □

Now also we give a function at a point $p^n x$ by binomial expression and (p,q)-derivative of order k.

Theorem 4 *Let the function $f : D_{p,q} \to \mathbb{C}$ be (p,q)-differentiable of order n. Then*

$$f(p^n x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k p^{\binom{k}{2}} (p-q)^k D_{p,q}^k\bigl(f(x)\bigr).$$

Proof We consider Newton's formula as follows:

$$\begin{aligned} f(z) &= \sum_{k=0}^{n-1} (z-x_0)(z-x_1)\cdots(z-x_{k-1})[x_0,x_1,\ldots,x_k;f] \\ &\quad + (z-x_0)(z-x_1)\cdots(z-x_{n-1})[x_0,x_1,\ldots,x_{n-1},z;f]. \end{aligned} \tag{2.4}$$

Upon setting

$$x_k = p^k q^{n-k} x \quad (k = 0,1,\ldots,n-1)$$

in equation (2.4) and $z = p^n x$, if we use equation (2.1), we find that

$$\begin{aligned} f(p^n x) &= \sum_{k=0}^{n-1} (p^n x - q^n x)(p^n x - q^{n-1} p x)\cdots(p^n x - q^{n-k+1} p^{k-1} x) \\ &\quad \cdot [q^n x, q^{n-1} p x, \ldots, q^{n-k} p^k x; f] \\ &\quad + (p^n x - q^n x)(p^n x - q^{n-1} p x)\cdots(p^n x - q p^{n-1} x) \\ &\quad \cdot [q^n x, q^{n-1} p x, \ldots, q p^{n-1} x, p^n x; f] \\ &= \sum_{k=0}^{n-1} (p^n x - q^n x)(p^n x - q^{n-1} p x)\cdots(p^n x - q^{n-k+1} p^{k-1} x) \frac{(D_{p,q}^k f)(x)}{[k]_{p,q}!} \\ &\quad + (p^n x - q^n x)(p^n x - q^{n-1} p x)\cdots(p^n x - q p^{n-1} x) \frac{(D_{p,q}^n f)(x)}{[n]_{p,q}!} \\ &= \sum_{k=0}^{n} (p^n x - q^n x)(p^n x - q^{n-1} p x)\cdots(p^n x - q^{n-k+1} p^{k-1} x) \frac{(D_{p,q}^k f)(x)}{[k]_{p,q}!} \\ &= \sum_{k=0}^{n} x^k p^{\binom{k}{2}} \frac{\frac{(p^n - q^n)(p^{n-1} - q^{n-1})\cdots(p-q)}{(p-q)^n}}{\frac{(p^{n-k} - q^{n-k})(p^{n-k-1} - q^{n-k-1})\cdots(p-q)}{(p-q)^{n-k}(p-q)^k}} \frac{(D_{p,q}^k f)(x)}{[k]_{p,q}!} \\ &= \sum_{k=0}^{n} x^k p^{\binom{k}{2}} (p-q)^k \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} (D_{p,q}^k f)(x), \end{aligned}$$

as asserted by Theorem 4. □

Finally, we are in a position to give the following result.

Corollary 2 *Let p and q be complex numbers such that*

$$0 < |q| < |p| \leqq 1.$$

Also let the function $f : D_{p,q} \to \mathbb{C}$ be (p,q)-differentiable of order n. Then

$$f(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{k(k-n)} p^{\binom{k+1}{2}} (qx - px)^k \bigl(D_{p,q}^k f\bigr)\left(\frac{xp^{n-k}}{q^k}\right).$$

Proof Since, for $k \in \{0, 1, \ldots, n\}$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{p},\frac{1}{q}} = \frac{[n]_{\frac{1}{p},\frac{1}{q}}!}{[n-k]_{\frac{1}{p},\frac{1}{q}}![k]_{\frac{1}{p},\frac{1}{q}}!} = \frac{(pq)^{-\binom{n}{2}}}{(pq)^{-\binom{n-k}{2}}(pq)^{-\binom{k}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q},$$

we have

$$(D_{\frac{1}{p},\frac{1}{q}} f)(x) = \frac{f(\frac{x}{q}) - f(\frac{x}{p})}{(p-q)x}(pq) = pq(D_{p,q}f)\left(\frac{x}{pq}\right)$$

and

$$\begin{aligned} \bigl(D^2_{\frac{1}{p},\frac{1}{q}} f\bigr)(x) &= \frac{pq(D_{p,q}f)(\frac{x}{pq}) - pq(D_{p,q}f)(\frac{x}{pq})}{(\frac{1}{p} - \frac{1}{q})x} \\ &= \frac{(pq)^2[(D_{p,q}f)(\frac{x}{pq^2}) - (D_{p,q}f)(\frac{x}{pq})]}{(p-q)x} \\ &= p^2q^2\bigl(D^2_{p,q}f\bigr)\left(\frac{x}{p^2q^2}\right). \end{aligned}$$

Continuing the process, we readily observe that

$$\bigl(D^n_{\frac{1}{p},\frac{1}{q}} f\bigr)(x) = p^n q^n \bigl(D^n_{p,q}f\bigr)\left(\frac{x}{p^n q^n}\right). \tag{2.5}$$

From Theorem 4, we thus conclude that

$$f(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{k(k-n)} p^{\binom{k+1}{2}} (qx - px)^k \bigl(D_{p,q}^k f\bigr)\left(\frac{xp^{n-k}}{q^k}\right),$$

which evidently proves Corollary 2. □

3 Conclusion

We have considered (p,q)-analogs of several results investigated recently by Sofonea [1]. We have also given the (p,q)-Leibniz rule and stated the (p,q)-Leibniz rule by means of divided differences. Moreover, we have shown that a function f at a point $q^n x$ can be generated by a linear combination of the (p,q)-derivatives of order k. In the case when $p = 1$, the results derived in this paper would correspond to those based upon the relatively more familiar q-numbers.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, Gaziantep, 27410, Turkey. [2]Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, Gaziantep, 27310, Turkey. [3]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada. [4]China Medical University, Taichung, 40402, Taiwan, Republic of China.

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