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On general partial Gaussian sums

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Abstract

Let $q \geq 2$ be a fixed integer, $A = A(q) \leq q$, $B = B(q) \leq q$, and $H = H(q) \leq q$. Define

$$\hbar(A,B,H) = \{a \in \mathbb{Z} \mid (a,q) = 1, ab \equiv 1 \pmod q, 1 \leq a \leq A, 1 \leq b \leq B, |a-b| \leq H\}.$$

With the aid of the estimates for the general Kloosterman sums and the properties of trigonometric sums, we obtain an upper bound of the general partial Gaussian sums over the number set $\hbar(A,B,H)$.

MSC: 11L05; 11L40

Keywords: partial Gaussian sums; Kloosterman sums

1 Introduction

Let q, N, H, n be integers with $q \geq 2$, $H > 0$, χ be a Dirichlet character mod q and $e(y) = e^{2\pi iy}$. The study of the following partial Gaussian sums:

$$\sum_{a=N+1}^{N+H} \chi(a) e\left(\frac{na}{q}\right)$$

is of great importance. By extending his well-known work on character sums, Burgess obtained the following.

Proposition 1 ([1]) *Let q be a prime and χ be a non-principal Dirichlet character mod q. Then, for any integers N, n, H, r with $0 < H < q$ and $r \geq 2$, we have*

$$\sum_{a=N+1}^{N+H} \chi(a) e\left(\frac{na}{q}\right) \ll H^{1-1/r} q^{1/4(r-1)} \log^2 q.$$

Proposition 2 ([2]) *Let $q \geq 2$ be an integer and χ be a primitive character mod q. Then, for any integers N, n, H with $0 < H$, we have*

$$\sum_{a=N+1}^{N+H} \chi(a) e\left(\frac{na}{q}\right) \ll H^{2/3} q^{1/8+\epsilon}. \tag{1.1}$$

Proposition 3 ([3]) *Let $q = p^\alpha$ ($\alpha > 1$) be a power of the prime $p > 3$ and χ be a non-principal Dirichlet character mod q. Then, for any integers N, n, H with $0 < H$, we have*

$$\sum_{a=N+1}^{N+H} \chi(a) e\left(\frac{na}{q}\right) \ll H^{3/4} q^{1/12} \log^3 q. \tag{1.2}$$

At almost the same time, Liu [4] showed independently the following.

Proposition 4 *Let q be a prime power, χ, ψ be a multiplicative and additive character mod q, respectively, with χ non-principal. Then, for any integers N, H with $0 < H$, we have*

$$\sum_{a=N+1}^{N+H} \chi(a)\psi(a) \ll H^{3/4} q^{1/12+\epsilon}.$$

Now let $q \geq 2$ be a fixed integer, $A = A(q) \leq q$, $B = B(q) \leq q$, and $H = H(q) \leq q$. Define

$$\hbar(A,B,H) = \left\{a \in \mathbb{Z} \mid (a,q) = 1, ab \equiv 1 \pmod{q}, 1 \leq a \leq A, 1 \leq b \leq B, |a-b| \leq H\right\}.$$

It is a direct generalization of the set of so-called H-flat numbers mod q, which was studied extensively by Xi (see [5] and references therein).

This paper deals with general partial Gaussian sums of the following type:

$$G_k(\chi, A, B, H; q) = \sum_{a \in \hbar(A,B,H)} a^k \chi(a) e\left(\frac{na}{q}\right),$$

where $k \geq 0$ is an arbitrary fixed integer. For the sake of periodicity of $e(\frac{na}{q})$ we can also restrict n to be $1 \leq n \leq q$. Then with the aid of the estimates for the general Kloosterman sums and the properties of trigonometric sums, we shall obtain upper bound estimates as follows.

Theorem *Let $q \geq 2$ be an integer and χ a non-principal Dirichlet character mod q. Then*

$$G_k(\chi, A, B, H; q) \ll A^k q^{1/2} d(q) \left(\frac{nABd(q)}{q^2} + \frac{Bd(q)(\log q)(\log H)}{q} + \log^3 q\right),$$

which is uniformly nontrivial for any positive integer n such that $n < q^{1/2}$.

Taking n a constant, we can immediately obtain the following.

Corollary 1 *Let $q \geq 2$ be an integer and χ a non-principal Dirichlet character mod q. Then we have*

$$G_k(\chi, A, B, H; q) \ll A^k q^{1/2} d(q) \left(\frac{ABd(q)}{q^2} + \frac{Bd(q)(\log q)(\log H)}{q} + \log^3 q\right).$$

Taking $k = 0$, $B = q$ in Corollary 1, we obtain the following.

Corollary 2 *Let $q \geq 2$ be an integer and χ a non-principal Dirichlet character mod q. Then, for any positive integers A, H such that $A, H \leq q$, we have*

$$G_0(\chi, A, q, H; q) \ll q^{1/2} d^2(q) \log^3 q. \tag{1.3}$$

Remark It is easy to see that (1.3) is stronger than (1.1) for any integer H such that $q^{9/16+\epsilon} < H \leq q$. It is also stronger than (1.2) for any integer H such that $q^{5/9} d^{8/3}(q) < H \leq q$. These results reveal that more cancelations occurred in the number set $\hbar(A, B, H)$.

Taking $A = H = q$ in Corollary 2, we obtain the following.

Corollary 3 *Let $q \geq 2$ be an integer and χ a non-principal Dirichlet character mod q. Then we have*

$$G_0(\chi, q, q, q; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{na}{q}\right) \ll q^{1/2} d^2(q) \log^3 q.$$

This is a little stronger than the classical result of the complete Gauss sums in the case when $(n, q) > 1$.

2 Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 1 *Let q, n, ℓ, r be integers with $q > 2$, $q > n$, and $\ell \geq 0$. Define $h(r, \ell; n) = \sum_{a=1}^{n} a^\ell e(\frac{ra}{q})$. Then we have*

$$h(r, \ell; n) \begin{cases} = \frac{n^{\ell+1}}{\ell+1} + O(n^\ell), & q \mid r, \\ \ll \frac{n^\ell}{|\sin(\pi s/q)|}, & q \nmid r, \end{cases}$$

where $s = \min(r, q-r)$ with $1 \leq r \leq q-1$.

Proof See Lemma 3 of [6] or Lemma 2.4 of [7]. □

Lemma 2 *Let q be a positive integer. Then we have*

$$\left|K_\chi(m, n; q)\right| \leq q^{1/2} (m, n, q)^{1/2} d(q),$$

where $K_\chi(m, n; q) = \sum_{a \pmod q} \chi(a) e(\frac{ma + n\bar{a}}{q})$ is the general Kloosterman sum, with $a\bar{a} \equiv 1 \pmod q$ and (m, n, q) the greatest common divisor of m, n, q.

Proof See Lemma 1 of [5]. □

3 Proof of the Theorem

Now we come to prove the theorem. Note that

$$G_k(\chi, A, B, H; q) = \sum_{\substack{t \leq H \\ a-b \equiv t \pmod q}} \ \sum_{\substack{a \leq A, b \leq B \\ ab \equiv 1 \pmod q}} a^k \chi(a) e\left(\frac{na}{q}\right).$$

Applying the trigonometric sums identity

$$\sum_{a=1}^{q} e\left(\frac{ma}{q}\right) = \begin{cases} q, & q \mid m, \\ 0, & q \nmid m, \end{cases}$$

we obtain

$$\begin{aligned}
&G_k(\chi, A, B, H; q) \\
&\quad = \sum_{a \in \hbar(A,B,H)} a^k \chi(a) e\left(\frac{na}{q}\right) \\
&\quad = \sum_{t \le H} \sideset{}{'}\sum_{\substack{a \le A, b \le B \\ a-b \equiv t \pmod q, ab \equiv 1 \pmod q}} a^k \chi(a) e\left(\frac{na}{q}\right) \\
&\quad = \frac{1}{q} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) \sideset{}{'}\sum_{\substack{a \le A, b \le B \\ ab \equiv 1 \pmod q}} a^k \chi(a) e\left(\frac{(m+n)a - mb}{q}\right) \\
&\quad = \frac{1}{q^3} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) \sideset{}{'}\sum_{\substack{a, b \le q \\ ab \equiv 1 \pmod q}} \chi(a) e\left(\frac{(m+n)a - mb}{q}\right) \\
&\qquad \times \sum_{c \le A} c^k \sum_{r \le q} e\left(\frac{r(a-c)}{q}\right) \sum_{d \le B} \sum_{s \le q} e\left(\frac{s(b-d)}{q}\right) \\
&\quad = \frac{1}{q^3} \sum_{r,s \le q} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) \sideset{}{'}\sum_{\substack{a, b \le q \\ ab \equiv 1 \pmod q}} \chi(a) e\left(\frac{(m+r+n)a - (m-s)b}{q}\right) \\
&\qquad \times \sum_{c \le A} c^k e\left(-\frac{rc}{q}\right) \sum_{d \le B} e\left(-\frac{sd}{q}\right) \\
&\quad = \frac{1}{q^3} \sum_{r,s \le q} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+r+n, s-m; q) h(-r, k; A) h(-s, 0; B) \\
&\quad = \frac{1}{q^3} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+n, -m; q) h(-q, k; A) h(-q, 0; B) \\
&\qquad + \frac{1}{q^3} \sum_{r \le q-1} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+r+n, -m; q) h(-r, k; A) h(-q, 0; B) \\
&\qquad + \frac{1}{q^3} \sum_{s \le q-1} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+n, s-m; q) h(-q, k; A) h(-s, 0; B) \\
&\qquad + \frac{1}{q^3} \sum_{r,s \le q-1} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+r+n, s-m; q) h(-r, k; A) h(-s, 0; B).
\end{aligned}$$

Then from Lemma 1 and Lemma 2, we have

$$\begin{aligned}
&\frac{1}{q^3} \sum_{m \le q} \sum_{t \le H} e\left(-\frac{mt}{q}\right) K_\chi(m+n, -m; q) h(-q, k; A) h(-q, 0; B) \\
&\quad \ll \frac{1}{q^3} \sum_{m \le q} \min\left(H, \left\| \frac{m}{q} \right\|^{-1}\right) \left| K_\chi(m+n, -m; q) \right| \cdot \left| h(-q, k; A) \right| \cdot \left| h(-q, 0; B) \right|
\end{aligned}$$

$$\begin{aligned}
&\ll HA^{k+1}Bq^{-5/2}d(q)\sum_{m\le q/H}(m+n,q)^{1/2}\\
&\quad + A^{k+1}Bq^{-3/2}d(q)\sum_{q/H<m\le q-1}\frac{(m+n,q)^{1/2}}{m},
\end{aligned}$$

where $\|x\| = \min_{a\in Z}|x-a|$.

Combining the estimates

$$\begin{aligned}
\sum_{m\le q/H}(m+n,q)^{1/2} &= \sum_{d|q}d^{1/2}\sum_{\substack{m\le q/H\\ d|(m+n)}}1\\
&= \sum_{d|q}d^{1/2}\sum_{m\le q/(dH)+n/d}1\\
&\ll H^{-1}qd(q)+nd(q)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{q/H<m\le q-1}\frac{(m+n,q)^{1/2}}{m} &= \sum_{d|q}d^{1/2}\sum_{\substack{q/H<m\le q-1\\ d|(m+n)}}\frac{1}{m}\\
&= \sum_{d|q}d^{1/2}\sum_{\frac{q}{Hd}+\frac{n}{d}<m\le\frac{q-1+n}{d}}\frac{1}{dm-n}\\
&\ll \sum_{d|q}d^{-1/2}\sum_{\frac{q}{Hd}+\frac{n}{d}<m\le\frac{q-1+n}{d}}\frac{1}{m}\left(1+\frac{n}{dm}\right)\\
&\ll d(q)\log H+nd(q),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{q^3}\sum_{m\le q}\sum_{t\le H}e\left(-\frac{mt}{q}\right)K_\chi(m+n,-m;q)h(-q,k;A)h(-q,0;B)\\
&\quad\ll nA^{k+1}Bq^{-3/2}d^2(q)+A^{k+1}Bq^{-3/2}d^2(q)\log H. \qquad (3.1)
\end{aligned}$$

Applying Lemma 1 and Lemma 2 again, we obtain

$$\begin{aligned}
&\frac{1}{q^3}\sum_{r\le q-1}\sum_{m\le q}\sum_{t\le H}e\left(-\frac{mt}{q}\right)K_\chi(m+r+n,-m;q)h(-r,k;A)h(-q,0;B)\\
&\quad\ll \frac{1}{q^3}\sum_{r\le q-1}\sum_{m\le q}\min\left(H,\left\|\frac{m}{q}\right\|^{-1}\right)\left|K_\chi(m+r+n,-m;q)\right|\cdot\left|h(-r,k;A)\right|\cdot\left|h(-q,0;B)\right|\\
&\quad\ll HBq^{-5/2}d(q)\sum_{r\le q-1}\sum_{m\le q/H}(m+r+n,-m,q)^{1/2}\cdot\frac{A^k}{|\sin(\frac{\pi r}{q})|}\\
&\qquad + Bq^{-3/2}d(q)\sum_{r\le q-1}\sum_{q/H<m\le q-1}\frac{(m+r+n,-m,q)^{1/2}}{m}\cdot\frac{A^k}{|\sin(\frac{\pi r}{q})|}
\end{aligned}$$

$$
\begin{aligned}
&\ll HA^kBq^{-3/2}d(q)\sum_{r\le q-1}\frac{1}{r}\sum_{m\le q/H}(m+r+n,-m,q)^{1/2}\\
&\quad + A^kBq^{-1/2}d(q)\sum_{r\le q-1}\frac{1}{r}\sum_{q/H<m\le q-1}\frac{(m+r+n,-m,q)^{1/2}}{m}.
\end{aligned}
$$

Combining

$$
\begin{aligned}
&\sum_{r\le q-1}\frac{1}{r}\sum_{m\le q/H}(m+r+n,-m,q)^{1/2}\\
&\quad=\sum_{d|q}d^{1/2}\sum_{\substack{r\le q\\ d|(r+n)}}\frac{1}{r}\sum_{\substack{m\le q/H\\ d|m}}1\\
&\quad=\sum_{d|q}d^{1/2}\sum_{(n+1)/d\le r\le(q+n-1)/d}\frac{1}{dr-n}\sum_{m\le q/(Hd)}1\\
&\quad\ll q/H\sum_{d|q}d^{-3/2}\sum_{(n+1)/d\le r\le(q+n-1)/d}\frac{1}{r}\left(1+\frac{n}{dr}\right)\\
&\quad\ll H^{-1}qd(q)\log\left(\frac{q+n-1}{n+1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{r\le q-1}\frac{1}{r}\sum_{q/H<m\le q-1}\frac{(m+r+n,-m,q)^{1/2}}{m}\\
&\quad=\sum_{d|q}d^{1/2}\sum_{\substack{r\le q-1\\ d|(r+n)}}\frac{1}{r}\sum_{\substack{q/H<m\le q-1\\ d|m}}\frac{1}{m}\\
&\quad=\sum_{d|q}d^{-1/2}\sum_{(n+1)/d\le r\le(q+n-1)/d}\frac{1}{dr-n}\sum_{q/(Hd)<m\le q/d}\frac{1}{m}\\
&\quad\ll(\log H)\sum_{d|q}d^{-3/2}\sum_{r\le q/d}\frac{1}{r(1-\frac{n}{dr})}\\
&\quad\ll d(q)(\log H)\log\left(\frac{q+n-1}{n+1}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{q^3}\sum_{r\le q-1}\sum_{m\le q}\sum_{t\le H}e\left(-\frac{mt}{q}\right)K_\chi(m+r+1,-m;q)h(-r,k;A)h(-q,0;B)\\
&\quad\ll A^kBq^{-1/2}d^2(q)(\log H)\log\left(\frac{q+n-1}{n+1}\right).
\end{aligned}
\tag{3.2}
$$

Similarly, we get the estimate

$$
\begin{aligned}
&\frac{1}{q^3}\sum_{s\le q}\sum_{m\le q}\sum_{t\le H}e\left(-\frac{mt}{q}\right)K_\chi(m+1,s-m;q)h(-q,k;A)h(-s,0;B)\\
&\quad\ll A^{k+1}q^{-3/2}d^2(q)\log\left(\frac{q-1}{n+1}\right).
\end{aligned}
\tag{3.3}
$$

Noting that

$$\begin{aligned}
&\frac{1}{q^3}\sum_{r,s\le q-1}\sum_{m\le q}\sum_{t\le H} e\left(-\frac{mt}{q}\right)K_\chi(m+r+n,s-m;q)h(-r,k;A)h(-s,0;B)\\
&\ll q^{-5/2}\tau(q)\sum_{r,s\le q-1}\sum_{m\le q}\min\left(H,\left\|\frac{m}{q}\right\|^{-1}\right)\cdot(m+r+n,s-m,q)^{1/2}\frac{A^k}{|\sin(\frac{\pi r}{q})|}\cdot\frac{1}{|\sin(\frac{\pi s}{q})|}\\
&\ll HA^kq^{-1/2}d(q)\sum_{r,s\le q-1}\frac{1}{rs}\sum_{m\le q/H}(m+r+n,s-m,q)^{1/2}\\
&\quad+A^kq^{1/2}d(q)\sum_{r,s\le q-1}\frac{1}{rs}\sum_{q/H<m\le q-1}\frac{(m+r+n,s-m,q)^{1/2}}{m}.
\end{aligned}$$

Using the estimates

$$\begin{aligned}
&\sum_{r,s\le q-1}\frac{1}{rs}\sum_{m\le q/H}(m+r+n,s-m,q)^{1/2}\\
&=\sum_{d\mid q}d^{1/2}\sum_{m\le q/H}\sum_{\substack{r\le q-1\\ d\mid(m+r+n)}}\frac{1}{r}\sum_{\substack{s\le q-1\\ d\mid(s-m)}}\frac{1}{s}\\
&=\sum_{d\mid q}d^{1/2}\sum_{m\le q/H}\sum_{(m+n+1)/d\le r\le(q+m+n-1)/d}\frac{1}{dr-m-n}\sum_{(1-m)/d\le s\le(q-m-1)/d}\frac{1}{ds+m}\\
&\ll H^{-1}qd(q)(\log q)\log(2q+n-1)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{r,s\le q-1}\frac{1}{rs}\sum_{q/H<m\le q-1}\frac{(m+r+n,s-m,q)^{1/2}}{m}\\
&=\sum_{d\mid q}d^{1/2}\sum_{q/H<m\le q-1}\frac{1}{m}\sum_{\substack{r\le q-1\\ d\mid(m+r+n)}}\frac{1}{r}\sum_{\substack{s\le q-1\\ d\mid(s-m)}}\frac{1}{s}\\
&=\sum_{d\mid q}d^{1/2}\sum_{q/H<m\le q-1}\frac{1}{m}\sum_{(m+n+1)/d\le r\le(q+m+n-1)/d}\frac{1}{dr-m-n}\sum_{(1-m)/d\le s\le(q-m-1)/d}\frac{1}{ds+m}\\
&\ll d(q)\left(\log^2 q\right)\log(2q+n-1),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{q^3}\sum_{r,s\le q-1}\sum_{m\le q}\sum_{t\le H} e\left(-\frac{mt}{q}\right)K_\chi(m+r+n,s-m;q)h(-r,k;A)h(-s,0;B)\\
&\ll A^kq^{1/2}d(q)\left(\log^2 q\right)\log(2q+n-1).
\end{aligned}\tag{3.4}$$

Now combining (3.1)-(3.4), we have

$$G_k(\chi,A,B,H;q)\ll A^kq^{1/2}d(q)\left(\frac{nABd(q)}{q^2}+\frac{Bd(q)(\log q)(\log H)}{q}+\log^3 q\right).$$

This completes the proof of the theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DH drafted the manuscript. GR and TZ participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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